§3 Properties of Varieties

fix k = field

① Def: (i) We say X is a variety

$\iff$ X is separated and of finite type over k

(ii) We say X is projective $/k$

$\iff$ $X \subset \mathbb{P}^n_k$ is closed

i.e. $X = \operatorname{Proj} \dfrac{k[x_0, x_1, \ldots, x_n]}{I}$

$I \subset k[x_0, \ldots, x_n]$ homogeneous ideal

② Rmk

X Proj $= \operatorname{Proj} \underbrace{\frac{k[x_0,\dots,x_n]}{I}}_{=:S}$, I homog. ideal

(graded ring)

$$\Rightarrow \quad X = \bigcup_{i=0}^{n} X \cap \mathcal{U}_i$$

where $\mathcal{U}_i := \mathbb{P}^n_k \setminus V^+(x_i) = \operatorname{Spec} k[y_0,\dots,\overset{\text{without}}{\widehat{y_i}},y_{i+1},\dots,y_n]$

standard open cover $\qquad y_j = \frac{x_j}{x_i}$

$X \cap \mathcal{U}_i = \operatorname{Spec} S_{(x_i)} \; ; = \operatorname{Spec} \frac{k[y_0,\dots,\widehat{y_i},y_{i+1},\dots,y_n]}{I_i}$

elts in S_{x_i} of degree 0

where

$$I_i = \{ f(y_0,\dots,\underset{i\text{-th spot}}{1},y_{i+1},\dots,y_n) \mid f \in I \}$$

$\Rightarrow$ X is of finite type

and $\mathbb{P}^n_k$ is separated $\Rightarrow$ X separated

Thus X proj$/k$ $\Rightarrow$ X variety$/k$

③ Def: $k \subset \bar{k}$ alg closure, $X/_k$ variety

(i) We say X is connected

if the topological space underlying X is connected

(i.e. $\underset{\text{open}}{X = X_1 \amalg X_2}$, $X_2 \neq \emptyset$ $\Rightarrow$ $X_1 = \emptyset$, $X = X_2$)

(ii) We say X is irreducible

if $\underset{\neq \emptyset}{Z_1}, Z_2 \subset X$ closed with $X = Z_1 \cup Z_2$

$\Rightarrow$ $X = Z_1$

(iii) We say X is geometrically connected

$\Longleftrightarrow$ $\bar{X} := X \otimes_k \bar{k} := X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$ is connected.

(iv) We say X is geometrically irreducible

$\Longleftrightarrow$ $\bar{X}$ is irreducible

(v) X is integral $:\Longleftrightarrow$ $[\forall\ U = \operatorname{Spec} A \subset X \Rightarrow A$ is integral dom$]$

$\Longleftrightarrow$ irred + red (Exc 2)

$\Longleftrightarrow$ X connected and $\mathcal{O}_{X,x}$ integral $\forall x$

(4) Ex:

$$X = \operatorname{Spec} \underbrace{\frac{\mathbb{R}[X]}{x^2+1}}_{=\mathbb{C}}$$ is integral, in part conn.

But $X \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{Spec}(\mathbb{C} \times \mathbb{C}) = (\operatorname{Spec} \mathbb{C}) \amalg \operatorname{Spec}(\mathbb{C})$

is not conn

$\Rightarrow X$ not geom int / irred / conn.

(5) Def (1) $k = \bar{k}$, X affine k-var, $x \in X$ closed point.

Then X is smooth at x

$\Leftrightarrow \exists$ presentation $X = \operatorname{Spec} \frac{k[t_1, \dots, t_n]}{(f_1, \dots, f_s)}$

s.t. if we write $x = (t_1 - a_1, \dots, t_n - a_n)$ (HNS $k = \bar{k}$)

then matrix

$$\underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a), & \frac{\partial f_1}{\partial x_2}(a), & \dots, & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a), & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_s}{\partial x_1}(a), & \dots & & \frac{\partial f_s}{\partial x_n}(a) \end{pmatrix}}_{\text{Jacobi matrix}}$$ has rank $= n - \dim \mathcal{O}_{X,x}$

, $a = (a_1, \dots, a_n)$

(2) $k = \bar{k}$, we say a k-var X is smooth$_{/k}$

$\iff \forall x \in X$ closed $\exists_x\ \mathcal{U}_x \subset X$ open affine

s.t. $\mathcal{U}$ is smooth at x

(3) k any field, $X_{/k}$ variety

We say X is sm$_{/k}$ $\iff \bar{X} = X \otimes_k \bar{k}$ is smooth$_{/\bar{k}}$

⑥ Bsp: char $k \neq 2, 3$

$$E = \operatorname{Spec} \frac{k[x,y]}{\underbrace{y^2 - x(x-1)(x-\lambda)}_{f}}, \quad \lambda \in k \setminus \{0,1\}$$

$\Rightarrow$ E is integral dim $E = \operatorname{trdeg}_k \left(\frac{k(x)[y]}{f} \right) = 1$
gen.

and

$$\frac{\partial f}{\partial x} = -(x-1)(x-\lambda) - x(x-\lambda) - x(x-1)$$

$$\frac{\partial f}{\partial y} = 2y$$

Then E is smooth

since $\forall (a,b) \in \bar{k}^2$ with $f(a,b) = 0$: $\left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right) \neq (0,0)$

(7) **Def.** (A, m) noetherian local ring, $k = A/m$

Then we say

$$A \text{ regular} \iff \underset{\substack{\uparrow \\ k\text{- V.sp dim}}}{\dim_k \frac{m}{m^2}} = \underset{\substack{\parallel \\ \text{Krull dim.}}}{\dim A}$$

(8) **Thm** $k = \bar{k}$, X var$/k$, $x \in X$ closed

Then X is smooth at x

$\iff$ $\mathcal{O}_{X,x}$ is a regular local ring.

Pf: Choose $x \in U = \operatorname{Spec} \dfrac{k[t_1, \dots t_n]}{I}$

$x = (t_1 - a_1, \dots, t_n - a_n)$, $a_i \in k$

$a = (a_1, \dots, a_n)$

define $\theta : k[t_1, \dots, t_n] \longrightarrow k^n$

$f \mapsto \left(\frac{\partial f}{\partial t_1}(a), \dots, \frac{\partial f}{\partial t_n}(a)\right)$

k- linear map

Have

$$\{ \theta(t_i - a_i) \ , \ i = 1, \dots, n \} \text{ is a basis of } k^n$$

and

$$\theta((t_i - a_i)(t_j - a_j)) = 0 \qquad \forall i,j$$

$\Rightarrow$ get induced surj

$$\bar{\theta} : \frac{m}{m^2} \longrightarrow k^n$$

where $m = (t_1 - a_1, \dots, t_n - a_n) \subset k[t_1 \dots t_n]$

and $\dim_k \left(\frac{m}{m^2} \right) = n$

$\Rightarrow \quad \bar{\theta}$ isom.

Write $I = (f_1, \dots, f_s) \subset m$
$\qquad\qquad\qquad\qquad\qquad\quad \updownarrow$
$\qquad\qquad\qquad\qquad\qquad\quad x \in X$

$$J = \left(\frac{\partial f_i}{\partial x_j}(a) \right) = \text{Jacobian matrix}$$

$r := \operatorname{rank} J$

$$\Rightarrow \quad r = \dim_k \theta(I) = \dim_k \bar{\theta}\left(\frac{I+m^2}{m^2}\right) \underset{\theta \text{ isom}}{=} \dim_k \frac{I+m^2}{m^2} \qquad (*)$$

set $m_x = m\,\mathcal{O}_{X,x}$ = max'l ideal of $\mathcal{O}_{X,x}$

$$\Rightarrow \quad \begin{array}{ccccccccc} 0 \to & \dfrac{m}{I+m^2} & \to & \dfrac{k[t_1 \ldots t_n]}{I+m^2} & \longrightarrow & \dfrac{k[t_1, \ldots, t_n]}{m} \overset{\displaystyle k}{\scriptstyle \|} & \to 0 \\ & \downarrow & & \downarrow \wr\| & & \downarrow \wr\| & \\ 0 \to & \dfrac{m_x}{m_x^2} & \to & \dfrac{\mathcal{O}_{X,x}}{m_x^2} & \longrightarrow & \dfrac{\mathcal{O}_{X,x}}{m_x} & \to 0 \end{array}$$

$$\Rightarrow \quad \frac{m_x}{m_x^2} \cong \frac{m}{I+m^2}$$

and

$$0 \to \frac{I+m^2}{m^2} \to \frac{m}{m^2} \to \frac{m}{I+m^2} \to 0$$

ex seq of k-v-sp.

$\Rightarrow$

$$\dim_k \frac{m_x}{m_x^2} = \dim_k \frac{m}{m^2+I} = \dim_k \frac{m}{m^2} - \dim_k \frac{m^2+I}{m^2} \underset{(*)}{=} n - r$$

$$\Longleftrightarrow \quad r = n - \dim_k \frac{m_x}{m_x^2} \quad (2*)$$

This holds for any $x \in U = \operatorname{Spec} \frac{k[t_1, \dots t_n]}{(f_1, \dots f_s)}$ chosen at the beginning.

Thus

X sm at $x \underset{\text{defn}}{\Longleftrightarrow} \exists\ x \in U = \operatorname{Spec} \frac{k[t_1 \dots t_n]}{(f_1, \dots, f_s)}$ s.t. $r = n - \dim \mathcal{O}_{X,x}$

$\underset{(2*)}{\Longleftrightarrow} \quad \dim_k \frac{m_x}{m_x^2} = \dim \mathcal{O}_{X,x} \Longleftrightarrow \mathcal{O}_{X,x}$ regular

$\square$

(9) Facts from Comm Alg

- A fin type k-alg, A_m regular $\forall$ max'l ideals m
  $\Rightarrow$ $A_{\mathfrak{p}}$ regular $\forall\ \mathfrak{p} \in \operatorname{Spec} A$ (i.e. A regular)
- A regular local ring $\Rightarrow$ A integral
- k perfect, A fin type$/k$. Then
  A regular $\Longleftrightarrow$ $A \otimes_k \bar{k}$ regular

(10) Cor: (i) $k = \bar{k}$ X smooth at x

$$\iff \quad \operatorname{rank}\left(\frac{\partial f_i}{\partial t_j}(x)\right) = n - \dim \mathcal{O}_{X,x}$$

$\forall$ presentations $\quad x \in U = \operatorname{Spec} \frac{k[t_1, \dots, t_n]}{I}$

$\quad X^{\text{open}}$

(ii) (uses (9)) $k =$ perf, $X = \operatorname{Spec} A$. Then [X smth $\iff$ A regular]

(iii) X smth $\implies$ X locally integral

(11) Lemma (Euler identity) $\quad k =$ field

$F \in k[T_0, \dots, T_n]$ homogeneous of degree d

$$\implies \quad d \cdot F = \sum_i T_i \frac{\partial F}{\partial T_i} \qquad (*)$$

Pf: $\quad S_d$: k-v-sp of homog polynomials in $T_0 \dots T_n$ of degree d.

Both sides of $(*)$ are k-linear map

$$S_d \to S_d$$

$\implies$ it suffices to check $(*)$ on a basis of S_d

set $F = T_0^{j_0} \cdots T_n^{j_n}$ with $j_0 + \dots + j_n = d$

$\implies T_i \frac{\partial F}{\partial T_i} = j_i F \qquad \implies$ OK $\qquad \square$

(12) **Thm** (smoothness crit for proj var)

$k = \bar{k}$, $X = \mathrm{Proj}\ \dfrac{k[T_0, \ldots, T_n]}{I} \subset \mathbb{P}^n_k$

$x \in X$ closed pt $\longleftrightarrow$ $(a_i T_j - a_j T_i \mid i,j \in \{0, \ldots, n\})$ $\quad a_i \in k$

$\left[\hat{=} (a_0 : \ldots : a_n) \text{ (representative) } \right]$

Let $I = (F_1, \ldots, F_s)$, F_i homogen

Then X is smooth at x

$\Longleftrightarrow$ $\operatorname{rk} J(\underline{a}) = n - \dim \mathcal{O}_{X,x}$

where $J = \left(\dfrac{\partial F_i}{\partial T_j} \right)_{\substack{i = 1, \ldots, s \\ j = 0, \ldots, n}}$

and $J(\underline{a}) = \left(\dfrac{\partial F_i}{\partial T_j}(\underline{a}) \right)$ $\quad \underline{a} = (a_0 : \ldots : a_n)$

[$J(\underline{a})$ depends on the representative for $\underline{a}$ but $\operatorname{Rg} J(\underline{a})$ does not]

<u>Pf:</u> Wlog $x \in \underset{T_0 \neq 0}{X \cap U_0} = \mathrm{Spec}\, \dfrac{k[t_1, \dots, t_n]}{(f_1, \dots, f_s)}$

where $t_i = \dfrac{T_i}{T_0}$

$f_j = F_j(1, t_1, \dots, t_n)$

and $x = (1 : a_1, \dots, : a_n)$

<u>for</u> $i \neq 0$:

$$\frac{\partial F_j}{\partial T_i}(1, a_1, \dots, a_n) = \frac{\partial f_j}{\partial t_i}(a_1, \dots, a_n) \quad \text{(check on monomials)}$$

$$d_j = \deg F_j \xRightarrow{\text{Euler}} d_j F_j = \sum_{i=0}^{n} T_i \frac{\partial F_j}{\partial T_i}$$

$$\xRightarrow{\text{plug in } (1, a_1, \dots, a_n)} \frac{\partial F_j}{\partial T_0}(1, a_1, \dots, a_n) = -\sum_{i=1}^{n} a_i \frac{\partial f_j}{\partial t_i}(a_1, \dots, a_n)$$

$$\Rightarrow J(1, a) = \begin{pmatrix} \frac{\partial F_1}{\partial T_0}(1,a) & \cdots & \frac{\partial F_s}{\partial T_0}(1,a) \\ \multicolumn{3}{c}{\underbrace{\boxed{\begin{matrix} \frac{\partial f_1}{\partial t_1}(a) & \cdots & \frac{\partial f_1}{\partial t_1}(a) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial t_n}(a) & & \frac{\partial f_n}{\partial t_n}(a) \end{matrix}}}_{= \tilde{J}(a)}} \end{pmatrix}$$

thus rk $= \mathrm{rk}\, \tilde{J}(a)$

By defn

$$X \text{ sm at } x \iff \operatorname{rk}(\bar{J}(x)) = n - \dim \mathcal{O}_{X,x}$$

$$\| $$

$$\operatorname{rk}(J(x))$$

$\square$