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Exercise sheet 1
Algebraic curves and the Weil conjectures

Let $\bar{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}_p$ and let $\mathbb{F}_q \subset \bar{\mathbb{F}}$ be the field with $q = p^r$ elements. Let X be a scheme of finite type over $\mathbb{F}_q$. Recall that the $\mathbb{F}_{q^n}$-rational points of X are given by

$$X(\mathbb{F}_{q^n}) := \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{F}_q}(\mathrm{Spec}\,\mathbb{F}_{q^n}, X).$$

(Note that the notation $X(\mathbb{F}_{q^n})$ is imprecise, as it does not only depend on the abstract scheme X but on the morphism $X \to \mathrm{Spec}\,\mathbb{F}_q$; however if this morphism is fixed, we usually drop it from the notation.)

Exercise 1.1. Let X be as above and denote by X_0 the set of closed points in X. For $x \in X_0$ we define $\deg(x) = [\kappa(x) : \mathbb{F}_q]$(= vector space dimension of $\kappa(x)$ over $\mathbb{F}_q$), where $\kappa(x)$ is the residue field associated with x. The aim of this exercise is to show:

$$|X(\mathbb{F}_{q^n})| = \sum_{\substack{x \in X_0 \\ \deg(x)|n}} \deg(x). \tag{1.1}$$

To this end proceed as follows:

(1) Show (recall) that $\kappa(x)/\mathbb{F}_q$ is a finite Galois extension, for all $x \in X$. In particular, $\kappa(x) \cong \mathbb{F}_{q^d} \subset \bar{\mathbb{F}}$ with $d = \deg(x)$.
(2) For x and d as above, denote by $\sigma_x \in X(\mathbb{F}_{q^d})$ the rational point induced by the composition

$$\sigma_x : \mathrm{Spec}\,\mathbb{F}_{q^d} \cong \mathrm{Spec}\,\kappa(x) \hookrightarrow \mathrm{Spec}\,\mathcal{O}_{X,x} \to X.$$

Show that $|\sigma_x \circ \mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)| = d$, where $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q)$ denotes the Galois group, which we can identify with the automorphisms of $\mathrm{Spec}\,\mathbb{F}_{q^d}$ over $\mathrm{Spec}\,\mathbb{F}_q$.
(3) Given d, n, then: $\mathbb{F}_{q^d} \subset \mathbb{F}_{q^n} \iff d|n$.
(4) Show (1.1).

Exercise 1.2. Give a product formula for $\zeta(\mathbb{A}^n/\mathbb{F}_q, s)$ and $\zeta(\mathbb{P}^n/\mathbb{F}_q, s)$.

For X as above, set

$$Z(X/\mathbb{F}_q, t) := \exp(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \cdot \frac{t^n}{n}) \in \mathbb{Q}[[t]]. \tag{1.2}$$

Exercise 1.3. Let X be the affine variety over $\mathbb{F}_2$ given by

$$X = \operatorname{Spec} \mathbb{F}_2[x, y]/(x^2 + x + y^2 + y + 1) \subset \mathbb{A}^2_{\mathbb{F}_2}$$

and denote by $\bar{X}$ its closure in $\mathbb{P}^2_{\mathbb{F}_2}$. Show (with the notation from (1.2))

(1) $Z(X/\mathbb{F}_2, t) = 1 + 4t^2 + \text{higher terms}\ldots$
(2) $Z(\bar{X}/\mathbb{F}_2, t) = 1 + t + 5t^2 + \text{higher terms}\ldots$

Exercise 1.4. Let X be a finite type scheme over $\mathbb{F}_q$. Then, the inclusion $\mathbb{F}_p \hookrightarrow \mathbb{F}_q$ defines a morphism $\operatorname{Spec} \mathbb{F}_q \to \operatorname{Spec} \mathbb{F}_p$ and we can view X as a finite type $\mathbb{F}_p$-scheme via this morphism. Give a formula relating $Z(X/\mathbb{F}_q, t)$ and $Z(X/\mathbb{F}_p, t)$.