§7 Riemann-Roch

① Recall (e.g. from "coh of sheaves on schemes")

X scheme, $\mathcal{F}$ sheaf on X of ab gps

$\rightsquigarrow$ $H^i(X, \mathcal{F}) = R^i\Gamma(X, -)(\mathcal{F})$

i-th cohomology gp of $\mathcal{F}$

- functorial in $\mathcal{F}$
- l.e.s.

- k = field, X proj $/k$, $\mathcal{F}$ coherent $\mathcal{O}_X$-mod

$\Longrightarrow$ $H^i(X, \mathcal{F})$ is a finite dim'l k-vsp $\forall i \geq 0$

and $= 0$ $\forall$ $i > \dim X$

② Def: X proj$/k$, $\mathcal{F}$ coh on X

$$\chi(X/k, \mathcal{F}) := \sum_{i=0}^{\dim X} \underbrace{\dim_k}_{k\text{-v.sp-dim}} H^i(X, \mathcal{F})$$

③ Lemma: X proj$/k$

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ s.e.s of coherent $\mathcal{O}_X$-mod

$$\Rightarrow \chi(X/k, \mathcal{F}) = \chi(X/k, \mathcal{F}') + \chi(X/k, \mathcal{F}'')$$

Pf: follows from the associated long exact cohomology sequence and the following lemma:

(4) Lemma: Let

$$0 \to V_0 \xrightarrow{d^0} V_1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} V_n \to 0$$ be an

exact sequence of k-v.sp.'es

$$\Rightarrow \sum_{i=0}^{n} (-1)^i \dim_k V_i = 0$$

Pf: $n=0$: ✓, $n=1$ ✓. Assume $n \geq 2$.

$n-1 \to n$:

$$0 \to V_0 \to \dots \to \operatorname{Im} d^{n-2} \to 0$$

$$0 \to \operatorname{Im} d^{n-2} \to V_{n-1} \xrightarrow{d^{n-1}} V_n \to 0$$

$$\text{Ind} \Rightarrow 0 = \sum_{i=0}^{n-2} (-1)^i \dim_k V_i + (-1)^{n-2} \dim_k \operatorname{Im} d^{n-2}$$

$$\dim_k \operatorname{Im} d^{n-2} \underset{\text{dim formula}}{=} \dim_k V_n - \dim_k V_{n-1}$$

$\Rightarrow$ OK $\square$

(5) Def: k field, C smooth proj geometrically connected curve $/k$

$g = g(C)$:= genus of C

$:= \dim_k H^1(C, \mathcal{O}_C)$

(6) Ex:

i) $g(\mathbb{P}^1_k) = 0$ (see CSS, §8)

ii) an elliptic curve $/k$ is a pair $(E, 0)$

where E is a smooth proj curve $/k$ of genus $g(E) = 1$

and $0 \in E(k)$ is a fixed rational points

(These are exactly the curves s.t.

$Sch_k \ni T \longmapsto E(T) = \mathrm{Hom}_k(T, E)$ is a functor into ab gps (with 0 inducing the neutral elt)

(7) Lemma: $F \in k[X_0, X_1, X_2]$ form of degree d

$C := V_+(F) \subset \mathbb{P}^2_k$ smooth, geom. conn.

$$\Rightarrow \quad g(C) = \frac{(d-1)(d-2)}{2}$$

Pf.: $P := \mathbb{P}^2_k$

$$\mathcal{O}_P(-C) \cong \mathcal{O}_P(-d) \qquad (\text{cf. } §5, (10))$$

$\overset{\text{s.e.s}}{\underset{§5, (11)}{\Longrightarrow}}$ $0 \to \mathcal{O}_P(-d) \longrightarrow \mathcal{O}_P \longrightarrow i_* \mathcal{O}_C \to 0$

$\underset{\text{l.e.s.}}{\Longrightarrow}$ $H^1(P, \mathcal{O}_P) \longrightarrow H^1(P, i_* \mathcal{O}_C) \longrightarrow H^2(P, \mathcal{O}(-d)) \longrightarrow H^2(P, \mathcal{O}_P)$

$H^1(P, \mathcal{O}_P) = 0$ (CSS, §8), $\quad H^1(P, i_*\mathcal{O}_C) = H^1(C, \mathcal{O}_C)$, $\quad H^2(P, \mathcal{O}_P) = 0$

$$H^2(P, \mathcal{O}(-d)) \overset{\text{CSS, §8}}{=} \left(\frac{1}{X_0X_1X_2} k\left[\frac{1}{X_0}, \frac{1}{X_1}, \frac{1}{X_2}\right]\right)_{-d} \cong k[Y_0, Y_1, Y_2]_{d-3}$$

has dim $\dfrac{(d-1)(d-2)}{2}$ $\square$

8 Thm (Riemann-Roch, preliminary version)

k field, C smooth proj geometrically connected curve $/k$ ($C_k^{\otimes \bar{k}}$ is con)

L inv. sheaf on C

$$\Rightarrow \chi(C/_k, L) = \dim_k H^0(C, L) - \dim_k H^1(C, L)$$

$$= \deg L - g(C) + 1$$

($\deg L$ defined as in §6, 11)

Pf: Write $L \cong \mathcal{O}_C(D)$ with D a Weil divisor

$$\Rightarrow \chi(C/_k, L) = \chi(C/_k, \mathcal{O}_C(D))$$

1. case: $D = 0$, i.e. $L = \mathcal{O}_C$

$\Rightarrow \deg L = 0$

$\dim_k H^0(C, \mathcal{O}_C) = 1$:

indeed: we know

$H^0(C, \mathcal{O}_C)$ is a k-subalgebra of $k(C)$ which is finite dim'l over k

$\Rightarrow H^0(C, \mathcal{O}_C)$ is an integral Artin ring$/k$

i.e. $H^0(C, \mathcal{O}_C) = K$ with K/k fin field ext

$$\Rightarrow K \otimes_k \bar{k} = H^0(C, \mathcal{O}_C) \otimes_k \bar{k} = H^0(C_{\bar{k}}, \mathcal{O}_{C_{\bar{k}}}) \underset{\text{(see above } C_{\bar{k}} \text{ geom con)}}{=} \bar{k}$$

$\Rightarrow K = k$

Thus

$$\chi(C/k, \mathcal{O}_C) = 1 - g$$

<u>2. case</u> $D = -A$ with A effective (i.e. $A = \sum n_i P_i$, $P_i \in C$ closed pt, $n_i \geq 0$)

$\Rightarrow$ A defines a closed subscheme of dim 0 and we obtain a s.e.s

$$0 \to \mathcal{O}_C(-A) \to \mathcal{O}_C \to i_* \mathcal{O}_A \to 0$$

(see §5, (21))

$$\underset{(3)}{\Longrightarrow} \quad \chi(C/k, L) = \chi(C/k, \mathcal{O}_C(-A))$$

$$= \chi(C/k, \mathcal{O}_C) - \chi(C/k, i_* \mathcal{O}_A)$$

$$\underset{1.\text{case}}{=} 1 - g - \dim_k H^0(A, \mathcal{O}_A)$$

$A \subset C$ is the subscheme

$$A = \coprod_i n_i P_i \quad \text{where } n_i P_i = \operatorname{Spec} \frac{\mathcal{O}_{C,P_i}}{m_{P_i}^{n_i}} \subset C$$

$$\Rightarrow \dim_k H^0(A, \mathcal{O}_A) = \sum_i \dim_k \left(\frac{\mathcal{O}_{C,P_i}}{m_{P_i}^{n_i}} \right)$$

$$\dim_k \left(\frac{\mathcal{O}_{C,P_i}}{m_{P_i}^{n_i}} \right) = n_i \, [k(P_i) : k]$$

since: $\dim_k \left(\frac{\mathcal{O}_{C,P_i}}{m_{P_i}} \right) = [k(P_i) : k]$ by defn.

$$0 \to \frac{m_{P_i}^{n_i-1}}{m_{P_i}^{n_i}} \to \frac{\mathcal{O}_{C,P_i}}{m_{P_i}^{n_i}} \to \frac{\mathcal{O}_{C,P_i}}{m_{P_i}^{n_i-1}} \to 0$$

$m_{P_i} = t\, \mathcal{O}_{C,P_i}$

$$\frac{m_{P_i}^{n_i-1}}{m_{P_i}^{n_i}} = \frac{t^{n_i-1}\mathcal{O}_{C,P_i}}{t^{n_i}\mathcal{O}_{C,P_i}} \cong \frac{\mathcal{O}_{C,P_i}}{m_{P_i}}$$

$$t^{n_i-1}\tilde{a} \longleftarrow \bar{a}$$

$$\underset{\text{ind.}}{\Longrightarrow} \quad \dim_k \frac{\mathcal{O}_{C,P_i}}{m_{P_i}^{n_i}} = [k(P_i):k] + (n_i-1)\,[k(P_i):k]$$

$\Rightarrow$ $\chi(C_{/k}, \mathcal{O}_C(-A)) = 1 - g - \sum n_i [k(P_i):k]$

$= 1 - g + \deg D$

$D = -A$

OK

<u>3. case</u> D = Weil div.

$D = A - B$, A, B effective

$0 \to \mathcal{O}_C(-A) \to \mathcal{O}_C \to i_* \mathcal{O}_A \to 0$ $(*)$ exact.

Have $(i_* \mathcal{O}_A) \otimes_{\mathcal{O}_C} \mathcal{O}_C(A) \cong i_* \mathcal{O}_A$

[<u>since</u> $A = \sum n_j \cdot P_j$, $P_j \in C$ closed.

$\Rightarrow$ $i_* \mathcal{O}_A = \bigoplus i_{j*}\left(\mathcal{O}_{C,P_j}/m_{P_j}^{n_j}\right)$ $\leftarrow$ supported at P_j

$i_j : \operatorname{Spec} \mathcal{O}_{C,P_j} \to C$]

and $$j_{j*}\left(\frac{\mathcal{O}_{C,p_j}}{m_j^{n_j}}\right) \otimes_{\mathcal{O}_C} \mathcal{O}_C(A) = i_{j*}\left(\frac{\mathcal{O}_{C,p_j}}{m_j^{n_j}} \otimes_{\mathcal{O}_{C,p_j}} \mathcal{O}_{C,p_j}(A)\right)$$

$$\cong i_{j*}\left(\frac{\mathcal{O}_{C,p_j}}{m_j^{n_j}}\right)$$

$\llcorner$ $\lrcorner$

$\Longrightarrow$ exact seq.
$i_* \otimes \mathcal{O}_C(A)$

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(A) \to i_* \mathcal{O}_A \to 0$$

$\Longrightarrow$ $\otimes\, \mathcal{O}_C(-B)$ (s.o.) $$0 \to \mathcal{O}_C(-B) \to \mathcal{O}_C(D) \to i_* \mathcal{O}_A \to 0$$

$\Longrightarrow$ (3) $$\chi(C/_k, \mathcal{O}_C(D)) = \chi(C/_k, \mathcal{O}_C(-B)) + \dim_k H^0(A, \mathcal{O}_A)$$

$$= 1 - g - \deg B + \deg A$$

2. case

$$= 1 - g + \deg D$$

$\square$

(9) Excursion (Kähler differentials, see e.g. CSS):

k field, A k-algebra.

$$\Omega^1_{A/k} = \frac{\{\text{free } A\text{-mod gen by } da,\ a \in A\}}{R}$$

$R =$ A-mod gen by

- $d\lambda$, $\lambda \in k$
- $d(a+b) - da - d(b)$
- $a\,db + b\,da - d(ab)$

$\Rightarrow$ $\Omega^1_{A/k}$ is an A-mod and

$d: A \to \Omega^1_{A/k}$, $a \mapsto da$ is a k-linear map

(and it is a derivation)

(i) Have $B \overset{\varphi}{\twoheadrightarrow} A$ with kernel I
$/k$

$\Rightarrow$ $I/I^2 \to A \otimes_B \Omega^1_{B/k} \twoheadrightarrow \Omega^1_{A/k} \to 0$ exact seq.

$z \mapsto d_c$ $\qquad b_0\, db_1 \longmapsto \varphi(b_0)\, d\varphi(b_1)$

(ii) $\Omega^1_{k[x_1, \dots, x_n]/k} \cong \bigoplus_{i=1}^{n} k[x_1, \dots, x_n]\, dx_i$

(iii) $A = \dfrac{k[x_1, \dots, x_n]}{(f_1, \dots, f_s)}$

$\underset{(i)\,(ii)}{\Rightarrow}$ $\Omega^1_{A/k} = \dfrac{\bigoplus_{i=1}^{n} k[x_1, \dots x_n]\, dx_i}{\langle df_1, \dots, df_s \rangle}$

where $df_i = \sum_{j=1}^{n} \dfrac{\partial f_i}{\partial x_j}\, dx_j$

(iv) X finite type k-scheme

define $\Omega^1_{X/k}$ as sheaf assoc to

$$\mathcal{U} \longrightarrow \Omega^1_{\mathcal{O}(\mathcal{U})/k}$$

this is a coherent sheaf on X

with $\Gamma(\operatorname{Spec} A, \Omega^1_{X/k}) = \Omega^1_{A/k}$

with $\operatorname{Spec} A \subset X$ open.

(v) X/k smooth of dim n

$\Rightarrow$ $\Omega^1_{X/k}$ is locally free of rk n

(this is a consequence of Jacobi crit. an (iii) above)

10 The canonical sheaf on a smooth curve

C/k smooth curve $/k$

set $\omega_{C/k} := \Omega^1_{C/k}$

. it is locally free of rank 1

(see 9 (v))

$\Rightarrow \omega_{C/k} \cong \mathcal{O}_C(K_C)$

where K_C is a Weil divisor
called a "canonical divisor of C/k"

- Note K_C is only well-defined in $CH^1(C)$
  i.e. up to rational equiv, (i.e.
  $\forall f \in k(C)$: $K_C + \mathrm{div}_C f$ is also a canonical divisor)

- Note $\deg K_C = \deg \omega_{C/k}$ is well-defined

(11) Thm (Serre duality for curves)

C smooth proj curve $/k$

E locally free sheaf of finite rk on C

set $E^{\vee} := \mathcal{H}om_{\mathcal{O}_C}(E, \mathcal{O}_C)$

Then there is a canonical isomorphism.

$$H^0(C, E^{\vee} \otimes \omega_{C/k}) \cong \mathrm{Hom}_k(H^1(C, E), k)$$

Pf:
- see Ha, III, § 7, CSS
- Serre, Algebraic curves and class field theory

(12) Cor (final version of RR)

C smooth proj. geom conn. $/k$

L inv on C $\Rightarrow$

$$\dim_k H^0(C, L) - \dim_k H^0(C, L^{\vee} \otimes \omega_C) = 1 - g + \deg L$$

(12') Cor: D Weil div. on $C_{/k}$ as above, K_C can div

$$h^0(D) := \dim_k H^0(C, \mathcal{O}_C(D))$$

$$\Rightarrow \quad h^0(D) - h^0(K_C - D) = 1 - g + \deg D$$

(13) Cor: $\deg K_C = 2g - 2$

Pf: $h^0(K_C) \underset{SD}{=} h^1(K_C - K_C) = h^1(\mathcal{O}_C) = g$

$$\underset{(12'),\, D = K_C}{\Rightarrow} \quad \underbrace{h^0(K_C)}_{= g} - \underbrace{h^0(\mathcal{O})}_{= 1} = 1 - g + \deg K_C$$

$\square$

(14) Lemma: X smooth proper$/k$

L inv $\mathcal{O}_X$-mod, $\deg L < 0$

$$\Rightarrow \quad H^0(X, L) = 0$$

Pf: $L = \mathcal{O}_X(D)$

$f \in H^0(C, \mathcal{O}_X(D))$

$\iff \quad \mathrm{div}(f) \geq -D$

$\Rightarrow \quad \underbrace{\deg(\mathrm{div}(f))}_{0} \geq \underbrace{-\deg D > 0}_{0}$ ↯

15 Cor: C sm proj, geom conn $/k$, $g = g(C)$

D Weil divisor

Assume $\deg D \geq 2g - 1 \Rightarrow$

$h^0(D) = 1 - g + \deg D \geq g$

Pf: $\deg(K_C - D) = 2g - 2 - (2g-1) = -1 < 0$

$\Rightarrow h^0(K_C - D) = 0$ $\square$