

§9 Intersection Theory on Surfaces

① Recall A ring, $M = A$ -mod.

M has finite length ($l_A(M) < \infty$): (\Leftrightarrow)

$$\left[\begin{array}{l} M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_r = 0 \quad (*) \end{array} \right.$$

Chain of proper submodules s.t.

$$\frac{M_i}{M_{i+1}} \cong \frac{A}{m_i} \quad \text{for some } m_i \subset A \text{ max'id}$$

(as A -mod)

$$(\Leftrightarrow M_i/M_{i+1} \text{ is simple})$$

Facts (see e.g. Fulton, Intersection Theory, App A)

i) the number r is independent of the chosen chain $(*)$

We set $l_A(M) := \text{length of } M := r$

ii) If $I \subset A$ ideal and M is an A/I -mod $\Rightarrow l_{A/I}(M) = l_A(M)$
of finite length

iii) A artinian ring, $M = \text{fin gen } A\text{-mod} \Rightarrow l_A(M) < \infty$

iv) $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ exact seq of A -mod
and two of them have finite length,
then so has the third and

$$l_A(M) = l_A(M') + l_A(M'')$$

v) $l(M) < \infty \Rightarrow l_A(M) = \sum_{\mathfrak{p} \in \text{Spec}(A)} l_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$

vi) $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ hom of local Rings

$$d := [B_{\mathfrak{n}} : A_{\mathfrak{m}}] < \infty$$

$M = B$ -mod.

Then $l_A(M) < \infty \Leftrightarrow l_B(M) < \infty$ and

$$l_A(M) = d \cdot l_B(M)$$

vii) $(A, \mathfrak{m}) = 1$ -dim'l local domain

$\tilde{A} =$ integral closure of A in $\text{Frac}(A)$ (assume \tilde{A} fin over A)

$0 \neq a \in A \setminus A^{\times} \Rightarrow \dim A_{(a)} = 0$ by Krull \Rightarrow Artinian.

$$\text{Then } \text{ord}_A(a) := l_A(A_{(a)}) = \sum_{\substack{\mathfrak{n} \in \tilde{A} \text{ max'l} \\ \mathfrak{n} \cap A = \mathfrak{m}}} v_{\mathfrak{n}}(a) [\tilde{A}_{\mathfrak{n}} : A_{\mathfrak{m}}]$$

(see Fulton, Example A3.1)

and $v_{\mathfrak{n}}: K^{\times} \rightarrow \mathbb{Z}$ discrete val assoc to $\mathfrak{n} \in \tilde{A}^{(1)}$

(2) Cor: W integral finite type \mathbb{A}^1 -scheme
 $f \in \mathbb{Z}(W)^{\times}$, $x \in W^{(1)}$

$$\Rightarrow \text{ord}_x(f) = \ell_{\mathcal{O}_{W,x}} \left(\frac{\mathcal{O}_{W,x}}{a} \right) - \ell_{\mathcal{O}_{W,x}} \left(\frac{\mathcal{O}_{W,x}}{b} \right)$$

where $f = \frac{a}{b}$, $a, b \in \mathcal{O}_{W,x} \setminus \{0\}$

Pf: $\bar{W} \rightarrow W$ normal

By def $\text{ord}_x(f) = \sum_{\substack{z \in \bar{W}^{(1)} \\ \gamma \mapsto x}} v_z(f) [\mathbb{Z}(z) : \mathbb{Z}(x)]$

$\Rightarrow \text{OK} \quad \square$
 (7.vii)

In the following X is smooth and equidim'l / \mathbb{Z}

(3) Def: X smooth/ \mathbb{Z}

D, E effective Cartier divisors on X
 $x \in X^{(2)}$ z -codim'l pt (i.e. $\dim \mathcal{O}_{X,x} = 2$)

Assume $\exists \mathcal{Z} \subset X$ open s.t. $\underbrace{|D| \cap |E| \cap \mathcal{Z}}_{\substack{\text{reduced closed subscheme supported on } D \\ \text{is irred with generic pt } x}}$

(we say D and E meet properly at x)

set $i_x(D, E) := \ell_{\mathcal{O}_{X,x}} \left(\frac{\mathcal{O}_{X,x}}{(d, e)} \right)$ where

$d, e \in \mathcal{O}_{X,x}$ are local equations of D, E

(4) Prop: $X \text{ sm } 1/2$, $x \in X^{(2)}$
 E, D eff Cart div meeting properly at x

Then

i) $i_x(D, E) \in \mathbb{N}$ is well-defined
 (i.e. indep of the local eq of D, E at x)

ii) $i_x(D, E) = i_x(E, D)$

iii) D', D'' eff Cart div s.t. D' and E meet properly
 D'' and E

$$\Rightarrow i_x(D' + D'', E) = i_x(D', E) + i_x(D'', E)$$

Pf: i) since $D \cap E \cap \mathcal{U}$ has generic pt x for some open $\mathcal{U} \subset X$
 $\Rightarrow 0 = \dim \frac{\mathcal{O}_{X,x}}{(d, e)} \Rightarrow \frac{\mathcal{O}_{X,x}}{(d, e)}$ Artin
 \Rightarrow finite length

well-defined: \checkmark

ii) \checkmark

(iii) locally $A := \mathcal{O}_{X,x}$ UFD
 (since X sm $\Rightarrow A$ reg)

$$0 \rightarrow \frac{(d''_1, e)}{(d''_1 d'_1, e)} \rightarrow \frac{A}{(d''_1 d'_1, e)} \rightarrow \frac{A}{(d''_1, e)} \rightarrow 0$$

$$\parallel$$

$$\frac{d''_1 A}{d''_1 A \cap (d''_1 d'_1, e)}$$

$$\parallel \longleftarrow A \text{ UFD}$$

and d''_1 and e have no common prime divisor

$$\frac{d''_1 A}{d''_1 \cdot (d'_1, e)}$$

$$\cong \uparrow \cdot d''_1$$

$$\frac{A}{(d'_1, e)}$$

\rightarrow have s.e. \rightarrow

$$0 \rightarrow \frac{A}{(d'_1, e)} \xrightarrow{d''_1} \frac{A}{(d''_1 d'_1, e)} \rightarrow \frac{A}{(d''_1, e)} \rightarrow 0$$

\Rightarrow

$$\ell_A \left(\frac{A}{d''_1 d'_1, e} \right) = \ell_A \left(\frac{A}{d'_1, e} \right) + \ell_A \left(\frac{A}{d''_1, e} \right)$$

(1, iv)

$$i_{\mathbb{C}}(D''_1 + D'_1, E)$$

Def: X sm g

$Z \hookrightarrow X$ integral closed subscheme of codim 1
(prime Weil Divisor)

(Cartier divisor on X)

Assume Z and $|D|$ have no common component

Write $\mathcal{O}_X(D)|_Z := i^*(\mathcal{O}_X(D))$

and $c_1(\mathcal{O}_X(D)|_Z) := v_* c_1(v^*\mathcal{O}_X(D)) \in CH^1(Z)$

where $v: \tilde{Z} \rightarrow Z$ normalization

$c_1: Pic(\tilde{Z}) \rightarrow CH^1(\tilde{Z})$ from §4, (7)

and $v_*: CH^1(\tilde{Z}) \rightarrow CH^1(Z)$ pushforward

Then

$$c_1(\mathcal{O}_X(D)|_Z) = \sum_{x \in Z} i_x(D, Z) [x] \in CH^1(Z)$$

[- c₁]

Pf.:

$D \leftrightarrow \{ (z_i, f_i) \}$ mit

$$X = \bigcup_i z_i, \quad f_i \in$$

$$f_i|_{z_i} \in \mathcal{O}(z_i)^{\times}$$

