

§ 6 Chow groups and proper pushforward

$k = \text{field}$, X/k variety
(i.e. X is separated and of finite type / k)

We assume throughout that

X is equidimensional of dim n ,
i.e., any irreducible cpt $X_1 \subset X$
has $\dim X_1 = n$

① Def:

a) $Z^i(X) =$ free abelian group generated
by irred closed subsets $Z \subset X$
with $\text{codim}(Z, X) = i$

$\Leftrightarrow \dim Z = \dim X - i = n - i$
(under our general assumptions)

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$$= \bigoplus_{x \in X^{(i)}} \mathbb{Z} \cdot [x]$$

b) $Z_j(X) =$ free ab group gen
by irred closed subsets
 $Z \subset X$ with $\dim Z = j$

$$= Z^{n-j}(X)$$

c) W irred integral variety
 $f \in \mathbb{Z}(W)^{\times}$

define

$$\operatorname{div}_W(f) \in Z^1(W)$$

as follows:

$$\nu: \widehat{W} \rightarrow W \text{ normalization}$$

$$\Gamma \text{ if } W = \operatorname{Spec} A \Rightarrow \widehat{W} = \operatorname{Spec} \widetilde{A}$$

with $\widetilde{A} =$ integral closure of A in

$\mathbb{Z}(W)$

ν is a finite map (i.e. \widetilde{A} is a finitely A -mod)

for $x \in W^{(1)}$ - 77-

we have $v^{-1}(x) = \{y_1, \dots, y_r\} \subset \tilde{W}^{(1)}$
 fin many pts.

$$\text{set } \text{ord}_x(f) := \sum_{y \in v^{-1}(x)} [\bar{g}(y) : g(x)] v_y(f)$$

Γ Note \tilde{W} normal $\Rightarrow \mathcal{O}_{\tilde{W}, y}$ DVR Γ
 and v_y is the assoc. discrete val. Γ
 \perp

We define

$$\text{div}_W(f) := \sum_{x \in W^{(1)}} \text{ord}_x(f) [x]$$

$$d) \quad CH^i(X) = \text{coker} \left(\bigoplus_{W \in X^{(i-1)}} \mathcal{K}(W)^{\times} \xrightarrow{\oplus \text{div}_W} \bigoplus_{x \in X^{(i)}} \mathbb{Z} \right)$$

Γ Note $W \subset X$ codim $i-1$ Γ

$\Rightarrow \text{div}_W(f) \subset W \subset X$ has codim i in X Γ
 \perp

② \mathbb{F}_X : i) $X = \bigcup_{i=1}^r X_i$

$$X_i \text{ indd, } \dim X_i = n$$

$$\Rightarrow CH^0(X) = CH_n(X) = \bigoplus_{i=1}^r \mathbb{Z} \cdot [X_i]$$

ii) $CH_0(X) = CH^n(X)$

$$= \bigoplus_{\substack{x \in X \\ \text{closed pts}}} \mathbb{Z} [x]$$

$\left\langle \begin{array}{l} \text{div}_c f \\ \text{curve} \\ \text{indd} \end{array} \right\rangle \subset \mathbb{C} \subset X$

iii) $r > n \Rightarrow CH_r(X) = 0 = CH_r(X)$
or $r < 0$

iv) $CH^1(\mathbb{A}^n) = 0$, $CH^1(\mathbb{P}^n) = \mathbb{Z}$
(see § 6)

③ Def:

$$f : X \rightarrow Y$$

proper morphism

between equidimensional varieties

Let $Z \subset X$ irred, closed, $\dim Z = j$

$$\rightarrow [Z] \in Z_j(X)$$

$$\text{set } f_* [Z] := \begin{cases} 0, & \text{if } \dim f(Z) \neq j \\ [k(Z) : k(f(Z))] \cdot [f(Z)], & \text{if } \dim f(Z) = j \end{cases}$$

Note: f proper $\Rightarrow f(Z) \subset Y$ irred closed.

$$\dim f(Z) = \dim(Z) \Rightarrow \text{trdeg}_{k_f} k(f(Z)) = \text{trdeg}_k k(Z)$$

$$\Rightarrow k(Z) / k(f(Z)) \text{ fin. field extension.}$$

\rightarrow Def makes sense.

We can extend f_* linearly to obtain a group homomorphism

$$f_* : z_j(X) \longrightarrow z_j(Y)$$

$$z_j(X) \longrightarrow z_j(Y) \quad m = \dim Y$$

$$\{z_i, [z_i]\} \longmapsto \{z_i, f_*[z_i]\}$$

(4) Lemma: $X \xrightarrow{f} Y \xrightarrow{g} Z$ proper maps between equidimensional \mathbb{Z} -var

$$\Rightarrow g_* f_* = (g \circ f)_* : z_j(X) \longrightarrow z_j(Z)$$

Pf: $W \subset X$ irred closed, $\dim W = j$

1. case $\dim W > \dim f(W) \Rightarrow \dim W > \dim g f(W)$

$$\Rightarrow g_* \underbrace{f_* W}_{=0} = 0 = (g \circ f)_* W$$

2. case $\dim W = \dim f(W) > \dim g f(W) : \text{same}$

3. case: $\dim W = \dim f(W) = \dim g(f(W))$.

$$f_* f_* [W] = g_* \left([g(W) : g(f(W))] \cdot [f(W)] \right)$$

$$= [g(W) \cdot g(f(W))] [g(f(W) : g(f(W)))] \cdot [g(f(W))]$$

$$= (g \circ f)_* [W] \quad \square$$

(5) Recollection of the norm:

L/K finite field extension

$f \in L$

$\mu_f: L \xrightarrow{ef} L$ is a K -linear endomorphism of the fin. dim^l K -vsp L

def $N_{L/K}(f) := \det(\mu_f) \in K$

⑥ Properties: (see e.g. Bosch, Algebra)

(i) $L/K/E$: $N_{L/E} = N_{K/E} N_{L/K}$

(ii) $L = K(\alpha)$, $f = x^n + a_{n-1}x^{n-1} + \dots + a_0$
 $\in K[x]$ minimal poly of α

$\Rightarrow N_{L/K}(\alpha) = (-1)^n a_0$

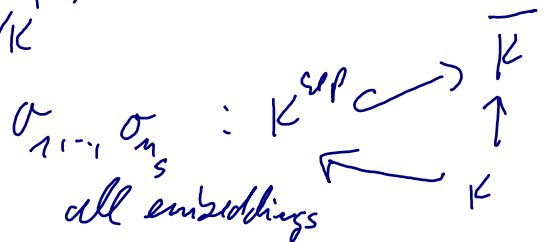
$\{1, \alpha, \dots, \alpha^{n-1}\}$ is a basis of L/K

\rightarrow matrix of M_α in this basis

$$\begin{pmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & & -a_1 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & 1 & -a_{n-1} \end{pmatrix} \rightarrow \det() = (-1)^n a_0$$

(iii) $a \in K \Rightarrow N_{L/K}(a) = [L:K] \cdot a$

(iv) $L \supset K' \supset K$
 purely inseparable in sep of degree p^r sep degree m_s



$\Rightarrow N_{L/K}(f) = \prod_{i=1}^{m_s} \sigma_i(f)^{p^r}$

O Lemma: A finite type R -algebra
integrally closed (normal)
in $K = \text{Frac}(A)$

\mathbb{B} integrally closed

