§8 Rationality of the zeta function and functional equation

In this section we fix the following notation

- $k = \mathbb{F}_q$, $q = p^r$
- C smooth proj., geom. connected curve $/\mathbb{F}_q$

Recall:

$$Z(C/\mathbb{F}_q, t) := \exp\left(\sum_{n=1}^{\infty} |C(\mathbb{F}_{q^n})| \frac{t^n}{n}\right)$$

$$= \prod_{\substack{x \in C \\ \text{closed}}} \frac{1}{1-t^{\deg(x)}}$$

$$= 1 + \sum_{d \geq 1} b_d t^d \quad \in 1 + t\mathbb{Z}[\![t]\!]$$

where $b_d = |\{\mu \in Z_0(C)^{\mathrm{eff}} \mid \deg \mu = d\}|$

(see §2, ③)

① Notation

set $m := \gcd\{[\kappa(x):\kappa] \mid x \in C \text{ closed}\}$

i.e. $m\mathbb{Z} = ([\kappa(x):\kappa] \mid x \in C \text{ closed})$

$= \deg(CH_0(C))$

Note if $C(\mathbb{F}_q) \neq \emptyset \Rightarrow m = 1$

(we will see m is always $= 1$)

② Prop Assumptions and Notations as above

$$\Rightarrow \quad Z(C/\mathbb{F}_q, t) = \frac{f(t)}{(1-t^m)(1-q^m t^m)}$$

where $f \in \mathbb{Q}[t]$ with $f(0) = 1$

and $\deg f \le 2g-2+2m$

We need some preliminaries before we start the proof.

③ Lemma:

Fix $D \in Z^1(C) = Z_0(C)$

$\deg D = n$

$$\Rightarrow \Big| \{ \underset{\substack{\updownarrow \\ D \geq 0}}{D' \in Z^1(C)^{\text{eff}}} \mid \underset{\substack{\updownarrow \\ D' = D + \mathrm{div}(f)}}{D' \sim D} \} \Big| = \frac{q^{\ell^0(D)} - 1}{q - 1}$$

$[\ell^0(D) = \dim_k H^0(C, \mathcal{O}_C(D))]$

Pf: $0 \leq D' \sim D \iff 0 \leq D' = D + \mathrm{div}(f)$, $f \in k(C)^\times$

and $\mathrm{div}(f) = \mathrm{div}(g) \iff \mathrm{div}(f/g) = 0$

$\iff f/g \in H^0(C, \mathcal{O}_C) = \mathbb{F}_q$

↑ (C geom. conn.

(see pf of §7.②)

Thus

$$\Big| \{ D' \in Z^1(C) \mid D' \sim D \} \Big| = \Big| \{ f \in K(C)^\times \mid \mathrm{div}(f) \geq -D \} / \mathbb{F}_q^\times \Big|$$

$$H^0(C, \mathcal{O}_C(D)) = \{ f \in K(C)^\times \mid \mathrm{div}(f) \geq -D \} \cup \{0\}$$

$\Rightarrow$ □

④ Lemma: $CH_0(C)^0 := \operatorname{Ker}(\deg : CH_0(C) \longrightarrow \mathbb{Z})$ is finite

Pf: Fix $D \in Z^1(C) = Z_0(C)$ with $\deg D = d \geq 2g$

Take $\alpha \in CH_0(C)^0$, $\tilde{\alpha} \in Z_0(C)$ rep of α, $\deg(\tilde{\alpha}) = 0$

$\Rightarrow \deg(\tilde{\alpha} + D) = d \geq 2g \geq 2g-1$

$\underset{§7, (15)}{\Longrightarrow} \quad h^0(\tilde{\alpha}+D) = 1 - g + d \geq 1+g$

$\Rightarrow \quad \exists_{0 \neq} f \in H^0(C, \mathcal{O}(\tilde{\alpha}+D))$

Thus $\underbrace{\tilde{\alpha} + D + \operatorname{div}(f)}_{\deg(\ ) = d} \geq 0$

$\Rightarrow$ get surjection

$$\underbrace{Z_0(C)^{\mathrm{eff}}_{\deg = d}}_{\text{finite}} \twoheadrightarrow CH_0(C)^0, \quad \beta \mapsto \beta - D$$

$\square$

Proof of Prop 2:

$$Z(C,t) = 1 + \sum_{d \geq 1} b_d t^d \quad , \quad b_d = \left| Z_0(C)^{\text{eff}}_{\deg = d} \right|$$

Have $m\mathbb{Z} = \deg\left(CH_0(C)\right)$

$$\Rightarrow Z(C,t) = 1 + \sum_{d \geq 1} b_{md}\, t^{md}$$

Fix $D_d \in Z_0(C)^{\text{eff}}$ with $\deg D_d = md$ (exists)

$$\Rightarrow Z_0(C)^{\text{eff}}_{\deg = md} = \coprod_{\alpha \in CH_0(C)^0} \left\{ \beta \in Z_0(C)^{\text{eff}} \mid \beta = D_d + \alpha \text{ in } CH_0(C) \right\}$$

$$\underset{③}{\Rightarrow} \quad b_{md} = \sum_{\alpha \in CH_0(C)^0} \frac{q^{h^0(D_d + \alpha)} - 1}{q - 1} \qquad (*)$$

if $md \geq 2g-1 \underset{§7,(15)}{\Longrightarrow} \ell^0(\alpha + D_d) = 1-g+md$

$$\Rightarrow \quad b_{md} = \underbrace{|\,CH_0(C)^0\,|}_{=:A} \frac{q^{1-g+md}-1}{q-1}$$

Note $\deg(CH_0(C)^0) = m\mathbb{Z} \Rightarrow m \mid \underset{\text{can. div}}{\deg K_C} = 2g-2 \quad (§7,(13))$

$$\Rightarrow Z(C,t) = \underbrace{1 + \sum_{d=1}^{\frac{2g-2}{m}} b_{md}\, t^{md}}_{\substack{\ell(t) \in \mathbb{Z}[t] \\ \text{of } \deg \leq 2g-2}} + \underbrace{\sum_{d > \frac{2g-2}{m}} A \frac{q^{1-g+md}-1}{q-1} t^{md}}_{=:\, h(t) \in \mathbb{Z}[[t]]}$$

$\ell(0) = 1$

$$(1-t^m)(1-q^m t^m)\cdot h(t) = \frac{A}{q-1}\, t^{2g-2}(1-t^m)(1-q^m t^m) \sum_{\ell \geq 1} \left(q^{g-1+m\ell}-1\right) t^{m\ell}$$

$$\left[\ell = d - \frac{2g-2}{m} \Rightarrow 1-g+md = g-1+m\ell\right]$$

$$= \frac{A}{q-1}\, t^{2g-2}\Big((1-t^m)\, q^{g-1} \underbrace{(1-(qt)^m) \sum_{\ell \geq 1} (qt)^{m\ell}}_{q^m t^m} - (1-q^m t^m) \underbrace{(1-t^m) \sum_{\ell \geq 1} t^{m\ell}}_{t^m} \Big)$$

$$= \frac{A}{q-1}\, t^{2g-2}\Big((1-t^m)\, q^{g-1} (qt)^m - (1-q^m t^m)\, t^m \Big) \in \mathbb{Q}[t]$$

$$\deg(\;) = 2g-2+2m, \qquad = 1 \text{ at } t=0$$

$$\Rightarrow (1-t^m)(1-q^m t^m)\, Z(C,t) = \underbrace{(1-t^m)(1-q^m t^m)\,(\ell(t)+h(t))}_{=f(t)} \in \mathbb{Q}[t]$$

of deg $\leq 2g-2+2m$

$$f(0) = 1 \qquad \square$$

5 **Lemma** K = field, X = smooth proj curve $/K$
dim $X = 1$

$L \in \mathrm{Pic}(X)$, $\deg L = 0$

Then: either $L \cong \mathcal{O}_X$

or $H^0(X, L) = 0$

Pf: Assume $H^0(X, L) \neq 0$

Write $L \cong \mathcal{O}_X(D)$ (see §5, 9)

with $D \in Z^1(X)$ and $\deg D = 0$

$$\Rightarrow \exists\ f \in K(X)^\times \text{ s.t. } \left.\begin{array}{l} \mathrm{div}(f) + D \geq 0 \\ \text{and } \deg(\mathrm{div}(f) + D) = 0 \end{array}\right\}$$

$$\Rightarrow \mathrm{div}(f) + D = 0 \Rightarrow L \cong \mathcal{O}_X \qquad \square$$

6 Thm (Function equation)

$$Z\left(C, \frac{1}{qt}\right) = q^{1-g}\, t^{2-2g}\, Z(C,t)$$

Pf: We have seen (see Pf of Prop 2)

$$Z(C,t) = 1 + \sum_{d=1}^{\frac{2g-2}{m}} b_{md}\, t^{md} + \frac{A}{q-1}\, t^{2g-2}\left(\frac{q^{g-1}(qt)^m}{1-(qt)^m} - \frac{t^m}{1-t^m}\right)$$

and

$$b_{md} = \sum_{\alpha \in CH_0(C)^0} \frac{q^{h^0(D_d+\alpha)} - 1}{q-1} \qquad \text{(see (*))}$$

$\deg D_d = md$

5 $\Rightarrow$ $\underset{\alpha \in CH_0(C)^0}{h^0(\alpha)} = \begin{cases} 0 & , \alpha \neq 0 \\ 1 & , \alpha = 1 \end{cases}$

set $D_0 := 0$

$$\Rightarrow \quad 1 = \sum_{\alpha \in CH_0(C)^0} \frac{q^{h^0(D_0+\alpha)} - 1}{q-1}$$

$$\Rightarrow \quad (q-1)\, Z(C,t) = \underbrace{\sum_{d=0}^{\frac{2g-2}{m}} \sum_{\alpha \in CH_0(C)^0} q^{\ell^0(D_d+\alpha)}\, t^{md}}_{=: F(t)} - \underbrace{\sum_{d=0}^{\frac{2g-2}{m}} \underbrace{\Big(\sum_{\alpha \in CH_0(C)^0} 1\Big)}_{=A}\, t^{md}}_{A\,\frac{1-t^{2g-2+m}}{1-t^m}}$$

$$+ A\, t^{2g-2}\, \frac{q^{g-1}(qt)^m}{1-(qt)^m} - A\, \frac{t^{2g-2+m}}{1-t^m}$$

$$= F(t) + \underbrace{A\left(t^{2g-2}\, \frac{q^{g-1}(qt)^m}{1-(qt)^m} - \frac{1}{1-t^m}\right)}_{=: R(t)}$$

<u>Check directly</u>

$$R\left(\tfrac{1}{qt}\right) = q^{1-g}\, t^{2-2g}\, R(t)$$

It remains to show function eq for F:

$$F(t) = \sum_{d=0}^{\frac{2g-2}{m}} \sum_{\alpha \in CH_0(C)^0} q^{h^0(\alpha + D_d)} t^{md}$$

$$= \sum_{d=0}^{\frac{g-1}{m}-1} \sum_{\alpha} q^{h^0(\alpha + D_d)} t^{md} + \overbrace{\sum_{\alpha \in CH_0(C)^0} q^{h^0(\alpha + D_{\frac{g-1}{m}})} t^{g-1}}^{\text{only appears if } m \mid g-1}$$

$$+ \sum_{d=\frac{g-1}{m}+1}^{\frac{2g-2}{m}} \sum_{\alpha} q^{h^0(K_C - \alpha - D_d)} q^{1-g} (qt)^{md}$$

$$\left[RR: \quad h^0(\alpha + D_d) = h^0(K_C - \alpha - D_d) + 1 - g + md \right]$$

$$\deg(K_C - \alpha - D_d) = 2g-2-md = md' \quad \text{with } d' = \frac{2g-2-md}{m}$$
$d \in [\frac{g-1}{m}+1, \frac{2g-2}{m}]$ $\qquad \in [0, \frac{g-1}{m}-1]$

$$K_C - \alpha - D_d = \alpha' + D_{d'} \qquad , \alpha' \in CH_0(C)^0$$

$$\Rightarrow F(t) = \sum_{d=0}^{\frac{g-1}{m}-1} \sum_{\alpha \in CH_0(C)^0} q^{h^0(\alpha + D_d)} \underbrace{\left(t^{md} + q^{1-g} (qt)^{2g-2-md} \right)}_{F_d(t)}$$

$$+ \sum_{\alpha \in CH_0(C)^0} q^{h^0(\alpha + D_{\frac{g-1}{m}})} \underbrace{t^{g-1}}_{= F_{\frac{g-1}{m}}(t)}$$

- $$F_d(t) = t^{md} + q^{g-1} \frac{t^{2g-2}}{(qt)^{md}}$$

$$\Rightarrow F_d\left(\frac{1}{qt}\right) = \frac{1}{(qt)^{md}} + q^{g-1} \frac{(t)^{md}}{(qt)^{2g-2}}$$

$$= \frac{1}{q^{g-1}} \frac{1}{t^{2g-2}} \underbrace{\left[q^{g-1} \frac{t^{2g-2}}{t^{md}} + t^{md} \right]}_{= F_d(t)} \quad \rightarrow OK$$

- $$F_{\frac{g-1}{m}}\left(\frac{1}{qt}\right) = \frac{1}{q^{g-1}} \frac{1}{t^{g-1}} = \frac{1}{q^{g-1}} \frac{1}{t^{2g-2}} F_{\frac{g-1}{m}}(t) \quad \rightarrow OK \qquad \square$$

(7) Main Thm $C / \mathbb{F}_q$ as above

Set $a_n := 1 + q^n - |C(\mathbb{F}_{q^n})|$

$$\Longrightarrow \quad |a_n| \leq 2g\sqrt{q^n}$$

i.e. $1+q^n - 2g\sqrt{q^n} \leq |C(\mathbb{F}_{q^n})| \leq 2g\sqrt{q^n} + 1 + q^n$

We will prove this theorem in the remainder of the course

Assuming (7) we get the Weil conjectures for curves:

⑧ Thm (Weil conj., see §2) $C/\mathbb{F}_q$ as above

(0) $m=1$, i.e. $\deg(CH_0(C)) = \mathbb{Z}$

i.e. $\exists\ \alpha \in Z_0(C)$ with $\deg(\alpha) = 1$

(but maybe $C(\mathbb{F}_q) = \emptyset$)

(1) $$Z(C,t) = \frac{f(t)}{(1-t)(1-qt)}$$

$f \in \mathbb{Z}[t]$, $\deg f(t) = 2g$, $f(0) = 1$

(2) $Z(C, 1/qt) = q^{1-g}\, t^{2-2g}\, Z(C,t)$

(3) Write $f(t) = \prod_{i=1}^{2g} (1-\alpha_i t)$, $\alpha_i \in \mathbb{C}$

$\Rightarrow \quad |\alpha_i| = \sqrt{q}$

<u>Proof of (8) assuming (7):</u> (2) $\Leftarrow$ (6)

set $H(t) := (1-t)(1-qt)\, Z(C,t)$

$\underset{(6)}{\Rightarrow}$ $H\left(\frac{1}{qt}\right) = \left(1-\frac{1}{qt}\right)\left(1-\frac{1}{t}\right) t^{2-2g} q^{1-g}\, Z(C,t)$

$= q^{-g} t^{-2g}\, H(t)$

(2) $\Rightarrow$ $H(t) \in \mathbb{Q}(t)$, in part it is a meromorphic fct on $\mathbb{C}$

and $\frac{H'(t)}{H(t)} = -\frac{1}{1-t} - \frac{q}{1-qt} + \frac{Z'(C,t)}{Z(C,t)}$

$\frac{Z'(C,t)}{Z(C,t)} = \frac{d\log}{dt}\left(\exp \sum_{n\geq 1} |C(\mathbb{F}_{q^n})| \frac{t^n}{n}\right)$

$= \sum_{n\geq 1}^{\infty} |C(\mathbb{F}_{q^n})|\, t^{n-1}$

$= -\sum_{n=1}^{\infty} a_n t^{n-1}$ a_n as in (8)

For $|t| < q^{-1/2} \underset{(7)}{\Rightarrow} |a_n t^{n-1}| \leq 2g\sqrt{q^n}\,|t|^{n-1}$

$$= 2g\sqrt{q}\,\underbrace{(\sqrt{q}\,|t|)}_{<1}^{n-1}$$

$\Rightarrow \dfrac{H'(t)}{H(t)}$ converges for $|t| < q^{-1/2}$

$\Rightarrow H(t)$ has no poles or zeros for $|t| < q^{-1/2}$

$H\left(\frac{1}{qt}\right) = q^{-g} t^{-2g} H(t)$

$|t| > \frac{1}{\sqrt{q}} \Rightarrow \frac{1}{|qt|} < \frac{1}{\sqrt{q}}$

$\Rightarrow H(t)$ has no poles or zeros in $|t| > q^{-1/2}$

$\Rightarrow Z(C, t) = \dfrac{H(t)}{(1-t)(1-qt)}$ has at most a simple pole at 1 and q^{-1} and all other poles or zeros are in $|t| = \frac{1}{\sqrt{q}}$

Pf of ⑥ $\Longrightarrow$

$$(q-1)\,Z(C,t) = \sum_{d=0}^{\frac{2g-2}{m}} \sum_{\alpha \in CH_0(C)^0} q^{\ell^0(D_d+\alpha)}\, t^{md}$$

$$+ A\left(\frac{1}{t^{-m}-1} + \frac{(qt)^m}{1-(qt)^m}\, q^{g-1} t^{2g-2} \right)$$

$\Longrightarrow$ $m=1$ and $H(t) \in \mathbb{Q}[t] \cap \mathbb{Z}[[t]] = \mathbb{Z}[t]$

and $H\left(\frac{1}{qt}\right) = q^{-g} t^{-2g} H(t)$ $\Rightarrow$ $\deg H(t) = 2g$

$\Rightarrow$ $H(t) = f(t)$ from ⑦

$(\Rightarrow f(0)=1)$

Write $f(t) = \prod (1-\alpha_i t)$

$\Rightarrow$ $\frac{1}{\alpha_i}$ is a root of $f(t) = H(t)$

$\underset{\text{see above}}{\Rightarrow}$ $\left|\frac{1}{\alpha_i}\right| = \frac{1}{\sqrt{q}}$ $\Rightarrow$ $|\alpha_i| = \sqrt{q}$ $\square$

It remains to prove Thm ⑦. First some preliminaries...