§9 Intersecting divisors

(1) Recall A ring, $M = A$-mod.

M has finite length ($\ell_A(M) < \infty$) $:\Longleftrightarrow$

$\exists$

$$M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_r = 0 \quad (*)$$

chain of proper submodules s.t.

$$M_i / M_{i+1} \cong A/m_i \quad \text{for some } m_i < A \text{ max'l}$$

(as A-mod)

$(\Leftrightarrow M_i/M_{i+1}$ is simple$)$

Facts (see e.g. Fulton, Intersection Theory, App A)

i) the number r is independent of the chosen chain $(*)$

We set $\ell_A(M) :=$ length of $M := r$

ii) If $I < A$ ideal and M is an A/I-mod of finite length $\Rightarrow \ell_{A/I}(M) = \ell_A(M)$

iii) A artinian Ring, M = fin gen A-mod $\Rightarrow \ell_A(M) < \infty$

iv) $0 \to M' \to M \to M'' \to 0$ exact seq of A-mod and two of them have finite length, then so has the third and

$$\ell_A(M) = \ell_A(M') + \ell_A(M'')$$

v) $\ell(M) < \infty \Rightarrow \ell_A(M) = \sum_{\mathfrak{p} \in \operatorname{Spec}(A)} \ell_{A_\mathfrak{p}}(M_\mathfrak{p})$

vi) $(A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ hom of local Rings

$$d := [B/\mathfrak{n} : A/\mathfrak{m}] < \infty$$

$M = B$-mod.

Then $\ell_A(M) < \infty \Leftrightarrow \ell_B(M) < \infty$ and

$$\ell_A(M) = d \cdot \ell_B(M)$$

vii) $(A, \mathfrak{m})$ = 1-dim'l local domain

$\tilde{A}$ = integral closure of A in $\operatorname{Frac}(A)$ (assume $\tilde{A}$ fin over A)

$0 \neq a \in A \setminus A^\times \Rightarrow \dim A/(a) = 0$ (by Krull) $\Rightarrow$ Artinian.

Then

$$\operatorname{ord}_A(a) := \ell_A(A/a) = \sum_{\substack{\mathfrak{n} \subset \tilde{A} \text{ max'l} \\ \mathfrak{n} \cap A = \mathfrak{m}}} v_\mathfrak{n}(a) \, [\tilde{A}/\mathfrak{n} : A/\mathfrak{m}]$$

(see Fulton, Example A 3.1)

and $v_\mathfrak{n} : K^\times \to \mathbb{Z}$ discrete val assoc to $\mathfrak{n} \in \tilde{A}^{(1)}$

② Cor: W integral finite type k-scheme
$f \in k(W)^\times$, $x \in W^{(1)}$

$$\Rightarrow \quad \mathrm{ord}_x(f) = \ell_{\mathcal{O}_{W,x}}\left(\mathcal{O}_{X,x}/a\right) - \ell_{\mathcal{O}_{W,x}}\left(\mathcal{O}_{X,x}/b\right)$$

where $f = \frac{a}{b}$, $a, b \in \mathcal{O}_{W,x} \setminus \{0\}$

Pf: $\widetilde{W} \to W$ normal

By def $\mathrm{ord}_x(f) = \sum_{\substack{y \in \widetilde{W}^{(1)} \\ y \mapsto x}} v_y(f) \, [k(y) : k(x)]$

$\overset{}{\Rightarrow}$ OK (⑦, vii) □

In the following X is smooth and equidim'l $/k$

③ Def: X smooth$/k$

D, E effective Cartier divisors on X
$x \in X^{(2)}$ 2-codim'l pt (i.e. $\dim \mathcal{O}_{X,x} = 2$)

Assume $\exists_{x \in} \mathcal{U} \subset X$ open s.t. $|D| \cap |E| \cap \mathcal{U}$ is irred with generic pt x

$|D|$: reduced closed subscheme supported on D

(we say D and E meet properly at x)

set $\quad i_x(D, E) := \ell_{\mathcal{O}_{X,x}}\left(\dfrac{\mathcal{O}_{X,x}}{(d, e)}\right)$ where

$d, e \in \mathcal{O}_{X,x}$ are local equations of D, E

④ **Prop:** X sm $/k$, $x \in X^{(2)}$

E, D eff Cart div meeting properly at x

Then

i) $i_x(D, E) \in \mathbb{N}$ is well-defined (i.e. indep of the local eq of D, E at x)

ii) $i_x(D, E) = i_x(E, D)$

iii) D', D'' eff Cart div s.t. D' and E meet properly, D'' and E

$\Rightarrow$ $i_x(D'+D'', E) = i_x(D', E) + i_x(D'', E)$

Pf: i) Since $D \cap E \cap \mathcal{U}$ has generic pt x for some open $\underset{\ni x}{\mathcal{U}} \subset X$

$\Rightarrow$ $0 = \dim \dfrac{\mathcal{O}_{X,x}}{(d,e)}$ $\Rightarrow$ $\dfrac{\mathcal{O}_{X,x}}{(d,e)}$ Artin
$\Rightarrow$ finite length

well-defined: ✓

ii) ✓

(iii) locally $A := \mathcal{O}_{X,x}$ UFD
(since X sm $\Rightarrow A$ reg)

$$0 \to \frac{(d'',e)}{(d'd'',e)} \to \frac{A}{(d'd'',e)} \longrightarrow \frac{A}{(d'',e)} \to 0$$

$$\frac{(d'',e)}{(d'd'',e)} = \frac{d''A}{d''A \cap (d''d',e)} = \frac{d''A}{d''\cdot(d',e)} \underset{\cdot d''}{\overset{\uparrow}{\cong}} A/(d',e)$$

A UFD and d'' and e have no common prime divisor

$\to$ have s.e.s

$$0 \to \frac{A}{(d',e)} \xrightarrow{d''} \frac{A}{(d'd'',e)} \longrightarrow \frac{A}{(d'',e)} \to 0$$

$\Rightarrow$

$$\ell_A\left(\frac{A}{d'd'',e}\right) = \ell_A\left(A/_{d',e}\right) + \ell_A\left(A/_{d'',e}\right)$$

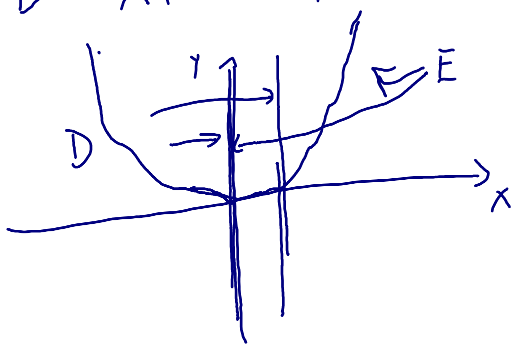
(1), iv) $\quad i_x(D'+D'',E) \qquad i_x(D',E) \qquad i_x(D'',E)$ □

(5) Example

$X = \mathbb{A}^2_k \ni O = (x, y)$

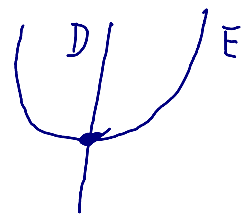
$= \operatorname{Spec} k[x,y]$

a) $D = x(x-1)$, $E = x(y-x^2)$



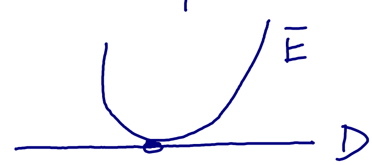
$\Rightarrow$ D and E don't meet properly at 0

b) $D = x$, $E = y - x^2$



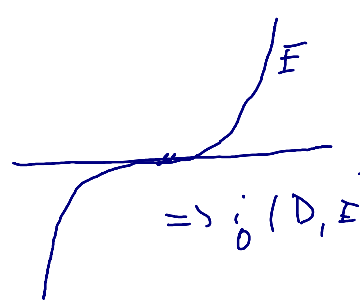
$\Rightarrow i_0(D, E) = 1$

c) $D = y$, $E = y - x^2$



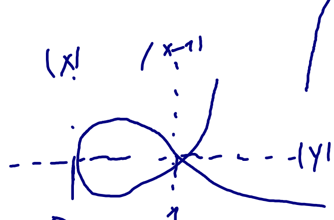
$\Rightarrow i_0(D, E) = 2$

d) $D = y$, $E = y - x^5$



$\Rightarrow i_0(D, E) = 5$

e) $D = y^2 - x(x-1)^2$



$i_{(0,0)}(D, (x)) = 2$, $i_{(1,0)}(D, (x-1)) = 2$, $i_{(0,0)}(D, (y)) = 1$

$i_{(1,0)}(D, (y)) = 2$

6 Cor: X sm $/k$

$Z \overset{i}{\hookrightarrow} X$ integral closed subscheme of codim 1

(prime Weil Divisor)

D eff Cartier divisor on X

Assume Z and $|D|$ have no common component

Write $\mathcal{O}_X(D)|_Z := i^*(\mathcal{O}_X(D))$

and $c_1(\mathcal{O}_X(D)|_Z) := \nu_* \, c_1(\nu^* \mathcal{O}_X(D)) \in CH^1(Z)$

where $\nu: \tilde{Z} \to Z$ normalization

$c_1: \mathrm{Pic}(\tilde{Z}) \to CH^1(\tilde{Z})$ from §4, 7

and $\nu_*: CH^1(\tilde{Z}) \to CH^1(Z)$ pushforward

Then

$$c_1(\mathcal{O}_X(D)|_Z) = \sum_{x \in Z^{(1)}} i_x(D, Z) \, [x] \in CH^1(Z)$$

$[Z^{(1)} \subset X^{(2)}]$

Pf:

$$D \leftrightarrow (\{U_i\}, f_i) \text{ with}$$

eff Cart $\qquad X = \bigcup_i U_i \ , \ f_i \in \mathcal{O}(U_i) \setminus \{0\}$

$$f_i / f_j \in \mathcal{O}(U_{ij})^\times$$

and assume $Z \cap U_i = \operatorname{Spec}(\mathcal{O}_{U_i} / \mathcal{I}_i)$

$|D|$ and Z no common cpt

$$\Rightarrow \quad \mathcal{O}_X(U_i) \longrightarrow \mathcal{O}_Z(U_i \cap Z) = \mathcal{O}(U_i) / \mathcal{I}_i$$

$$f_i \longmapsto \overline{f_i} \neq 0$$

and $\quad c_1(\mathcal{O}_X(D)|_Z) \,|\, U_i$

given by $\displaystyle \sum_{x \in (Z \cap U_i)^{(1)}} \Big(\underbrace{\sum_{\substack{y \in \tilde{Z}^{(1)} \\ y \mapsto x}} v_y(\overline{f_i}) \, [\kappa(y) : \kappa(x)]}_{= \operatorname{ord}_x(\overline{f_i})} \Big) \cdot [x]$

$$= \operatorname{ord}_x(\overline{f_i}) \underset{(2)}{=} \ell_{\mathcal{O}_{Z,x}}\left(\frac{\mathcal{O}_{Z,x}}{\overline{f_i}} \right) \underset{(1), ii}{=} \ell_{\mathcal{O}_{X,x}}\left(\frac{\mathcal{O}_{X,x}}{(f_i, \mathcal{I}_i)} \right) = i_x(D, Z)$$

$\square$

(7) **Def.:** X sm/k

$D \in Z^1(X)$, $\quad Z \overset{i}{\hookrightarrow} X$ integral, $\operatorname{codim}(Z, X) = 1$

define
$$D \cdot Z := i_*\Big(c_1\big(\mathcal{O}_X(D)|_Z\big)\Big) \in CH^2(X)$$

→ extend. to

$$Z^1(X) \times Z^1(X) \longrightarrow CH^2(X)$$
$$\Big(D, \alpha = \sum n_i [Z_i]\Big) \longmapsto D \cdot \alpha := \sum n_i \, D \cdot Z_i$$

(8) Prop: X sm/k

1) the pairing from (7) induces well defined morph.

$$CH^1(X) \times CH^1(X) \longrightarrow CH^2(X)$$
$$(D, E) \longmapsto D \cdot E$$

2) $D \cdot E = E \cdot D$

3) $(D+D') \cdot E = D \cdot E + D' \cdot E$

4) D, E eff without common cpt's ($\Leftrightarrow$ the meet properly at all $x \in |D| \cap |E|$)

$$\Rightarrow \quad D \cdot E = \sum_{x \in |D| \cap |E| \cap X^{(2)}} i_x(D, E) \cdot [x]$$

Pf.: $D \in Z^1(X)$, $Z \subset X$ prim div.

$D \sim D'$, i.e. $D = D' + \mathrm{div}(f)$, $f \in K(X)^\times$

$\Rightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$

$\Rightarrow \mathcal{O}_X(D)|_Z \cong \mathcal{O}_X(D')|_Z$

$\Rightarrow D.Z = i_* c_1(\mathcal{O}_X(D)|_Z) = i_* c_1(\mathcal{O}_X(D')|_Z) = D'.Z$ in $CH^2(X)$

$\Rightarrow$ get well defined map

$$CH^1(X) \times Z^1(X) \longrightarrow CH^2(X)$$

bilinear: . linear in second variable by defn
. linear in first variable by
$\mathcal{O}_X(D+D') = \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$
and $c_1 : \mathrm{Pic}(Z) \to CH_0(Z)$ gp hom.
and i_* also. $(\Rightarrow 3))$

2) Take

$D = \sum m_i D_i$, D_i integral

$E = \sum n_j E_j$, E_j integral

in $Z^1(X)$

$$\Rightarrow D \cdot E \underset{\text{(bilinearity)}}{=} \sum_{i,j} m_i n_j \; D_i \cdot E_j$$

if $D_i = E_j \Rightarrow D_i \cdot E_j = E_j \cdot D_i$

if $D_i \neq E_j \Rightarrow D_i \cdot E_j \underset{(6)}{=} \sum_{x \in |D_i| \cap |E_j| \cap X^{(2)}} i_x(D_i, E_j) \; [x]$

$$= \sum_{x \in |E_j| \cap |D_i| \cap X^{(2)}} i_x(E_j, D_i) \cdot [x]$$

$$\underset{(6)}{=} E_j \cdot D_i$$

$\Rightarrow D \cdot E = E \cdot D \Rightarrow 2) \Rightarrow 1)$

4) $D, E \geq 0$, $E = \sum n_i E_i$

no common cpts

$$\Rightarrow D \cdot E = \sum n_i \; D \cdot E_i$$

$$\underset{(6)}{=} \sum n_i \sum_{x \in |D| \cap |E_i| \cap X^{(2)}} i_x(D, E_i) \; [x]$$

$$= \sum_{x \in |D| \cap |E| \cap X^{(2)}} \Big(\sum_{\substack{i \\ \text{with } x \in |E_i|}} n_i \, i_x(D, E_i) \Big) \; [x]$$

$$\underset{(4), iii), ii)}{=} \sum_{x \in |D| \cap |E| \cap X^{(2)}} i_x(D, E) \cdot [x]$$

$\square$

(9) Def: X sm, proper surface $/k$
$\uparrow$
$\dim X = 2$

$\pi : X \longrightarrow \operatorname{Spec} k$ str. map.

$$CH^1(X) \times CH^1(X) \longrightarrow CH^2(X) \underset{\dim X = 2}{=} CH_0(X) \xrightarrow{\pi_* = \deg} \mathbb{Z}$$

Set $(E \cdot D) := \pi_*(E \cdot D) = \deg(E \cdot D)$

Intersection number.

$\ulcorner$ if $|E|$ and $|D|$ have no common cpt

$$\Rightarrow \quad E \cdot D = \sum_{x \in |E| \cap |D| \cap X^{(2)}} i_x(E, D)\,[x]$$

$$\Rightarrow (E \cdot D) = \sum_{x \in |E| \cap |D| \cap X_{(0)}} i_x(E, D)\,[k(x) : k]$$ $\lrcorner$

(10) Cor (Bezout's Thm)

$X = \mathbb{P}^2_k$, $C_i = V(F_i)$
with $F_i \in k[X, Y, Z]$ hom of deg d_i $i = 1, 2$

Then $(C_1 \cdot C_2) = d_1 \cdot d_2$

If $k = \bar{k}$ and C_1, C_2 have no common cpt
this means
$$d_1 d_2 = \sum_{x \in |C_1| \cap |C_2|} i_x(C_1, C_2)$$
i.e. C_1 and C_2 intersect at $d_1 d_2$ many pts (counted with multipl.)

Pf: $C_1 \sim d_1 H_1$ $H_1 = V(X)$ $H_1 \cap H_2 = 0 \in \mathbb{P}^1_k$
$C_2 \sim d_2 H_2$ $H_2 = V(Y)$

$\Rightarrow (C_1 \cdot C_2) = d_1 d_2 (H_1 \cdot H_2) = d_1 d_2$ $\square$