§2 The Weil conjectures

$\mathbb{F}_q$ = finite field with $q = p^r$ elts

$X/\mathbb{F}_q$ separated finite type scheme

($\Rightarrow X$ arith scheme)

① Def: Define

$$Z(X/\mathbb{F}_q, t) := \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{t^n}{n} \right)$$

We call $Z(X/\mathbb{F}_q, t)$ also the zeta function of $X/\mathbb{F}_q$

Rmk: §1 ⑫, iii) $\Rightarrow$

$$Z(X/\mathbb{F}_q, q^{-s}) = \zeta(X, s)$$

② **Def:** k field

$X/_k$ fin. type, sep.

$X_0 := \{$ closed points of $X \}$

(i) $Z_0(X) :=$ free abelian group with generators the closed points of X

$$= \bigoplus_{x \in X_0} \mathbb{Z} \cdot \underset{\uparrow \text{formal symbol}}{[x]}$$

$$\alpha \in Z_0(X) \implies \alpha = \sum_{i=1}^{r} n_i [x_i] \quad , \; n_i \in \mathbb{Z}, \; x_i \in X_0$$

$Z_0(X)$ is called the <u>group of zero cycles of X</u>

We denote by $Z_0(X)^{\text{eff}}$ the <u>monoid of effective zero cycles</u>

i.e. $$\sum n_i [x_i] \in Z_0(X)^{\text{eff}} \iff n_i \geq 0 \;\; \forall \, i$$

(ii) $\alpha \in Z_0(X)$, $\alpha = \sum n_i [x_i]$

set $$\deg \alpha := \sum n_i \deg(x_i) := \sum n_i \underset{\substack{\uparrow \\ \text{field degree}}}{[k(x_i) : k]}$$

= degree of α

Rmk.: $\alpha \in Z_0(X)^{\text{eff}}$

$\Rightarrow$ $\deg \alpha \geq 0$ and

[$\deg \alpha = 1 \iff \alpha = [x]$, $x \in X(k)$ ← k-rational point]

③ Lemma: $X/\mathbb{F}_q$

$$Z(X/\mathbb{F}_q) \in \mathbb{Z}[[t]]$$

more precisely

$$Z(X/\mathbb{F}_q, t) = 1 + \sum_{d \geq 1} b_d t^d$$

where $b_d = |\{\alpha \in Z_0(X/\mathbb{F}_q)^{\text{eff}} \mid \deg \alpha = d\}| < \infty$

Pf: $$\zeta(X, s) = \prod_{x \in X_0} \frac{1}{1 - N(x)^{-s}} = \prod_{x \in X_0} \frac{1}{1 - q^{-\deg(x)s}}$$

$$\left[N(x) = |\kappa(x)| = q^{[\kappa(x):\mathbb{F}_q]} = q^{\deg(x)} \right]$$

$$\Rightarrow \quad Z(X/\mathbb{F}_q, t) = \prod_{x \in X_0} \frac{1}{1 - t^{\deg(x)}}$$

$q^{-s} = t$

$$= \prod_{x \in X_0} \left(1 + t^{\deg(x)} + t^{2\deg(x)} + \dots\right) \in \mathbb{Z}[[t]]$$

$\Uparrow$

coeff before t^d:

$$\sum_{d = \sum n_i \deg(x_i)} 1 = b_d \qquad \square$$

④ Weil conjectures (= Thm's of Weil, Dwork, Grothendieck, Deligne)

X smooth proj scheme $/\mathbb{F}_q$, $n = \dim X$
geom connected (i.e. $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is connected)

⌜ smooth is recalled later ⌝

proj $/\mathbb{F}_q$: $X \subset \mathbb{P}^N_{\mathbb{F}_q}$ closed subscheme

i.e. $X = \mathrm{Proj}\left(\dfrac{\mathbb{F}_q[X_0, \dots, X_N]}{I}\right)$ ← homogeneous ideal

$$\Rightarrow \quad X(\mathbb{F}_{q^r}) = \{(\lambda_0, \dots, \lambda_N) \in \mathbb{F}_{q^r}^{N+1} \setminus \{0\} \mid F(\lambda_0, \dots, \lambda_N) = 0 \ \forall F \in I\} / \sim$$

where $(\lambda_0, \dots, \lambda_n) \sim (\mu_0, \dots, \mu_n)$
$:\Longleftrightarrow \exists\, \nu \in \mathbb{F}_{q^r}^\times : \lambda_i = \nu \mu_i \ \forall i$ ⌟

Then

(1) (Rationality) $Z(X/\mathbb{F}_q, t) \in \mathbb{Q}(t)$ rational fct

More precisely

$$Z(X/\mathbb{F}_q, t) = \frac{P_1(t) P_3(t) \cdot \ldots \cdot P_{2n-1}(t)}{P_0(t) P_2(t) \cdot \ldots \cdot P_{2n}(t)} \qquad n = \dim X$$

where $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$

and $\forall\ 1 \le i \le 2n-1$: $P_i(t) \in 1 + t\mathbb{Z}[t]$

(2) (Functional equation) $\quad b_i := \deg P_i(t)$

$$E := \sum_{i=0}^{2n} (-1)^i b_i$$

$$\Rightarrow \quad Z\left(X/\mathbb{F}_q, \frac{1}{q^n t}\right) = \pm\, q^{nE/2}\, t^E\, Z(X/\mathbb{F}_q, t)$$

(3) Analog of Riemann hypothesis.

Write

$$P_i(t) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} t)$$

(the α_{ij} are the inverses of the roots of $P_i(t)$ in $\bar{\mathbb{Q}}$).

$$\Rightarrow \quad \alpha_{ij} \in \bar{\mathbb{Z}} \subset \bar{\mathbb{Q}}$$

and $\forall\ \tau : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$

we have $\quad |\tau(\alpha_{ij})| = q^{i/2}$

Rmk: $\dim X = 1$: proved by Weil ~ 1949

(1), (2) : Dwork 1960
rationality
Grothendieck et al ~ 1970

(3) Deligne 1974

The Goal of this course is to do the curve case i.e. to prove

(5) Thm (Weil)

$C/\mathbb{F}_q$ smooth proj geom connected curve of genus g

to be explained. dim = 1 to be explained

Then

(i) $Z(C/\mathbb{F}_q, t) = \dfrac{P(t)}{(1-t)(1-qt)}$

where $P \in \mathbb{Z}[t]$ is a polynomial of degree $2g$ with $P(0) = 1$

(ii) $Z(C/\mathbb{F}_q, 1/qt) = q^{1-g} t^{2-2g} Z(C/\mathbb{F}_q, t)$

(iii) If we write $P(t) = \prod_{i=1}^{2g} (1 - \alpha_i t)$, $\alpha_i \in \bar{\mathbb{Z}} \subset \bar{\mathbb{Q}}$

then $\forall\, \iota : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ we have

$|\iota(\alpha_i)| = \sqrt{q} \quad \forall i$

6 Ex: § 1, 12, (iii)

$$Z(\mathbb{P}^1/\mathbb{F}_q, t) = \frac{1}{(1-t)(1-qt)}$$

and $g(\mathbb{P}^1) = 0$ (see later)

$$Z\left(\mathbb{P}^1/\mathbb{F}_q, \frac{1}{qt}\right) = \frac{1}{(1-\frac{1}{qt})(1-\frac{1}{t})} = q t^2 \frac{1}{(1-qt)(1-t)}$$ $\Rightarrow$ OK

7 Cor: $\zeta(C, s) = Z(C/\mathbb{F}_q, q^{-s})$

is holomorphic on $\mathbb{C} \setminus \{0, 1\}$

and if $\zeta(C, s) = 0 \Rightarrow \mathrm{Re}(s) = 1/2$

Pf: $\zeta = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})} \Rightarrow$ hol

$\zeta = 0 \iff P(q^{-s}) = 0 \underset{\alpha_i \text{ as above}}{\iff} q^{s} = \alpha_i$

$\Rightarrow \underset{q^{\mathrm{Re}(s)}}{\underset{\|}{|z(q^s)|}} = |\alpha_i| = \sqrt{q} \Rightarrow \mathrm{Re}(s) = 1/2$ $\square$