§6 Chow groups and proper pushforward

k = field, $X/_k$ variety

(i.e. X is separated and of finite type $/k$)

We assume throughout that

X is equidimensional of dim n,

i.e., any irreducible cpt $X_1 \subset X$ has $\dim X_1 = n$

① Def:

a) $Z^i(X)$ = free abelian group generated by irred closed subsets $Z \subset X$ with $\operatorname{codim}(Z, X) = i$

$\iff \dim Z = \dim X - i = n - i$

(under our general assumptions)

$$= \bigoplus_{x \in X^{(i)}} \mathbb{Z} \cdot [x]$$

b) $Z_j(X)$ = free ab group gen by irred closed subsets $Z \subset X$ with $\dim Z = j$

$$= Z^{n-j}(X)$$

c) W irred integral variety
$f \in \mathcal{R}(W)^\times$

define
$$\mathrm{div}_W(f) \in Z^1(W)$$
as follows:
$\nu : \hat{W} \to W$ normalization
[if $W = \operatorname{Spec} A \Rightarrow \hat{W} = \operatorname{Spec} \hat{A}$ with $\hat{A}$ = integral closure of A in $\mathcal{R}(W)$]
ν is a finite map (i.e. $\hat{A}$ is a finite A-mod)

for $x \in W^{(1)}$

we have $\nu^{-1}(x) = \{y_1, \dots, y_r\} \subset \tilde{W}^{(1)}$

fin many pts.

set $\operatorname{ord}_x(f) := \sum_{y \in \nu^{-1}(x)} [k(y) : k(x)]\, v_y(f)$

[Note $\tilde{W}$ normal $\Rightarrow \mathcal{O}_{\tilde{W},y}$ DVR and v_y is the assoc. discrete val.]

We define

$$\operatorname{div}_W(f) := \sum_{x \in W^{(1)}} \operatorname{ord}_x(f)\, [x]$$

d) $CH^i(X) = \operatorname{coker}\left(\bigoplus_{w \in X^{(i-1)}} k(w)^\times \xrightarrow{\oplus \operatorname{div}_W} \bigoplus_{x \in X^{(i)}} \mathbb{Z} \right)$

[Note $W \subset X$ codim $i-1$

$\Rightarrow |\operatorname{div}_W(f)| \underset{\substack{\uparrow \\ \text{codim } 1}}{\subseteq} W \subset X$ has codim i in X]

② Ex: i) $X = \bigcup_{i=1}^{n} X_i$

X_i irred , $\dim X_i = n$

$$\Rightarrow CH^0(X) = CH_n(X) = \bigoplus_{i=1}^{n} \mathbb{Z} \cdot [X_i]$$

ii) $CH_0(X) = CH^n(X)$

$$= \bigoplus_{\substack{x \in X \\ \text{closed pts}}} \mathbb{Z}\,[x] \Big/ \langle \operatorname{div}_C f \;|\; C \subset X \text{ curve irred} \rangle$$

iii) $r > n$ or $r < 0$ $\Rightarrow CH_r(X) = 0 = CH_r(X)$

iv) $CH^1(\mathbb{A}^n) = 0$, $CH^1(\mathbb{P}^n) = \mathbb{Z}$

(see § 6)

③ Def:

$f : X \longrightarrow Y$

proper morphism

between equidimensional varieties

Let $Z \subset X$ irred, closed, $\dim Z = j$

$\rightsquigarrow [Z] \in Z_j(X)$

set

$$f_*[Z] := \begin{cases} 0 & , \text{ if } \dim f(Z) \neq j \\ [k(Z) : k(f(Z))] \cdot [f(Z)], & \text{if } \dim f(Z) = j \end{cases}$$

Note: f proper $\Rightarrow f(Z) \subset Y$ irred closed.

$\dim f(Z) = \dim(Z) \Rightarrow \operatorname{trdeg}_k k(f(Z)) = \operatorname{trdeg}_k k(Z)$

$\Longrightarrow k(Z)/k(f(Z))$ fin. field extension.

$\rightarrow$ Def makes sense.

We can extend f_* linearly to obtain a group homomorphism

$$f_* : Z_j(X) \longrightarrow Z_j(Y)$$
$$\qquad \quad \shortparallel$$
$$Z^{n-j}(X) \longrightarrow Z^{m-j}(Y) \qquad m = \dim Y$$

$$\sum n_i [Z_i] \longmapsto \sum n_i\, f_*[Z_i]$$

④ Lemma: $X \xrightarrow{f} Y \xrightarrow{g} Z$ proper maps between equidim'l k-var

$$\Rightarrow \quad g_* f_* = (g\circ f)_* : Z_j(X) \to Z_j(Z)$$

Pf: $W \subset X$ irred closed, $\dim W = j$

1. case $\underline{\dim W > \dim f(W)} \Rightarrow \dim W > \dim gf(W)$

$$\Rightarrow \quad g_* \underbrace{f_* W}_{=0} = 0 = (g\circ f)_* W$$

2. case $\underline{\dim W = \dim f(W) > \dim g f(W)}$: same

3. case: dim W = dim $f(W)$ = dim $gf(W)$.

$$g_* f_* [W] = g_* \left([\mathcal{R}(W) : \mathcal{R}(f(W))] \cdot [f(W)] \right)$$

$$= [\mathcal{R}(W) : \mathcal{R}(f(W))] \, [\mathcal{R}(f(W)) : \mathcal{R}(gf(W))] \cdot [gf(W)]$$

$$= (g \circ f)_* [W] \qquad \square$$

(5) Recollection of the norm:

L/K finite field extension

$f \in L$

$\mu_f : L \xrightarrow{\cdot f} L$ is a K-linear endomorphism of the fin. dim'l K-vsp L

def $\quad \mathrm{Nm}_{L/K}(f) := \det(\mu_f) \in K$

⑥ <u>Properties:</u> (see e.g. Bosch, Algebra)

(i) $L/K/E$: $\mathrm{Nm}_{L/E} = \mathrm{Nm}_{K/E}\,\mathrm{Nm}_{L/K}$

(ii) $L = K(\alpha)$, $f = X^n + a_{n-1}X^{n-1} + \dots + a_0$ $\in K[X]$ minimal poly of α

$$\Rightarrow \quad \mathrm{Nm}_{L/K}(\alpha) = (-1)^n a_0$$

⌈ $1, \alpha, \dots, \alpha^{n-1}$ is a basis of L/K ⌉

$\rightarrow$ matrix of μ_α in this basis

$$\begin{pmatrix} 0 & 0 & & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & & & \vdots \\ \vdots & 0 & & 0 & \\ 0 & \vdots & & 1 & -a_{n-1} \end{pmatrix} \rightarrow \det(\,) = (-1)^n a_0$$ ⌋

⌊

(iii) $a \in K \Rightarrow \mathrm{Nm}_{L/K}(a) = a^{[L:K]}$.

(iv) $L \underbrace{\supset}_{\substack{\text{purely}\\ \text{insep of}\\ \text{degree } p^r}} K' \underbrace{\supset}_{\substack{\text{sep}\\ \text{degree } n_s}} K$ $\qquad \sigma_1, \dots, \sigma_{n_s} : K^{sep} \hookrightarrow \overline{K}$

all embeddings

$$\begin{tikzcd} K^{sep} \arrow[r, hook] & \overline{K} \\ & K \arrow[u] \arrow[ul] \end{tikzcd}$$

$$\Rightarrow \quad \mathrm{Nm}_{L/K}(f) = \prod_{i=1}^{n_s} \sigma_i(f)^{p^r}$$

7 Lemma: A finite type k-algebra integrally closed (normal) in $K = \mathrm{Frac}(A)$

B integrally closed k-algebra
$L = \mathrm{Frac}(B)$

$A \hookrightarrow B$ inclusion of k-algebras s.t. the induced field extension L/K is finite

$\mathfrak{p} \subset A$ prime of ht 1 $\Rightarrow A_{\mathfrak{p}}$ = regular 1-dim'l local ring with valuation
$v_{\mathfrak{p}} : K \to \mathbb{Z} \cup \{\infty\}$

For $\mathfrak{q} \subset B$ height 1 with $\mathfrak{q} \cap A = \mathfrak{p}$

write

$e(\mathfrak{q}/\mathfrak{p}) := v_{\mathfrak{q}}(t)$, $t \in \mathfrak{p}A_{\mathfrak{p}}$ generator.

$f(\mathfrak{q}/\mathfrak{p}) := [k(\mathfrak{q}) : k(\mathfrak{p})]$ $\quad k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$

$n = [L:K]$

Then

i) $B \otimes_A A_\mathfrak{p}$ is finite and free of rank n over $A_\mathfrak{p}$ and there are only fin. many $\mathfrak{q} \subset B$ with $\mathfrak{q} \cap A = \mathfrak{p}$

ii) $$n = \sum_{\mathfrak{q}/\mathfrak{p}} e(\mathfrak{q}/\mathfrak{p})\, f(\mathfrak{q}/\mathfrak{p})$$

(iii) $b \in L^\times$

$$\Rightarrow \quad v_\mathfrak{p}\left(Nm_{L/K}(b)\right) = \sum_{\mathfrak{q}/\mathfrak{p}} f(\mathfrak{q}/\mathfrak{p})\, v_\mathfrak{q}(b)$$

Pf: See Serre, Local Fields

(i) $\Leftarrow$ LF, I, §4, Prop 8, Rmk 2), Prop 9

(ii) $\Leftarrow$ LF, I, §4, Prop 10

(iii) $\Leftarrow$ LF, I, §5, Prop 14

⑧ **Prop:** $f : X \to Y$ proper k-morph surj.

X, Y normal varieties/k

$L = k(X)$, $K = k(Y)$

$g \in L^\times$

Then

$$f_*[\mathrm{div}_X g] = \begin{cases} 0 & \dim Y \neq \dim X \\ \mathrm{div}_Y\left(\mathrm{Nm}_{L/K}(g)\right), & \dim Y = \dim X \end{cases}$$

[Note $\dim Y = \dim X \Rightarrow L/K$ fin field ext]

Pf: 1. case $\dim Y = \dim X$

Take $y \in Y^{(1)}$ $\quad Y_{(y)} := \mathrm{Spec}\, \mathcal{O}_{Y,y}$

define $X_{(y)}$ by the cartesian square

$$\begin{tikzcd} X_{(y)} \arrow[r] \arrow[d, "f_{(y)}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y_{(y)} \arrow[r] & Y \end{tikzcd}$$

(7) $\Rightarrow$ $f_{(y)}$ is finite

the coeff of $[y]$ in $f_*[\mathrm{div}_X(g)]$ is

$$\sum_{\substack{x \in X_{(y)}^{(1)} \\ x/y}} v_x(g)\,[k(x):k(y)]$$

and by (7), (iii) this is also the coefficient of $[y]$ in $\mathrm{div}_Y(\mathrm{Nm}_{L/K}(g))$

$\Rightarrow$ 1-case

2. case $\quad \dim X \neq \dim Y$

i.e. $\quad \dim X > \dim Y$

$x \in X^{(1)}$

if $\quad \dim \bar{x} > \dim \overline{f(x)}$

$\Rightarrow f_* [x] = 0$ by defn

if $\dim(\bar{x}) = \dim(\overline{f(x)})$

$$\Rightarrow \underset{\substack{\wedge\!\!\backslash \\ \dim Y}}{\operatorname{trdeg}_k k(f(x))} = \operatorname{trdeg}_k(k(x)) = n-1 \geq \dim Y$$

$\Rightarrow \quad \dim Y = n-1$ $^{(*)}$ and $\quad f(x) =$ generic pt of Y

Thus $\quad f_* : Z^1(X) \longrightarrow Z^0(Y) \cong \mathbb{Z}$

$$\{ n_i [x_i] \} \longmapsto \{ \underset{\substack{i \text{ with} \\ x_i \mapsto \text{generic pt} \\ \text{of } Y}}{n_i} [k(x_i) : K] \}$$

$\xi \in Y$ generic pt

$$\begin{tikzcd} X_\xi \arrow[r] \arrow[d, "f_\xi"'] & X \arrow[d, "\text{proper.}"] \\ \operatorname{Spec} K = \xi \arrow[r] & Y \end{tikzcd}$$

$$\{x \in X^{(1)} \mid f(x) = \xi\} = X_\xi^{(1)}$$

and $\operatorname{trdeg}_K L = 1$ $(*)$

$\Rightarrow$ $X_\xi \to \xi$ proper curve.

and it suff to show

$$f_{\xi *}\left(\operatorname{div}_{X_\xi}(g)\right) = 0$$

if $g \in L$ is algebraic $/K$ $\Rightarrow$ $g \in \mathcal{O}(X_\xi)^\times$
$\Rightarrow \operatorname{div}_{X_\xi}(g) = 0$

$\Rightarrow$ we can assume
$g \in L$ is transcendent$/_K$

$\Rightarrow$ get $\quad X_\xi \xrightarrow{\mu} \mathbb{P}^1_\xi \xrightarrow{\pi} \xi \qquad (\ast)$

defined by $\quad Z_+ = \{x \in X^{(1)}_\xi \mid v_x(g) > 0\}$

closed pts

$Z_- = \{x \in X^{(1)}_\xi \mid v_x(g) < 0\}$

$\Rightarrow \quad g \in \Gamma(X_\xi \setminus Z_-, \mathcal{O}_{X_\xi})$

and $\quad K[t] \longrightarrow \Gamma(X_\xi \setminus Z_-, \mathcal{O}_{X_\xi})$

$t \longmapsto g$

induces $\quad X_\xi \setminus Z_- \to \mathbb{A}^1_K$

$\frac{1}{g} \in \Gamma(X_\xi \setminus Z_+, \mathcal{O}_{X_\xi})$

and $\quad K[1/t] \to \Gamma(X_\xi \setminus Z_+, \mathcal{O}_{X_\xi})$

$1/t \longmapsto 1/g$

induces $\quad X_\xi \setminus Z_+ \longrightarrow \mathbb{P}^1_K \setminus \{0\} \ \underset{\text{glue}}{\Rightarrow} (\ast)$

$$(4) \Rightarrow \quad f_{\zeta *} \operatorname{div}_{X_\zeta}(g) = \pi_* \gamma_* \operatorname{div}_{X_\zeta}(\gamma^* t)$$

⌜ $x \in X_\zeta^{(1)}$, $z := \gamma(x) \in \mathbb{P}^1_\zeta$

$\Rightarrow$ coeff of $\gamma_* \operatorname{div}_{X_\zeta}(\gamma^* t)$ in front of z

is $\displaystyle\sum_{x/z} [k(x) : k(z)] \underbrace{v_x(\gamma^* t)}_{= e(x/z)\, v_z(t)} \overset{(7)}{=} n\, v_z(t)$

$\Rightarrow \gamma_* \operatorname{div}_{X_\zeta}(\gamma^* t) = n \operatorname{div}_{X_\zeta}(t)$ ⌟

$$\Rightarrow f_{\zeta *} \operatorname{div}_{X_\zeta}(g) = n\, \pi_* \operatorname{div}_{\mathbb{P}^1_\zeta}(t)$$

$$= n\, \pi_*([0] - [\infty])$$

$$= n([\zeta] - [\zeta]) = 0$$

$\square$

(9) Thm: $f : X \to Y$ proper between k-var.

then $f_* : Z_j(X) \to Z_j(Y) \to CH_j(Y)$

factors via to give a well-defined group hom

$$f_* : CH_j(X) \to CH_j(Y)$$

[or if X equi-dim'l of dim $X = n$
Y —— " —— $\dim Y = m$

$\Rightarrow f_* : CH^{n-j}(X) \to CH^{m-j}(Y)$]

functorial as in (4).

Pf - let $W \subset X$ irred of dim $W = j+1$
$g \in k(W)$

We have to show: $f_*(\mathrm{div}_W(g)) = 0$ in $CH_j(Y)$

$\nu : \tilde{W} \to W$ normalization

Recall $\quad \mathrm{div}_W(g) = \sum_{x \in W^{(1)}} \mathrm{ord}_x(g)\,[x]$

and $\quad \mathrm{ord}_x(g) = \sum_{\substack{\tilde{x} \in \tilde{W}^{(1)} \\ \tilde{x}/x}} [\kappa(\tilde{x}) : \kappa(x)]\, v_{\tilde{x}}(g)$

$$\Rightarrow \quad \mathrm{div}_W(g) = \nu_* \,\mathrm{div}_{\tilde{W}}(g)$$

$V := f(W) \subset Y$ closed integral subscheme

(use red scheme str. on the induced closed subset)

$\hat{V} \to V$ normalization

⇝ obtain

$$\begin{tikzcd}
\tilde{W} \arrow[r, "\tilde{\nu}"] \arrow[d, "\hat{f}_W"] & W \arrow[r, hook] \arrow[d, "f_W"] & X \arrow[d, "f"] \\
\hat{V} \arrow[r, "\mu"'] & V \arrow[r, hook, "i"'] & Y
\end{tikzcd}$$

(use (4))

$$f_* \,\mathrm{div}_W(g) = i_* f_{W*} \,\mathrm{div}_W(g) = i_* f_{W*} \nu_* \,\mathrm{div}_{\tilde{W}}(g)$$
$$= i_* \mu_* \hat{f}_{W*} \,\mathrm{div}_{\tilde{W}}(g)$$

(8) $\Longrightarrow$ $$\tilde{f}_{W*}(\mathrm{div}_{\tilde{W}}(g)) = \begin{cases} 0 & \dim \tilde{W} > \dim \tilde{V} \\ \mathrm{div}_{\tilde{V}}(\mathrm{Nm}(g)), & \text{else} \end{cases}$$

$$\Rightarrow \quad f_*(\mathrm{div}_W(g)) = \begin{cases} 0 & \dim W > \dim V \\ \mathrm{div}_V(\mathrm{Nm}(g)) & \dim W = \dim V \end{cases}$$

$$= 0 \text{ in } CH_j(Y)$$

$\square$

<u>(10) Cor-Def:</u> X proper variety $/k$, $\pi : X \to \operatorname{Spec} k$

$$\deg := \pi_* \quad CH_0(X) \longrightarrow CH_0(\operatorname{Spec} k) = \mathbb{Z}$$

$$\alpha = \sum n_i [x_i] \longmapsto \deg(\alpha) = \pi_*(\alpha) = \sum n_i \cdot [k(x_i) : k]$$

well defined.

(11) Def: C sm proper curve $/k$

(i.e. C variety of pure dim 1 and is proper and smooth)

L inv sheaf on C

We define the $\deg(L) \in \mathbb{Z}$ as follows:

$$\mathrm{Pic}(C) \underset{c_1}{\overset{\S 5}{\xrightarrow{\sim}}} CH^1(C) = CH_0(C) \xrightarrow{\deg} \mathbb{Z}$$

$\Rightarrow \deg L := \deg(c_1(L))$

Clearly if $L = \mathcal{O}_C(D)$, $D = \sum n_i x_i$

$\Longrightarrow \deg L = \deg D = \sum n_i [k(x_i):k]$

(12) Rmk: • By defn: $L \cong L' \Rightarrow \deg L = \deg L'$

• $C = \mathbb{P}^1_k \Rightarrow \deg \mathcal{O}_{\mathbb{P}^1}(r) = r$