§5 Weil Divisors and first Chern class of invertible sheaves

① (R1) and DVR

X scheme, $x \in X$

We say x has codimension c in X if $\operatorname{codim}(\overline{x}, X) := \dim \mathcal{O}_{X,x} = c$

closure of x in X

We write $X^{(c)} := \{ x \in X \mid \text{codim of } x = c \}$

We say <u>X is (R1)</u> $\iff$ $\mathcal{O}_{X,x}$ is a regular local ring $\forall x \in X^{(1)}$

Assume X is (R1) and $x \in X^{(1)}$

$\Rightarrow \dim \mathcal{O}_{X,x} = 1 = \dim_{\kappa(x)} \frac{m_x}{m_x^2}$, $\kappa(x) = \frac{\mathcal{O}_{X,x}}{m_x}$

$\Rightarrow$ Nakayamas Lemma $\quad m_x = t \cdot \mathcal{O}_{X,x}$ Principal ideal.

set

$$v_x : \mathcal{O}_{X,x} \setminus \{0\} \longrightarrow \mathbb{N}_0$$

$$v_x(a) = \max\{ n \geq 0 \mid a \in m_x^n \}$$

(where $m_x^0 := \mathcal{O}_{X,x}$)

i.e. $v_x(a) = n \iff a = t^n u$ for some $u \in \underbrace{\mathcal{O}_{X,x} \setminus m_x}_{= \mathcal{O}_{X,x}^\times}$

for $q = \frac{a}{b} \in K^\times = \mathrm{Frac}(\mathcal{O}_{X,x})^\times$ set

$$v_x(q) := v_x(a) - v_x(b)$$

$\Rightarrow$ get well-defined map

$$v_x : K^\times \longrightarrow \mathbb{Z}$$

s.t. $v_x(fg) = v_x(f) + v_x(g)$

$v_x(f+g) \geq \min\{v_x(f), v_x(g)\}$

extend to K via $v_x(0) := \infty$

We say $v_x : K \to \mathbb{Z} \cup \{\infty\}$ associated to $x \in X^{(1)}$ is the (discrete) valuation (if $\mathcal{O}_{X,x}$ is reg)

7) Ex: • $X = \operatorname{Spec} \mathbb{Z}$, $p \in X$

$v_p : \mathbb{Q} \longrightarrow \mathbb{Z}$

$v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b)$.

the number n s.t. $p^n \mid b$ and $p^{n+1} \nmid b$

$a, b \in \mathbb{Z}$

• $X = \operatorname{Spec} k[t]$ $(f) \in X$

irred polyn.

$g \in k[t]$ with $g(t) = f^n \cdot h$, $(f, h) = 1$

$\Rightarrow v_{(f)}(g) = n$

• $X = \operatorname{Spec} \frac{k[s,t]}{(s \cdot t)}$ not regular in $x = (s,t) \in X$

$\Rightarrow \mathcal{O}_{X,x}$ not regular $\to$ no valuation.

(3) Lemma: X noeth (R1) scheme and integral with function field K

$f \in K^\times \Rightarrow \exists$ only finitely many $x \in X^{(1)}$ s.t. $v_x(f) \neq 0$

Pf: Take $\mathcal{U} = \operatorname{Spec} A \subset X$ open

$\Rightarrow f = \frac{a}{b}$, $a, b \in A \setminus \{0\}$

$\Rightarrow f \in (A_{ab})^\times = (A[\frac{1}{ab}])^\times = \mathcal{O}(D(ab))^\times$

$\Rightarrow f \in \mathcal{O}_{X,x}^\times \quad \forall \; x \in D(ab) \subset \mathcal{U} \subset X$

$\Rightarrow v_x(f) = 0 \quad \forall \; x \in D(ab)^{(1)}$

X integral $\Rightarrow$ irred

$\Rightarrow Z = X \setminus D(ab)$ is closed subset in X with $\dim Z < \dim X$

$\Rightarrow$ the points in $X^{(1)} \cap Z$ are irred cpt of Z $\underset{Z \text{ noeth}}{\Rightarrow}$ $|X^{(1)} \cap Z| < \infty$

$\Rightarrow$ Claim $\square$

4 **Def**: X scheme

1) $Z^1(X) :=$ free abelian gp generated by $x \in X^{(1)}$

$$= \bigoplus_{x \in X^{(1)}} \mathbb{Z} \cdot [x]$$

2) A **Weil divisor** is an element

$$D \in Z^1(X)$$

i.e. $D = \sum_{\text{finite}} n_i [x_i]$, $n_i \in \mathbb{Z}$

Let $D_i = \overline{x_i} \subset X$ closure

we also identify $D = \sum n_i D_i$

($D_i \subset X$ irred closed subset)

3) X noeth, integral, (R1), $K = \mathcal{K}(X)$

$f \in K^\times$

set $\operatorname{div}(f) := \sum_{x \in X^{(1)}} v_x(f) \cdot [x] \in Z^1(X)$

(well defined by 3)

We obtain a group hom.

$$\mathrm{div}: K^\times \longrightarrow Z^1(X)$$

$$\Big(\mathrm{div}(fg) = \sum_x \underbrace{v_x(fg)}_{v_x(f)+v_x(g)} [x] = \mathrm{div}(f) + \mathrm{div}(g)\Big)$$

We set

$$CH^1(X) := \mathrm{coker}\,\big(\mathrm{div}(K^\times) \to Z^1(X)\big)$$

= 1st (codimensional) Chow group of X.

5 Example R UFD (e.g. $R = \mathbb{Z}$, $\underset{\text{field}}{k}[x_1, \dots, x_n]$)

$$\Rightarrow \quad CH^1(\operatorname{Spec} R) = 0$$

indeed $D = \sum_{i=1}^{r} n_i [x_i] \neq 0$, $n_i \in \mathbb{Z}$

$x_i \in (\operatorname{Spec} R)^{(1)} \Leftrightarrow x_i \subset R$ prime ideal of ht 1

$\underset{R\text{ UFD}}{\Rightarrow}$ $x_i = (f_i)$ with $f_i \in R$ prime elt.

set $h := \prod_{i=1}^{r} f_i^{n_i} \in K^\times$

$$\Rightarrow \mathrm{div}(h) = \sum_i n_i \,\mathrm{div}(f_i) = \sum_i n_i [x_i] = D.$$

$\rightsquigarrow$ in general $CH^1(X)$ can be very large and hard to compute.

(i) Prop: X noeth, int, (R1), $K = \mathcal{K}(X)$

$\exists$ canonical gp homomorphisms

$$\begin{array}{ccc} \Gamma(X, K^\times/\mathcal{O}_X^\times) & \xrightarrow{[\;]} & Z^1(X) \\ \downarrow & \% & \downarrow \\ CaCl(X) & \xrightarrow{[\;]} & CH^1(X) \end{array}$$

Pf: $D \in \Gamma(X, K^\times/\mathcal{O}_X^\times)$ Cartier div

$\longleftrightarrow$ $X = \bigcup_{i \in I} \mathcal{U}_i$ open cover, $f_i \in K^\times$ s.t. $f_i/f_j \in \mathcal{O}(\mathcal{U}_{ij})^\times$

$\forall\, x \in X^{(1)}$ pick $i(x) \in I$ s.t. $x \in \mathcal{U}_{i(x)}$

set $[D] := \sum v_x(f_{i(x)})\,[\overline{x_{i(x)}}]$

well defined: $x \in \mathcal{U}_i \cap \mathcal{U}_j \Rightarrow v_x\left(\frac{f_i}{f_j}\right) = 0$ i.e. $v_x(f_i) = v_x(f_j)$

also indep of choice $\{\mathcal{U}_i, f_i\}$

Clearly if $D = (f)$ principal Cartier div.

$$\Rightarrow [D] = \sum_{x \in X^{(1)}} v_x(f)\,[x] = \mathrm{div}(f)$$

$\rightsquigarrow$ commutative Diagram in the statement. $\square$

⑦ Cor-Def: X noeth, int, (R1)

The composition

$$c_1 : \mathrm{Pic}(X) \xrightarrow{\cong} \mathrm{CaCl}(X) \xrightarrow{[\ ]} CH^1(X)$$

$$L \longmapsto c_1(L)$$

is a group homomorphism

$c_1(L)$ is called the first Chern class of L $\square$

⑧ Ex $X = \mathbb{P}^n_k = \mathrm{Proj}\, k[X_0, \dots, X_n]$

$\mathcal{O}_X(n) \in \mathrm{Pic}(\mathbb{P}^n_k)$ in $\mathrm{CaCl}(X)$ $\left\{ U_i, \frac{X_0^n}{X_i^n} \right\}$ have $\mathrm{div}_{U_i}\left(\frac{X_0}{X_1}\right)^n = n(H_0 \cap U_i)$

$H_0 = \{X_0 = 0\}$

$$\Rightarrow c_1(\mathcal{O}_X(n)) = n H_0 \underset{\text{in } CH^1(X)}{=} n H_i \text{ any } i.$$

(9) **Thm:** X smooth integral variety $/k$

$\Rightarrow$ $$\mathrm{CaCl}(X) \xrightarrow[{[\cdot]}]{\cong} CH^1(X) \qquad (\text{from } ⑥)$$

is an isom

Furthermore under the composition

$$CH^1(X) \xrightarrow{[\cdot]^{-1}} \mathrm{CaCl}(X) \underset{\substack{\uparrow \\ §4,\, ⑫}}{\xrightarrow{\cong}} \mathrm{Pic}(X)$$

a Weil divisor D is sent to $\mathcal{O}_X(D)$

where $\mathcal{O}_X(D)(\mathcal{U}) := \{ f \in k(X)^* \mid \mathrm{div}(f)_{|\mathcal{U}} \geq -D_{|\mathcal{U}} \}$

Pf: We use three facts from commutative alg.

(0) X smooth $\Rightarrow$ X regular

(i) A regular local ring $\Rightarrow$ A UFD
(Auslander - Buchsbaum, see e.g. Matsumura, Comm Ring Thy Thm 20.3)

(ii) A regular domain (in part noeth.)
$\Rightarrow$ a) $0 \neq f \in A \setminus A^*$ $\Rightarrow$ if $\mathfrak{p} \in \mathrm{Spec}\, A$ minimal with $\mathfrak{p} \supset (f)$
then $\mathrm{ht}(\mathfrak{p}) = 1$
(i.e. $\mathrm{codim}(\bar{\mathfrak{p}}, \mathrm{Spec}\, A) = 1$)

b) $$A = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Spec}\, A \\ \mathrm{ht}(\mathfrak{p}) = 1}} A_{\mathfrak{p}}$$

(Mat. Thm 19.4 (reg $\Rightarrow$ normal), Thm 11.5)

For the first statement we construct

$$CH^1(X) \longrightarrow CaCl(X) \quad \text{inverse to } [-]$$

Let $D = \sum_{i=1}^{r} n_i x_i \in Z^1(X)$, $x_i \in X^{(1)}$

$D_i = \overline{x_i}$ closure of x_i in X

let $y \in X$ let $\underset{\substack{\Downarrow \\ y}}{U = \operatorname{Spec} A} \subset X$ open nbhd.

$\Rightarrow$ $D_i|_U \longleftrightarrow P_i \subset A$ prime ideal of ht 1, $P_i \subset P_y$

$\Rightarrow$ $P_i \cdot A_{P_y}$ prime ideal of ht 1

X sm $\Rightarrow$ $\mathcal{O}_{X,y} = A_{P_y}$ regular $\underset{\text{Fact (i)}}{\Rightarrow}$ UFD

$\Rightarrow$ $P_i A_{P_y}$ Principal $= (f_{i,y})$

$\Rightarrow$ $\exists$ $V = \operatorname{Spec} B \underset{\text{open}}{\subset} U$ s.t. $f_{i,y} \in B$

and $(f_{i,y}) \subset B$ prime ideal.

⌈ for "prim ideal" consider

$$\mathrm{div}(f_{ij}) = \mathfrak{p}_i + \sum m_j E_j$$

and take $V = \mathcal{U} \setminus (\cup E_j)$ ⌋

$y \in X$ arb and X quasi-compact

$\Rightarrow \exists\ X = \cup\, \mathcal{U}_j$ open cover $\mathcal{U}_j := \mathrm{Spec}\, A_j$

and $f_{ij} \in A_j$

s.t. $D_{i\,|\mathcal{U}_j} = \mathrm{Spec}\, A_j/(f_{ij})$

set $f_j = \prod_{i=1}^{r} f_{ij}^{n_i} \in \mathrm{Frac}(A_j) = k(X)$

$\Rightarrow \mathrm{div}(f_j)_{|\mathcal{U}_j} = D_{|\mathcal{U}_j}$

$\Rightarrow \mathrm{div}(f_j)_{|\mathcal{U}_{jj'}} = \mathrm{div}(f_{j'})_{\mathcal{U}_{jj'}}$

$$\Rightarrow \quad \operatorname{div}\left(f_j/f_{j'}\right) = 0 \quad \text{on } \mathcal{U}_{jj'}$$

i.e. $$f_j/f_{j'} \in \left(\mathcal{O}_{\mathcal{U}_{jj'},x}\right)^\times \quad \forall x \in \mathcal{U}_{jj'}^{(1)}$$

$$\Rightarrow \quad f_j/f_{j'} \in \left(\bigcap_{x \in \mathcal{U}_{jj'}^{(1)}} \mathcal{O}_{\mathcal{U}_{jj'},x}\right)^\times \underset{\text{Fact ii, b)}}{=} \mathcal{O}(\mathcal{U}_{jj'})^\times$$

$\Rightarrow \{\mathcal{U}_j, f_j\}_j$ defines a Cartier divisor.

It is direct to check that

- $$Z^1(X) \longrightarrow \Gamma(X, K^\times/\mathcal{O}_X^\times)$$
  $$D \longmapsto \{\mathcal{U}_j, f_j\}_j$$
  is independent of the chosen cover $X = \bigcup \mathcal{U}_j$ and the f_j as above
- $$\underset{f \in K^\times}{\operatorname{div}(f)} \longrightarrow f$$

$\Rightarrow$ We get induced well def hom
$CH^1(X) \longrightarrow \operatorname{CaCl}(X)$ which is inverse to $[\cdot]$
$\Rightarrow$ isom.

It remains to show the following: set $K := \mathcal{K}(X)$

denote by $\mathcal{O}_X(D)$

the image of D under

$$CH^1(X) \xrightarrow{\cong} \mathrm{Pic}(X)$$

then

$$\mathcal{O}_X(D)(\mathcal{U}) \cong \{ f \in K^\times \mid \mathrm{div}(f)_{|\mathcal{U}} \geq -D_{|\mathcal{U}} \} \cup \{0\}$$

pf: denote the RHS by

$$\mathcal{L}(D)(\mathcal{U})$$

$\Rightarrow$ $\mathcal{L}(D) \subset K$ subsheaf

By constr. $\mathcal{O}_X(D) \subset K$ subsheaf (see §4)

$\Rightarrow$ suff to show

$$\mathcal{L}(D)_{|\mathcal{U}_i} = \mathcal{O}_X(D)_{|\mathcal{U}_i} \quad \text{for } X = \bigcup \mathcal{U}_i \text{ some open cover.}$$

Let $\overset{\mathrm{Spec}(A)}{\overset{\shortparallel}{\mathcal{U}}} \subset X$ open s.t.

$$D_{|\mathcal{U}} = \mathrm{div}(g), \quad g \in K^\times$$

then

$$\begin{array}{ccccc} CH^1(\mathcal{U}) & \longrightarrow & \mathrm{Call}(\mathcal{U}) & \longrightarrow & \mathrm{Pic}(X) \\ D_{|\mathcal{U}} = \mathrm{div}(g) & \longmapsto & \{g, \mathcal{U}\} & \longmapsto & \mathcal{O}_{\mathcal{U}} \cdot \frac{1}{g} \end{array} \quad (\text{see } \S 4)$$

i.e. $\mathcal{O}_X(D)_{|\mathcal{U}} = \mathcal{O}_{\mathcal{U}} \cdot \frac{1}{g}$

$V \subset \mathcal{U}$ open.
affin

$$\Rightarrow \Gamma(V, L(D)_{|\mathcal{U}}) = \{ f \in K^\times \mid \underbrace{\mathrm{div}(f)_{|V} \geq -\overset{\mathrm{div}(g)}{\overset{\shortparallel}{D_{|V}}}}_{\iff} \} \cup \{0\}$$

$\iff$

$\mathrm{div}(fg)_{|V} \geq 0$

i.e.

$v_x(fg) \geq 0 \quad \forall x \in V^{(1)}$

i.e. $fg \in \bigcap_{x \in V^{(1)}} \mathcal{O}_{V,x}$

$\underset{\text{Fact (iii), b)}}{=} \mathcal{O}(V)$

$$= \{ f \in K^\times \mid f \in \mathcal{O}(V) \tfrac{1}{g} \} \cup \{0\}$$

$$\Rightarrow L(D)_{|\mathcal{U}} = \mathcal{O}_{\mathcal{U}} \cdot \tfrac{1}{g} = \mathcal{O}_X(D)_{|\mathcal{U}}$$

$\square$

k = field

(10) Bsp.:

$$\mathrm{Pic}(\mathbb{P}^n_k) \cong CH^1(\mathbb{P}^n_k) \cong \mathbb{Z}$$

$$\mathcal{O}_{\mathbb{P}^n}(d) \leftarrow\!\shortmid d H_0 \leftarrow\!\shortmid d$$

$H_0 = V^+(x_0)$

Pf.:

Let $Z \subset \mathbb{P}^n_k = \mathrm{Proj}\, k[x_0, \dots, x_n]$
irred, closed
codim 1

$\Rightarrow \exists\, i : Z \cap U_i \neq \emptyset$ $[U_i := \{x_i \neq 0\}]$

$U_i \cong \mathbb{A}^n_k$ factorial $\Rightarrow Z \cap U_i = V(f)$

$\Rightarrow Z = V^+(F)$
↑ homog. Polynomial
say of degree d

$\Rightarrow \frac{F}{x_0^d} \in k(x_1, \dots, x_n)^\times = k(\mathbb{P}^n)^\times$

$x_i := \frac{x_i}{x_0}$

$\Rightarrow \mathrm{div}\left(\frac{F}{x_0^d}\right) = Z - d H_0$ $H_0 = V^+(x_0)$

$\Rightarrow \mathbb{Z} \longrightarrow CH^1(\mathbb{P}^n_k)$ is surj. $\Rightarrow$ isom.

$d \longmapsto d H_0$ clearly inj $\mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}^n}(dH_0)$ EVL □

(11) Lemma: X smooth integral variety $/k$ = field

$D = \sum n_i D_i$ effective Weil divisor

i.e. $n_i \geq 1$

D_X also defines a closed subscheme of X

[locally $D_{|\mathcal{U}} = \mathrm{div}(f)_{|\mathcal{U}}$, $f \in A$
$\mathcal{U} = \mathrm{Spec}\, A$ $\quad$ D effective

then $D_{\mathcal{U}} = \mathrm{Spec}\, A/(f)$ $\rightsquigarrow$ glues D_X]

Then $\mathcal{O}_X(-D)$ = ideal sheaf of the closed immersion

$D_X \hookrightarrow X$

Pf: $\mathcal{U}$ as above

$\Rightarrow \mathcal{O}_X(-D) = \mathcal{O}_{\mathcal{U}} \cdot f = \ker(\mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}/f)$ $\square$