§4 Invertible sheaves and Cartier divisors

① **Def:** X scheme, $\mathcal{F}, \mathcal{G} = \mathcal{O}_X$-mod

(i) We denote by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ the sheaf associated to

$$X \supset \underset{\text{open}}{\mathcal{U}} \longmapsto \mathcal{F}(\mathcal{U}) \otimes_{\mathcal{O}_X(\mathcal{U})} \mathcal{G}(\mathcal{U})$$

(ii) We denote by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ the presheaf

$$\mathcal{U} \longmapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(\mathcal{U}) := \mathrm{Hom}_{\mathcal{O}_\mathcal{U}}(\mathcal{F}_{|\mathcal{U}}, \mathcal{G}_{|\mathcal{U}})$$

It is a sheaf of $\mathcal{O}_X$-mod (check)

(iii) $f: X \to Y$. We denote by $f_* \mathcal{F}$ the presheaf

$$\mathcal{U} \longmapsto (f_* \mathcal{F})(\mathcal{U}) := \mathcal{F}(f^{-1}(\mathcal{U}))$$

It is a sheaf of $\mathcal{O}_Y$-mod

(iv) $g: Y \to X$ We denote by $g^* \mathcal{F}$ the sheaf

$$g^* \mathcal{F} := \mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_X} g^{-1} \mathcal{F},$$

where $g^{-1}\mathcal{F}$ is the sheaf associated to

$$\underset{\text{open}}{Y \supset V} \longmapsto \varinjlim_{\substack{\mathcal{U} \subset X \text{ open} \\ \mathcal{U} \supset g(V)}} \mathcal{F}(\mathcal{U})$$

② Rmk:

(i) $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{H} \cong \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{H})$

$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{F}$

(ii) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$

(iii) $f : X \to Y$ $\mathcal{F} = \mathcal{O}_X$ - mod

$\mathcal{G} = \mathcal{O}_Y$ - mod.

$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$ (f^*, f_*) adjoint functors.

$(f_*\mathcal{F} \xleftarrow{f_*(\varphi)} f_*f^*\mathcal{G} \underset{\text{nat}}{\leftarrow} \mathcal{G}) \longleftarrow \varphi$

$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \cong f_*\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$

③ Def.: L $\mathcal{O}_X$-mod

We say

L invertible sheaf (or line bundle)

$:\Longleftrightarrow$ L is a locally free $\mathcal{O}_X$-mod of rank 1, i.e.,

$\exists$ open cover $X = \bigcup_i \mathcal{U}_i$ s.t.

$L|_{\mathcal{U}_i} \cong \mathcal{O}_{\mathcal{U}_i}$ isom of $\mathcal{O}_{\mathcal{U}_i}$-mod.

④ Lemma (i) L inv sh on X

$\Rightarrow$ $L^{-1} := \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$ invertible

and $L \otimes_{\mathcal{O}_X} L^{-1} \cong \mathcal{O}_X$

(ii) L, M inv sh $\Rightarrow$ $L \otimes_{\mathcal{O}_X} M$ inv sheaf.

Pf: (i) L inv $\Rightarrow \exists$ open covs $X = \bigcup_i U_i$ s.t.

$$L_{|U_i} \cong \mathcal{O}_{U_i}$$

$$\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)_{|U_i} = \mathcal{H}om_{\mathcal{O}_{U_i}}(L_{|U_i}, \mathcal{O}_{U_i})$$

$$\cong \mathcal{H}om_{\mathcal{O}_{U_i}}(\mathcal{O}_{U_i}, \mathcal{O}_{U_i}) \cong \mathcal{O}_{U_i}$$

$$\varphi \mapsto \varphi(1)$$

$\Rightarrow L^{-1}$ inv.

and

$$L \otimes_{\mathcal{O}_X} L^{-1} \longrightarrow \mathcal{O}_X$$

induced by $L(U) \otimes_{\mathcal{O}(U)} \mathrm{Hom}_{\mathcal{O}_U}(L_{|U}, \mathcal{O}_U) \longrightarrow \mathcal{O}(U)$

$$s \otimes \varphi \longmapsto \varphi(s)$$

$\rightarrow$ globally defined sheaf hom which on the U_i is an isom $\Rightarrow$ isom $\Rightarrow$ (i)

(ii) We find $X = \bigcup_i U_i$ s.t. $L_{|U_i} \cong \mathcal{O}_{U_i} \cong M_{|U_i} \Rightarrow (L \otimes_{\mathcal{O}_X} M)_{|U_i} \cong \mathcal{O}_{U_i}$ $\square$

(5) Example (Serre's twisting sheaf) A = comm ring with 1

$$S = A[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$$

= graded Ring

S_d = homogen poly of deg d with coeff in A

$m \in \mathbb{Z} \Rightarrow S(m) := \bigoplus_{d \in \mathbb{Z}} S(m)_d$ = graded S-module

with $S(m)_d = S_{m+d}$

Then $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n] = \operatorname{Proj} S$

and $\mathcal{O}_{\mathbb{P}^n_A}(m) := \widetilde{S(m)}$

= sheaf associated to the graded S-module $S(m)$ on $\operatorname{Proj} S$

(see, eg, Hartshorne, II, Prop 5.11)

It is called Serre's m-th twisted sheaf.

More explicit description

$\mathbb{P}^n_A = \bigcup_{i=0}^{n} \mathcal{U}_i$ with

$$\mathcal{U}_i = \mathbb{P}^n_A \setminus V^+(X_i) = \operatorname{Spec} A[X_0, \dots, X_n]_{(X_i)}$$
$$= \operatorname{Spec} A[x_{i0}, \dots, x_{in}]$$

where $x_{ij} = \dfrac{X_j}{X_i}$ $[x_{ii} = 1]$

$\Rightarrow \mathcal{O}_{\mathbb{P}^n_A}(m)|\mathcal{U}_i$ = sheaf associated to the module $(S(m))_{(X_i)}$ on $\mathcal{U}_i$

$$(S(m))_{(X_i)} = \left\{ \frac{F}{X_i^d} \;\middle|\; F \in S(m)_d = S_{m+d},\ d \geq 0 \right\}$$
$$= \{ X_i^m \cdot F(x_{i0}, \underset{\text{at } i}{\overset{1}{\dots}}, x_{in}) \mid F \in S_{m+d} \}$$
$$= X_i^m A[x_{i0}, \underset{i}{\overset{1}{\dots}}, x_{in}] \cong A[x_{i0}, \underset{i}{\overset{1}{\dots}}, x_{in}]$$

$\Rightarrow \mathcal{O}_{\mathbb{P}^n_A}(m)|_{\mathcal{U}_i} = X_i^m \cdot \mathcal{O}_{\mathcal{U}_i} \cong \mathcal{O}_{\mathcal{U}_i}$

$\Rightarrow \quad \mathcal{O}_{\mathbb{P}^n_X}(m)$ is invertible on $\mathbb{P}^n_X$ $\forall\, m \in \mathbb{Z}$

<u>Glueing function</u>:

$$\begin{tikzcd}
 & x_i^m & 1 \arrow[l, mapsto] \\
\left(\mathcal{O}_{\mathbb{P}^n_A}(m)|_{U_i}\right)|_{U_i\cap U_j} \arrow[d, equal] \arrow[r, equal] & x_i^m\, \mathcal{O}_{U_i}|_{U_{ij}} \arrow[d, equal, "x_i^m = x_j^m \frac{x_i^m}{x_j^m}"] \arrow[r, "\cong"] & \mathcal{O}_{U_{ij}} \arrow[d, "\cdot \frac{x_i^m}{x_j^m} = x_{ji}^m"] \\
\left(\mathcal{O}_{\mathbb{P}^n_A}(m)|_{U_j}\right)|_{U_i\cap U_j} \arrow[r, equal] & \left(x_j^m\, \mathcal{O}_{U_j}\right)|_{U_{ij}} \arrow[r, "\cong"] & \mathcal{O}_{U_{ij}} \\
 & x_j^m & 1 \arrow[l, mapsto]
\end{tikzcd}$$

$(U_{ij} = U_i \cap U_j)$

$\rightsquigarrow$ alternative description

$$\mathcal{O}_{\mathbb{P}^n_A}(m)|_{U_i} = \mathcal{O}_{U_i}$$

and $\mathcal{O}_{\mathbb{P}^n_A}(m)$ is obtained from $\{\mathcal{O}_{U_i}\}_{i=0}^{n}$ by glueing along

$$\mathcal{O}_{U_i}|_{U_{ij}} \xrightarrow{\cong} \mathcal{O}_{U_j}|_{U_{ij}}$$
$$a \longmapsto x_{ji}^m \cdot \tau_{ji}(a) = \frac{x_i^m}{x_j^m} \cdot \tau_{ji}(a)$$

where $\tau_{ji} : \mathcal{O}_{U_i}|_{U_{ij}} \to \mathcal{O}_{U_j}|_{U_{ij}}$ glueing fct from $\mathbb{P}^n$

⑥ Lem: $X = \mathbb{P}^n_A$

(0) $\mathcal{O}_X(0) = \mathcal{O}_X$

(i) $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m') \cong \mathcal{O}_X(m+m')$

(ii) $\mathcal{O}_X(m)^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(m), \mathcal{O}_X) \cong \mathcal{O}_X(-m)$

(iii) $\Gamma(X, \mathcal{O}_X(m)) = \begin{cases} 0 & , m \leq -1 \\ S_m & , m \geq 0 \end{cases}$

S_m: hom. poly in $A[x_0, \ldots, x_n]$ of deg m

Pf: use explicit description in ⑤ $\square$

⑦ Def: X scheme

$$\mathrm{Pic}(X) = \{ L \text{ inv. sheaf on } X \} / \text{isom.}$$

This becomes an abelian group by setting

$$[L] \cdot [M] := [L \otimes_{\mathcal{O}_X} M]$$

$[L]$: isom class of L

- neutral elt: $[\mathcal{O}_X]$
- inverse $[L]^{-1} := [L^{-1}] = [\mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)]$

8 Prop: X quasi-compact integral

K = function field of X

K_X = const sheaf defined by K

(= sheaf assoc to $X \supset \mathcal{U} \mapsto K$)

L invertible $\mathcal{O}_X$-mod

$\Rightarrow \exists$ invertible $\mathcal{O}_X$-mod $L' \subset K_X$ which is isomorphic to L

Pf: $\exists$ $X = \bigcup_{i=1}^{n} \mathcal{U}_i$ finite open cover (quasicomp) s.t.

$L|_{\mathcal{U}_i} = \mathcal{O}_{\mathcal{U}_i} \cdot e_i \quad , e_i \in L(\mathcal{U}_i)$

$\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$

Have $\mathcal{O}_{\mathcal{U}_{ij}} \cdot e_i|_{\mathcal{U}_{ij}} = (L|_{\mathcal{U}_i})|_{\mathcal{U}_{ij}} = (L|_{\mathcal{U}_j})|_{\mathcal{U}_{ij}} = \mathcal{O}_{\mathcal{U}_{ij}} e_j|_{\mathcal{U}_{ij}}$

$\Rightarrow \exists \; g_{ji} \in \Gamma(\mathcal{U}_{ij}, \mathcal{O}_{\mathcal{U}_{ij}}^{\times})$ s.t. $e_i|_{\mathcal{U}_{ij}} = g_{ji} \, e_j|_{\mathcal{U}_{ij}}$

$\Rightarrow$ Have $g_{ii} = 1$

$$g_{ij} = g_{ji}^{-1}$$

$$g_{ki}|_{U_{ijk}} = g_{kj}|_{U_{ijk}}\, g_{ji}|_{U_{ijk}} \qquad (*)$$

set $f_i := g_{i1} \in \Gamma(U_{i1}, \mathcal{O}_{U_{i1}}^{\times}) \subset K^{\times}$

$$\left(\Rightarrow \frac{1}{f_i} = g_{1i}\right)$$

set $L'_i := \mathcal{O}_{U_i} \cdot \frac{1}{f_i} \subset K$

define

$$\varphi_i : L|_{U_i} = \mathcal{O}_{U_i} e_i \xrightarrow{\cong} \mathcal{O}_{U_i} \cdot \frac{1}{f_i} \subset K$$

$$e_i \longmapsto \frac{1}{f_i} \qquad \text{isom}$$

$\Rightarrow$ obtain

$$\begin{tikzcd}
(L_{|U_i})_{|U_{ij}} \arrow[r, "\varphi_{i|U_{ij}}"] \arrow[d, equal] & (\mathcal{O}_{U_i} \cdot 1/f_i)_{|U_{ij}} = \mathcal{O}_{U_{ij}} \cdot \frac{1}{f_{i|U_{ij}}} \arrow[d, equal] \\
(L_{|U_j})_{|U_{ij}} \arrow[r, "\varphi_{j|U_{ij}}"] & (\mathcal{O}_{U_j}\, 1/f_j)_{|U_{ij}} = \mathcal{O}_{U_{ij}} \cdot \frac{1}{f_j}{}_{|U_{ij}}
\end{tikzcd}$$

commutes, since

$$\begin{tikzcd}
e_i \arrow[r, mapsto] \arrow[d, equal] & \frac{1}{f_i} = \frac{f_j}{f_i} \cdot \frac{1}{f_j} = g_{j1}\, g_{1i}\, \frac{1}{f_j} \arrow[d, equal, "(*)"] \\
g_{ji}\, e_j \arrow[r, mapsto] & g_{ji}\, \frac{1}{f_j}
\end{tikzcd}$$

$\Rightarrow \exists!$ invertible sheaf $L' \subset K_X$ s.t.

$$L'_{|U_i} = \mathcal{O}_{U_i} \frac{1}{f_i}$$

and the (φ_i) glue to an isomorphism

$$\varphi : L \longrightarrow L' \subset K_X \qquad \square$$

9) Lemma: X integral scheme with function field K.

$\mathcal{L}, \mathcal{M} \subset K_X$ two inv. subsheaves of K_X (constant sheaf assoc. to K)

Assume $\exists$ isom $\varphi : \mathcal{L} \xrightarrow{\sim} \mathcal{M}$

Then $\exists f \in K^{\times}$ s.t. $\forall s \in \mathcal{L}(U)$ we have

$$\varphi(s) = f \cdot s \quad , \text{ i.e.}$$

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\varphi"] & \mathcal{M} \\ K \arrow[u, phantom, "\cap"] \arrow[r, "\cdot f"] & K \arrow[u, phantom, "\cap"] \end{tikzcd}$$ commutes

<u>pf:</u> Write $X = \bigcup_{i=1}^{n} \mathcal{U}_i$ with $\mathcal{U}_i$ connected

$$L|_{\mathcal{U}_i} = \mathcal{O}_{\mathcal{U}_i} \cdot e_i \quad , \quad M|_{\mathcal{U}_i} = \mathcal{O}_{\mathcal{U}_i} \cdot \varepsilon_i$$

$e_i, \varepsilon_i \in K^*$

φ isom $\Rightarrow$ $\varphi_i(e_i) = \mu_i \varepsilon_i \quad , \mu_i \in \mathcal{O}(\mathcal{U}_i)^*$

set $f_i := \dfrac{\varepsilon_i \mu_i}{e_i} \in K^\times$

$\Rightarrow \varphi_{|\mathcal{U}_i}$ = multiplication by f_i

<u>Claim</u> $f_i = f_j \quad \forall i,j$ ($\Rightarrow$ statement)

<u>indeed:</u> $e_{i|\mathcal{U}_{ij}} = g_{ji}\, e_{j|\mathcal{U}_{ij}}$ for $g_{ji} \in \mathcal{O}(\mathcal{U}_{ij})^\times$

$\Rightarrow \mu_i \varepsilon_i = g_{ji} \mu_j \varepsilon_j \quad |\mathcal{U}_{ij}$

$$\Rightarrow f_i = \frac{\varepsilon_i \mu_i}{e_i} = \frac{g_{ji}\, \mu_j \varepsilon_j}{g_{ji}\, e_j} = \frac{\mu_j \varepsilon_j}{e_j} = f_j \quad \square$$

(10) **Def:** X integral scheme, K = function field.

We have a s.e.s of sheaves of ab groups

$$0 \to \mathcal{O}_X^\times \longrightarrow \underset{\substack{\uparrow \\ \text{const sheaf ass} \\ \text{to } K^\times}}{K_X^\times} \longrightarrow \frac{K_X^\times}{\mathcal{O}_X^\times} \longrightarrow 0$$

(i) A Cartier divisor on X is a global section $D \in \Gamma\left(X, \frac{K_X^\times}{\mathcal{O}_X^\times}\right)$

(ii) A Cartier divisor is called principal

$$\Longleftrightarrow \quad D \in \operatorname{Im}\left(K^\times \longrightarrow \Gamma\left(X, K_X^\times/\mathcal{O}_X^\times\right)\right)$$

(iii) The class group of X is

$$\operatorname{CaCl}(X) = \operatorname{coker}\left(K^\times \longrightarrow \Gamma(X, K_X^\times/\mathcal{O}_X^\times)\right)$$

$$= \frac{\{\text{Cartier divisors}\}}{\{\text{principal divisors}\}}$$

11) The invertible sheaf associated to a Cartier divisor :

$D \in \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$

Thus $\exists$ open cover $X = \bigcup \mathcal{U}_i$ and fct's

$f_i \in \mathcal{K}^\times$ s.t.

$$f_i / f_j \in \Gamma(\mathcal{U}_{ij}, \mathcal{O}_X^\times)$$

and $D|_{\mathcal{U}_i} = \overline{f_i}$ in $\Gamma(\mathcal{U}_i, \mathcal{K}^\times / \mathcal{O}^\times_{\mathcal{U}_i})$

define $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D)|_{\mathcal{U}_i} := \mathcal{O}_{\mathcal{U}_i} \cdot \frac{1}{f_i} \subset \mathcal{K}$$

$\Longrightarrow$

$$\left(\mathcal{O}_X(D)|_{U_i}\right)_{U_{ij}} = \left(\mathcal{O}_X(D)|_{U_j}\right)_{U_{ij}}$$

since $\frac{q}{f_i} = \frac{f_j}{f_i} \frac{q}{f_j}$

$\frac{f_j}{f_i} = g_{ji} \in \Gamma(U_{ij}, \mathcal{O}^*_{U_{ij}})$

$\Rightarrow$ it glues to

$$\mathcal{O}_X(D) \subset K$$

<u>well-defined</u>

Let $X = \bigcup_j V_j$ open cover

and $h_j \in K$ with $\frac{h_i}{e_j} \in \Gamma(V_{ij}, \mathcal{O}_X^*)$

and s.t. $\overline{h_j|V_j} = D$

on $\mathcal{U}_i \cap V_j$:

$$\bar{g}_j = D|_{V_j \cap \mathcal{U}_i} = \bar{f}_i$$

$$\Rightarrow \frac{g_j}{f_i} \in \Gamma(V_j \cap \mathcal{U}_i, \mathcal{O}_X^\times) \quad \forall i, j$$

$$\Rightarrow \left(\mathcal{O}_{\mathcal{U}_i} \cdot \frac{1}{f_i}\right)\Big|_{V_j \cap \mathcal{U}_i} = \left(\mathcal{O}_{V_j} \cdot \frac{1}{g_j}\right)\Big|_{V_j \cap \mathcal{U}_i}$$

$\Rightarrow$ $\mathcal{O}_X(D)$ constructed via $\{\mathcal{U}_i, f_i\}$

$= \mathcal{O}_X(D)$ constructed via $\{V_j, g_j\}$

(12) **Prop:** The above construction (X integral) induces a group isomorphism

$$\mathrm{CaCl}(X) \xrightarrow{\cong} \mathrm{Pic}(X)$$

$$D \longmapsto \mathcal{O}_X(D)$$

Pf. $\Gamma(X, K^\times/\mathcal{O}_X^\times) \longrightarrow \overline{\mathrm{Pic}}(X)$

$D \longmapsto \mathcal{O}_X(D)$

group hom:

. $1 \in \Gamma(X, K^\times/\mathcal{O}_X^\times) \longrightarrow \mathcal{O}_X$

. $D, E \to \mathcal{O}_X(D+E) = \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}(E)$

$\ulcorner$ $D = \{U_i, f_i\}$ $\urcorner$

$E = \{U_i, g_i\}$

$\Rightarrow D+E = \{U_i, f_i g_i\}$

and $\mathcal{O}_X(D+E)|_{U_i} = \mathcal{O}_{U_i} \frac{1}{f_i g_i} = \mathcal{O}_{U_i} \frac{1}{f_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_{U_i} \frac{1}{g_i}$

$= \mathcal{O}_X(D)|_{U_i} \otimes_{\mathcal{O}_{U_i}} \mathcal{O}_X(E)|_{U_i}$

id glues

$\llcorner$ $\lrcorner$

. $D = (f)$ principal

$\Rightarrow \mathcal{O}_X(D) = \mathcal{O}_X \cdot \frac{1}{f} \cong \mathcal{O}_X$ i.e. $\mathcal{O}_X(D) = \mathcal{O}_X$ in $\mathrm{Pic}(X)$

$\Rightarrow$ get well defined gp hom.

$$\mathrm{Coll}(X) \longrightarrow \mathrm{Pic}\, X$$

- surj by (8) (and its proof)

- inj by (9) $\square$