§1 Zeta function of arithmetic schemes

① **Def** An arithmetic scheme

is a scheme X of finite type over $\mathbb{Z}$, i.e.,

$\exists \quad X = \bigcup_{i=1}^{r} \mathcal{U}_i$ finite open cover

s.t. $\mathcal{U}_i = \mathrm{Spec}\left(\dfrac{\mathbb{Z}[x_1, \dots, x_{n_i}]}{\underset{\text{ideal}}{I_i}}\right)$

② **Examples** Arithmetic schemes are

1) $X = \mathrm{Spec}\, \mathbb{Z}$

1bis) $K/\mathbb{Q}$ number field, $\mathcal{O}_K$ = integral closure of $\mathbb{Z}$ in K

$\Rightarrow X = \mathrm{Spec}\, \mathcal{O}_K$

2) $\mathbb{F}_q$ = finite field with $q = p^n$ elements (p prime)

X finite type$/\mathbb{F}_q \implies X \to \mathrm{Spec}\, \mathbb{F}_q \to \mathrm{Spec}\, \mathbb{F}_p \hookrightarrow \mathrm{Spec}\, \mathbb{Z}$ fin type.

Non-examples

$\mathrm{Spec}\, \mathbb{C}[T]$, $\mathrm{Spec}\, \mathbb{F}_p(T)[X]$ are not arithmetic schemes.

③ Lemma: X arithm scheme, $x \in X$ closed point

$\Longrightarrow$ $K(x) :=$ residue field of x
$:= \mathcal{O}_{X,x}/\mathfrak{m}_x$

is a finite field (i.e. $= \mathbb{F}_q$ for some $q = p^n$)

pf: Let $U \subset X$ open affine $U = \operatorname{Spec} A$ with $x \in U$

$A = \frac{\mathbb{Z}[x_1, \dots, x_n]}{I}$

$\Rightarrow x \leftrightarrow \mathfrak{m} \subset A$ max'l ideal

and $K(x) = \frac{A_\mathfrak{m}}{\mathfrak{m}A_\mathfrak{m}} = \frac{A}{\mathfrak{m}} \xleftarrow{\ \varphi\ } \mathbb{Z}$

$\Rightarrow K(x)$ is of finite type$/\mathbb{Z}$

Note $\mathbb{Q} = \operatorname{Frac}(\mathbb{Z})$ not of fin type$/\mathbb{Z}$

$\Rightarrow \mathbb{Q} \not\subset K(x)$ $\Longrightarrow$ $\varphi^{-1}(\mathfrak{m}) = (p)$ p some prime

i.e. $K(x)$ fin gen $\mathbb{F}_p$-algebra

$\underset{\text{Algebra (HNS)}}{\Longrightarrow}$ $K(x)$ finite $\mathbb{F}_p$-vector space

$\triangledown$

④ Notation: X arith scheme $x \in X$ closed

set $N(x) = |K(x)|$

(5) **Def** (zeta function) X arith scheme

set
$$\zeta(X,s) := \prod_{\substack{x\in X \\ \text{closed}}} \frac{1}{1-N(x)^{-s}}$$

as a formal Dirichlet series

Ex: $X=\operatorname{Spec}\mathbb{Z} \Rightarrow \zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ Riemann zeta function

$\longrightarrow$ Disgression

(6) **Def**: • A formal Dirichlet series is
$$f = \sum_{n\geq 1} \frac{a_n}{n^s}, \quad a_n \in \mathbb{C}$$
(formal sum, s variable)

• $f = \sum_{n\geq 1} \frac{a_n}{n^s}$, $g = \sum_{n\geq 1} \frac{b_n}{n^s}$

$$\Rightarrow f+g := \sum_{n\geq 1} \frac{a_n+b_n}{n^s}$$

$$f\cdot g := \sum_{n\geq 1} \frac{c_n}{n^s}, \quad c_n = \sum_{i\cdot j=n} a_i b_j$$

$\rightsquigarrow$ $D(\mathbb{C}) :=$ ring of formal Dirichlet series (comm. ring with 1)

7 **Def.** $f = \sum_{n \geq 1} \frac{a_n}{n^s}$

set $w(f) := \begin{cases} \infty & , f = 0 \ \ (\text{i.e. all } a_n = 0) \\ \min\{n \mid a_n \neq 0\} & , f \neq 0 \end{cases}$

Note:
- $w(f) \geq 1$
- $w(f \cdot g) = w(f) \cdot w(g)$
- $w(f+g) \geq \min\{w(f), w(g)\}$

8 **Lem.** $N \in \mathbb{N}$

$\{ f \in D(\mathbb{C}) \mid w(f) \geq N \}$ is an ideal in $D(\mathbb{C})$

Pf: ✓

9 **Def.** Given $f_r = \sum_{n \geq 1} \frac{a_{r,n}}{n^s}$, $r \geq 0$

We say $\{f_r\}_{r \geq 0}$ is summable $:\Leftrightarrow$

$\forall n_0 \geq 1$ only finitely many of the $\{a_{r,n_0}\}_{r \geq 0}$ are $\neq 0$

Then we define $\sum_{r=0}^{\infty} f_r = \sum_{n \geq 1} \frac{\left(\sum_{r \geq 0} a_{r,n}\right)}{n^s} \in D(\mathbb{C})$

10 Lem: $\{f_r\}_{r\geq 0}$ in $D(\mathbb{C})$

(i) $\{f_r\}_{r\geq 0}$ is summable $\iff$ $w(f_r) \to \infty$, $r\to\infty$

(ii) Assume $w(f_r) > 1 \quad \forall \quad r\geq 0$

and $w(f_r) \to \infty$, $r\to\infty$

$$\Rightarrow \quad \prod_{r\geq 0}(1+f_r) \underset{\text{defn}}{:=} 1 + \sum_{N\geq 1}^{\infty} \sum_{0\leq i_1<\dots<i_N} f_{i_1}\cdots f_{i_N}$$

$\in D(\mathbb{C})$ (well-defined)

Pf: (i) ✓

(ii) Set $h_{i_1\dots i_N} := f_{i_1}\cdots f_{i_N} \in D(\mathbb{C})$

Have $w(h_{i_1\dots i_N}) \geq \prod_{j=1}^{N} w(f_{i_j}) \longrightarrow \infty$ $\quad i_1+\dots+i_N \to \infty$

$\Rightarrow \{h_{i_1\dots i_N}\}$ is summable

$$\Rightarrow \quad g_N := \sum_{0\leq i_1<\dots<i_N} h_{i_1\dots i_N} \in D(\mathbb{C})$$

and $\omega(g_N) \geq \min\{\omega(h_{i_1 \dots i_N})\} \to \infty \quad N \to \infty$

(since $\omega(f_r) > 1 \ \forall r$)

$\Rightarrow \quad \{g_N\}_{N \geq 1} \in D(\mathbb{C}) \quad \square$

(11) Cor: $a : \mathbb{N} \setminus \{0\} \to \mathbb{C}$ completely multiplicative function

(i.e. $a(mn) = a(m) \cdot a(n) \ \forall m, n$
$a(1) = 1$)

$$\Rightarrow \quad \sum_{n \geq 1} \frac{a(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{a(p)}{p^s}\right)^{-1}$$

Pf: $f_r := \begin{cases} 0 & , \ r \text{ not prime} \\ \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \frac{a(p^3)}{p^{3s}} + \dots & , \ r = p \text{ prime} \end{cases}$

$\Rightarrow \omega(f_r) \to \infty, \quad r \to \infty, \quad \omega(f_r) > 1 \ \forall r$

$$\Rightarrow \prod_{p \text{ prime}} \left(1 - \frac{a(p)}{p^s}\right)^{-1} = \prod_{r \geq 1} (1 + f_r) = 1 + \sum_{N \geq 1} \sum_{0 \leq i_1 < \dots < i_N} f_{i_1} \cdots f_{i_N}$$

$$= \dots = \sum_{N \geq 1} \frac{a(N)}{N^s} \quad \square$$

<u>Recall</u> $\zeta(X,s) = \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1-N(x)^{-s}}$

<u>(12) Prop:</u> (i) $\zeta(X,s) \in D(\mathbb{C})$ well-defined.

(ii) $X = \coprod_{r=1}^{\infty} X_r$ with $X_r \subset X$ locally closed
(disj union) (i.e. closed in an open)

$$\Rightarrow \zeta(X,s) = \prod_{r=1}^{\infty} \zeta(X_r, s)$$

In particular, let $X_p := X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p$ fiber over p

$$\Rightarrow \zeta(X,s) = \prod_p \zeta(X_p, s)$$

(iii) Assume X is an $\mathbb{F}_q$-scheme $(q = p^n)$

$$\Rightarrow \zeta(X,s) = \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{q^{-ns}}{n} \right)$$

where $X(\mathbb{F}_{q^n}) = \operatorname{Hom}_{\operatorname{Spec} \mathbb{F}_q}(\operatorname{Spec} \mathbb{F}_{q^n}, X) = \mathbb{F}_{q^n}$-rational points of $X/\mathbb{F}_q$

and $\exp(a) = \sum_{n \geq 0} \frac{a^n}{n!}$

(iv) $\zeta(\mathbb{A}^1_{\mathbb{F}_q}, s) = \frac{1}{1-q^{1-s}}$

$\zeta(\mathbb{P}^1_{\mathbb{F}_q}, s) = \frac{1}{(1-q^{-s})(1-q^{1-s})}$

(v) $\zeta(\mathbb{A}^1 \times X, s) = \zeta(X, s-1)$ (X arith scheme)

Pf: (i) $\frac{1}{1-N(x)^{-s}} = (1 + \underbrace{N(x)^{-s} + N(x)^{-2s} + \dots}_{f_x}) = 1 + f_x$

Note $\{x \in X \text{ closed}\}$ countable

Let $X = \bigcup_{i=1}^{m} U_i$, $U_i = \operatorname{Spec} \underbrace{\frac{\mathbb{Z}[x_1, \dots, x_{n_i}]}{I_i}}_{A_i}$

$x \in U_i$ closed and if $k(x) \subset \mathbb{F}_{q^m}$

$\Rightarrow$ $A_i \longrightarrow A_i/m_x = k(x) \subset \mathbb{F}_{q^m}$

$\mathbb{Z}[x_1, \dots, x_n] \ni x_i \longmapsto \lambda_i$

i.e. $x \in X$ is determined by a tuple $\underbrace{(\lambda_1, \dots, \lambda_n) \in (\mathbb{F}_{q^m})^n}_{\text{only finitely many}}$

$\Rightarrow$ only fin many $x \in X$ closed have $N(x) \leq q^m$

$\Rightarrow \quad w(f_x) = N(x) > 1 \quad \forall x$

$w(f_x) \longrightarrow \infty \quad , x \longrightarrow \infty$

(somehow ordered)

$\Rightarrow \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1 - N(x)^{-s}} = \prod_{\substack{x \in X \\ \text{closed}}} (1 + f_x) \in D(\mathbb{C})$ by 10,(ii)

$\Rightarrow$ (i)

(ii) ✓

(iv) $\zeta(\mathbb{A}^1_{\mathbb{F}_q}, s) \underset{(iii)}{=} \exp\left(\sum_{n=1}^{\infty} |\mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^n})| \frac{q^{-ns}}{n} \right)$

$= \exp\left(\sum_{n=1}^{\infty} \frac{q^{(1-s)n}}{n} \right) \qquad \left[\log(1-X) = -\sum_{l=1}^{\infty} \frac{X^l}{l} \right]$

$= \frac{1}{1-q^{1-s}}$

$\zeta(\mathbb{P}^1_{\mathbb{F}_q}, s) \underset{(ii)}{=} \zeta(\mathrm{pt}, s) \cdot \zeta(\mathbb{A}^1, s) = \frac{1}{1-q^{-s}} \cdot \frac{1}{1-q^{1-s}}$

(v) By (ii) $\to$ wlog $X/\mathbb{F}_p$

$$\Rightarrow \zeta(\mathbb{A}^1 \times X, s) \underset{(iii)}{=} \exp\Big(\sum_{n=1}^{\infty} \underbrace{|\mathbb{A}^1 \times X(\mathbb{F}_{p^n})|}_{|\mathbb{A}^1(\mathbb{F}_{p^n})| \cdot |X(\mathbb{F}_{p^n})| = p^n \cdot |X(\mathbb{F}_{p^n})|} \frac{p^{-ns}}{n} \Big)$$

$$= \exp\Big(\sum_{n=1}^{\infty} |X(\mathbb{F}_{p^n})| \frac{p^{(1-s)n}}{n} \Big)$$

$$\underset{(iii)}{=} \zeta(X, 1-s)$$

It remains to prove (iii)

first

<u>(13) Lemma:</u> $x \in X/\mathbb{F}_q$ closed, set $\deg x := [k(x) : \mathbb{F}_q]$

$$\Rightarrow |X(\mathbb{F}_{q^n})| = \sum_{\substack{x \in X \text{ closed} \\ \deg x \mid n}} \deg x$$

<u>Pf:</u> Exercise

Proof of (iii) (Prop 12): $X/\mathbb{F}_q$

$$\log(\zeta(X,s)) = \sum_{\substack{x\in X\\ \text{closed.}}} -\log(1-N(x)^{-s})$$

$$= \sum_x \sum_{m=1}^{\infty} \frac{N(x)^{-sm}}{m}$$

$$\underset{N(x)=q^{\deg(x)}}{=} \sum_{m=1}^{\infty} \sum_x \frac{q^{-\deg(x)ms}}{m}$$

$$\underset{n:=\deg(x)m}{=} \sum_{n=1}^{\infty} \sum_{\substack{x\in X \text{ closed.}\\ \deg(x)\mid n}} \frac{q^{-ns}}{n}\cdot \deg(x)$$

$$= \sum_{n=1}^{\infty} \underbrace{\Big(\sum_{\deg(x)\mid n} \deg(x)\Big)}_{\overset{(13)}{=}\ |X(\mathbb{F}_{q^n})|} \frac{q^{-ns}}{n} \quad \underset{\exp}{\Longrightarrow} \quad \square$$

(14) **Recall** X scheme

$\Longrightarrow \dim X = \sup\{ n \geq 0 \mid \exists$ chain $\underset{\neq \emptyset}{Z_0} \subsetneq Z_1 \subsetneq \dots \subsetneq Z_n$ irred closed subsets $\}$

- k field $\Rightarrow \dim(\operatorname{Spec} k) = 0$
- X integral, fin type$/k$ $\Rightarrow \dim X = \operatorname{trdeg}(k(X)/k)$
  function field
- A ring
  $\Longrightarrow \dim \operatorname{Spec} A = \dim A = \sup\{ n \geq 0 \mid \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n$ prime ideals $\}$
  Krull dimension

  $\Rightarrow \dim \mathbb{Z} = \dim \operatorname{Spec} \mathbb{Z} = 1$
- $\dim \mathbb{A}^n_X = \dim X + n = \dim \mathbb{P}^n_X$
  X noetherian

- $X' \xrightarrow{\pi} X$ finite surjective, i.e.
  $\mathcal{U} = \operatorname{Spec} A \subset X$ open $\Rightarrow \pi^{-1}(\mathcal{U}) = \operatorname{Spec} A'$ some A'
  s.t. $\pi^*: A \to A'$ makes A' a finite A-module

$\Longrightarrow \dim X' = \dim X$

X arithmetic scheme ($\Rightarrow \dim X < \infty$)

(15) Thm

$\zeta(X,s)$ converges absolutely

on $\{ z \in \mathbb{C} \mid Re(z) > \dim X \}$

complex numbers

i.e.

$$\prod_{\substack{x \in X \\ \text{closed}}} \left| \frac{1}{1 - N(x)^{-z}} \right| \quad \text{converges}, \neq 0$$

$\forall z \in \mathbb{C}$ with $Re(z) > \dim X$

Pf: $\underline{0\ \text{Step}}$: $X = \operatorname{Spec} \mathbb{F}_q$ $\Rightarrow \dim X = 0$

$$\Rightarrow \zeta(X,s) = \frac{1}{1-q^{-s}} \rightarrow \text{OK}$$

1. Step: $X = X_1 \cup X_2$, $X_i \subset X$ closed

$$\Rightarrow X = X_1 \setminus (X_1 \cap X_2) \amalg X_2 \setminus (X_1 \cap X_2) \amalg (X_1 \cap X_2)$$

$$\overset{(12),(ii)}{\Longrightarrow} \zeta(X,s) = \frac{\zeta(X_1,s)}{\zeta(X_1 \cap X_2, s)} \cdot \frac{\zeta(X_2,s)}{\zeta(X_1 \cap X_2, s)} \cdot \zeta(X_1 \cap X_2, s)$$

$$= \frac{\zeta(X_1,s) \cdot \zeta(X_2,s)}{\zeta(X_1 \cap X_2, s)}$$

ind/dim $\Rightarrow$ wlog X irreducible

<u>2 Step:</u>

$Z \subsetneq X$ closed , $U = X \setminus Z$

$\dim Z < \dim X$

$$\Rightarrow \zeta(X,s) = \zeta(U,s) \cdot \underbrace{\zeta(Z,s)}_{\text{OK by ind. / dim}}$$

$\Rightarrow$ wlog U affine. irred

<u>3 Step:</u>

Assume

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \| & & \| \\ \operatorname{Spec} B & & \operatorname{Spec} A \end{array}$$

surj morph between irred affine arith schemes

s.t. $\exists\, d \geq 1 : A^d \twoheadrightarrow B$ (B is a finite A-module)

$\Rightarrow \dim X = \dim Y$ (see (14))

and $\forall\, y \in Y$ we have

$$|\pi^{-1}(y)| = |X \times_Y y| = |\operatorname{Spec} B/m_y B| \leq d \qquad (*)$$

$\uparrow$

$\dim_{A/m_y} (B/m_y B) \leq d$

$\uparrow$

$\dim$ as A/m_y - vector space

and $\forall \quad x \in \pi^{-1}(y)$

we have $\mathcal{K}(y) \subset \mathcal{K}(x)$

$$\Rightarrow \quad N(y) \leq N(x)$$

$u = \mathrm{Re}\, z > 0$

$$\Rightarrow \quad \left| \frac{1}{1 - N(x)^{-z}} \right| \leq \frac{1}{1 - N(x)^{-u}} \leq \frac{1}{1 - N(y)^{-u}}$$

$$\left[\underline{z = u + iv} \cdot \quad \left| \frac{1}{1 - N(x)^{-z}} \right|^2 = \frac{N(x)^{2u}}{|N(x)^z - 1|^2} = \frac{N(x)^{2u}}{N(x)^{2u} - 2N(x)^u \cos(v \ln N(x)) + 1} \leq \frac{N(x)^{2u}}{(N(x)^u - 1)^2} \right]$$

$$\Rightarrow$$

$$1 \leq \prod_{\substack{x \in X \\ \text{closed}}} \left| \frac{1}{1 - N(x)^{-z}} \right| = \prod_{\substack{y \in Y \\ \text{closed}}} \prod_{x \in \pi^{-1}(y)} \left| \frac{1}{1 - N(x)^{-z}} \right|$$

$$\underset{(*)}{\leq} \prod_{\substack{y \in Y \\ \text{closed}}} \left(\frac{1}{1 - N(y)^{-u}} \right)^d = \zeta(Y, u)^d$$

i.e. if $\zeta(Y, z)$ is abs convergent for $\mathrm{Re}(z) > \dim Y$

then so is $\zeta(X, z)$

4. Step: X irred affine arith scheme

1. case Assume

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathrm{Spec}\,\mathbb{Z} \\ & \mathrm{Spec}\,\mathbb{F}_p \arrow[u] \end{tikzcd}$$

i.e. X of finite type over $\mathbb{F}_p$

Noether normalization $\Longrightarrow$

$\exists$ $X \to \mathbb{A}^n_{\mathbb{F}_p}$ fin surj. , $n = \dim X$

$\textcircled{12}$, iv) $\Longrightarrow$ $\zeta(\mathbb{A}^n_{\mathbb{F}_p}, s) = \dfrac{1}{1-q^{n-s}}$ abs cgt for $\mathrm{Re}(s) > n$

$\underset{\text{3. Step}}{\Longrightarrow}$ $\zeta(X, s)$ —— " ——

2. case

$$\begin{tikzcd} X \arrow[r, two heads] & \mathrm{Spec}\,\mathbb{Z} \\ X_{\mathbb{Q}} \arrow[u] \arrow[r] & \mathrm{Spec}\,\mathbb{Q} \arrow[u] \end{tikzcd}$$

$\underset{N.N.}{\Longrightarrow}$ $X_{\mathbb{Q}} \to \mathbb{A}^n_{\mathbb{Q}}$ fin surj

$\Longrightarrow$ $\exists$ $U \subset \mathrm{Spec}\,\mathbb{Z}$ open

s.t. $X_U \to \mathbb{A}^n_U$ fin surj $\Longrightarrow \dim X = n+1$

By 2nd step suff to show $\zeta(\mathbb{A}^n_U, z)$ abs cgt for $\mathrm{Re}(z) > n+1$

(12)(v) $\Rightarrow$ $\zeta(A^n_{\mathcal{U}}, z) = \zeta(\mathcal{U}, z-n)$

and $$\zeta(\mathcal{U}, z-n) = \prod_{p \in \mathcal{U}} \frac{1}{1-p^{n-z}}$$

and
$$\prod_{p \in \mathcal{U}} \left| \frac{1}{1-p^{n-z}} \right| = \prod_{p \in \mathcal{U}} \left| 1 + p^{n-z} + p^{2(n-z)} + p^{3(n-z)} + \dots \right|$$

$$\leq \prod_{p \in \mathcal{U}} 1 + p^{n-\mathrm{Re}(z)} + p^{2(n-\mathrm{Re}(z))} + p^{3(n-\mathrm{Re}(z))} + \dots$$

abs convergent for $\mathrm{Re}(z) > n+1$

$\square$
