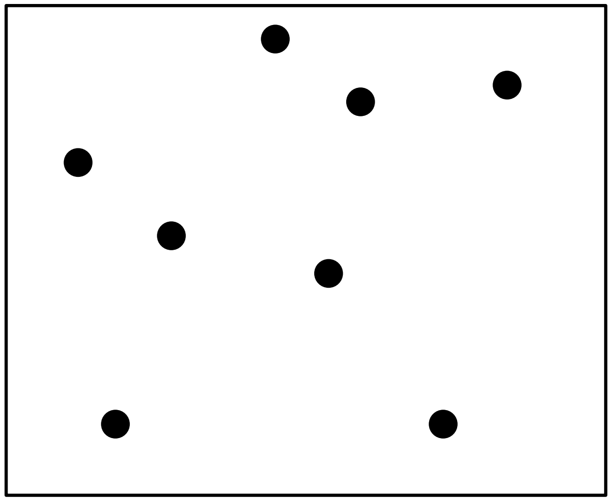
EuroCG 2020 Ph. D. School on Computational Geometry

Counting and Enumeration in Geometry

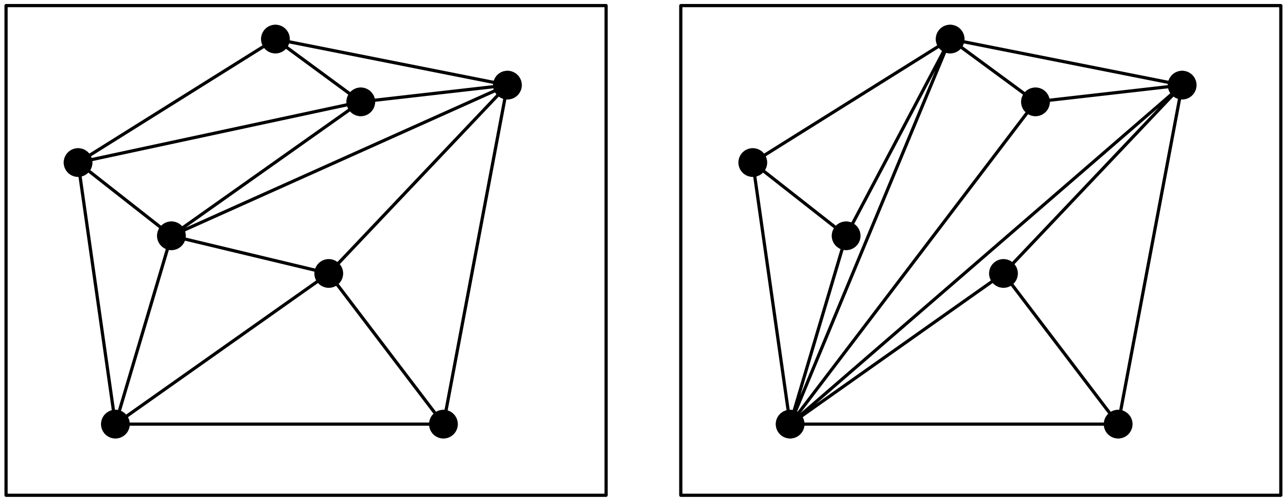
Günter Rote

Freie Universität Berlin

Triangulations of a point set

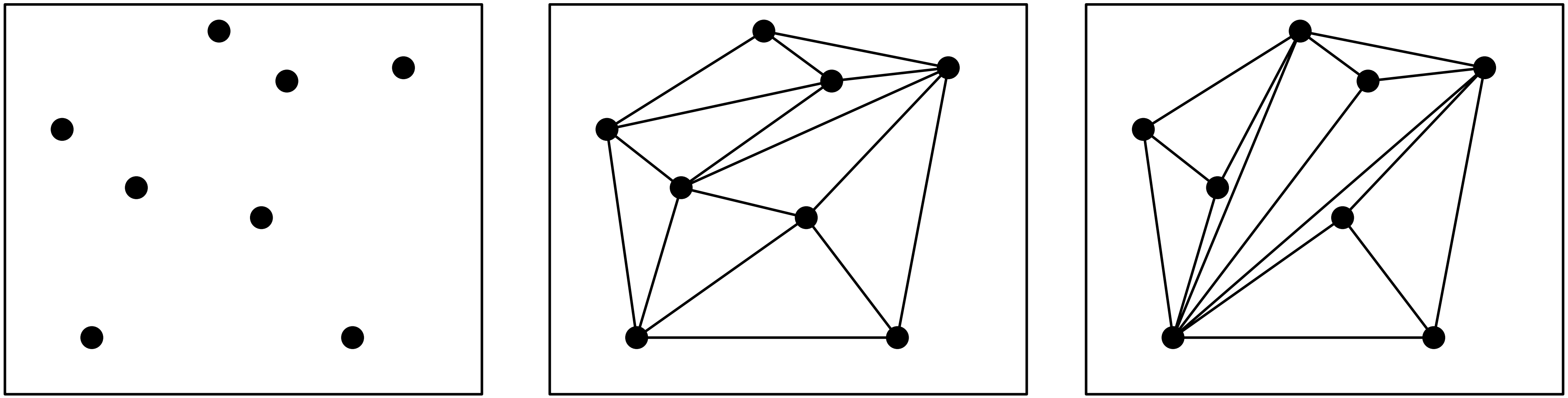


a point set



two triangulations

Triangulations of a point set



a point set — two triangulations

COUNT: How many triangulations does a given point set have?

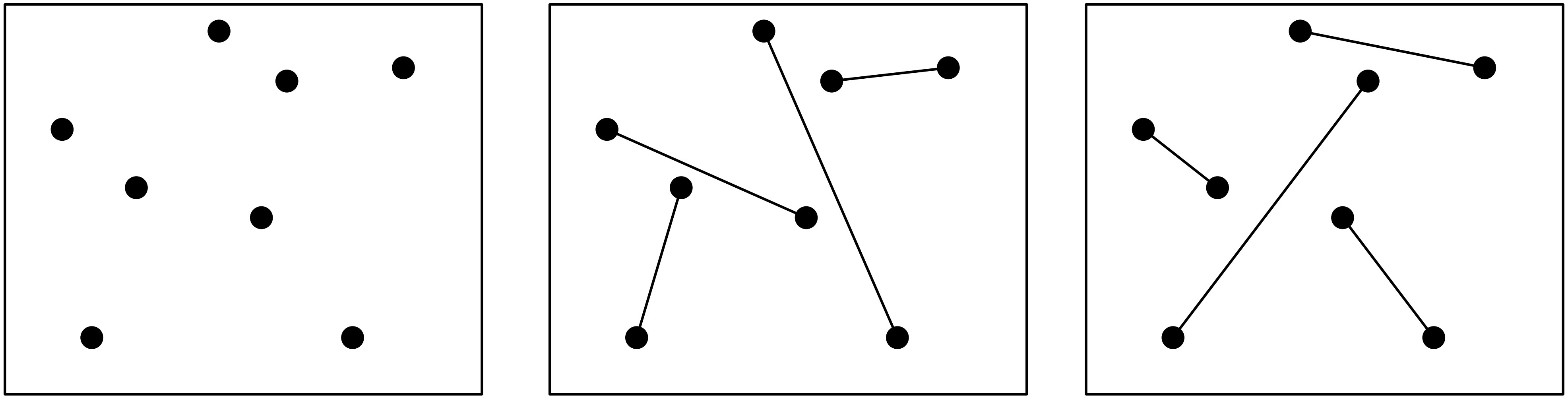
SAMPLE: Generate a random triangulation (uniformly)

ENUMERATE (list, visit) all triangulations of a given point set.

OPTIMIZE: Find the "best" triangulation of a given point set.

EXTREMAL QUESTION: How many triangulations can a set of n points have? at most? at least?

Other noncrossing geometric structures



a point set   two non-crossing perfect matchings

- triangulations
- non-crossing spanning trees
- non-crossing Hamiltonian cycles
- non-crossing matchings
- non-crossing perfect matchings
- . . .
- *[your favorite straight-line geometric graph structure]*

The extremal question

Given a set of n points in the plane (in general position), how many

- triangulations
- non-crossing spanning trees
- non-crossing Hamiltonian cycles
- non-crossing matchings
- non-crossing perfect matchings
- ...
- *[your favorite straight-line geometric graph structure]*

can it have, at most? (at least?)

`https://adamsheffer.wordpress.com/numbers-of-plane-graphs/`

The extremal question

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Numbers of Plane Graphs

https://adamsheffer.wordpress.com/numbers-of-plane-graphs/

We first consider the more popular variants – those with new works studying them every several years.

GRAPH TYPE	LOWER BOUND	REFERENCE	UPPER BOUND	REFERENCE
Plane Graphs	$\Omega(42.11^N)$	[HPS18]	$O(187.53^N)$	[SS12]
Triangulations	$\Omega(8.65^N)$	[DSST11]	30^N	[SS11]
Spanning Cycles	$\Omega(4.64^N)$	[GNT00]	$O(54.55^N)$	[SSW13]
Perfect Matchings	$\Omega(3.09^N)$	[AR15]	$O(10.05^N)$	[SW06]
Spanning Trees	$\Omega(12.52^N)$	[HM13]	$O(141.07^N)$	[HSSTW11; SS11]
Cycle-Free Graphs	$\Omega(13.61^N)$	[HM13]	$O(160.55^N)$	[HSSTW11; SS11]

Some less common variants:

New Horizons in Geomet... Micha Sharir

We're Hiring!

Recent Comments

Pascal on An Algorithms C... with Mini...

Ajmain Yamin... Teenagers do Mathematica...

Adam Sheffer... Points in General Posit...

r57shell on P... in General Po...

Incidences: O... Pro... on Incid... Lower Bound... (part...

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Think of some particular point set and COUNT its triangulations.

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Optimization

Given a set of points, find the triangulation that

- has the smallest total edge length
- minimizes the largest angle
- maximizes the smallest angle
- maximizes the total area of all triangles
- minimizes the total *squared* edge length
- is a good spanner
- ...

Enumerating all triangulations and taking the best one always works.

Optimization

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Given a set of points, find the triangulation that

- has the smallest total edge length NP-hard, quasipolynomial
- minimizes the largest angle polynomial
- maximizes the smallest angle Delaunay
- maximizes the total area of all triangles easy
- minimizes the total *squared* edge length ??
- is a good spanner ??
- . . .

Enumerating all triangulations and taking the best one always works.

Overview

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0. Introduction

1. Count triangulations [Alvarez and Seidel, 2013]
   - and perfect matchings [Wettstein, 2014]
   - Optimal triangulations

2. Coordinated primal-dual sweep
   [Biedl, Chambers, Kostitsyna, Rote, Felsner, 2020]

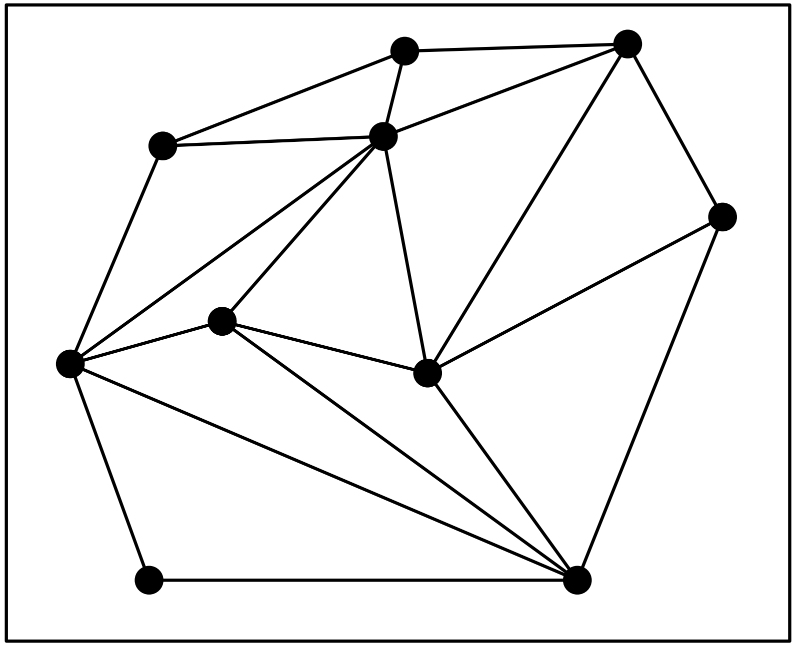
3. Count perfect matchings of structured point sets
   [Asinowski and Rote, 2018]

4. Production matrices [Huemer, Pilz, Seara, Silveira, 2016]

1. Count Triangulations

Count, sample, enumerate [V. Alvarez, R. Seidel, 2013]

triangulation



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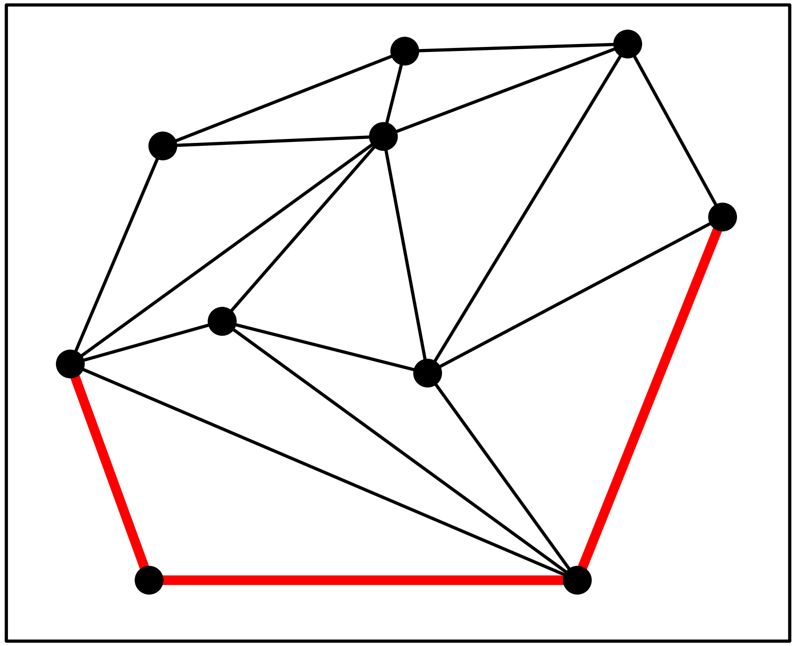
triangulation $\rightarrow$ sequence of x-monotone *ropes*



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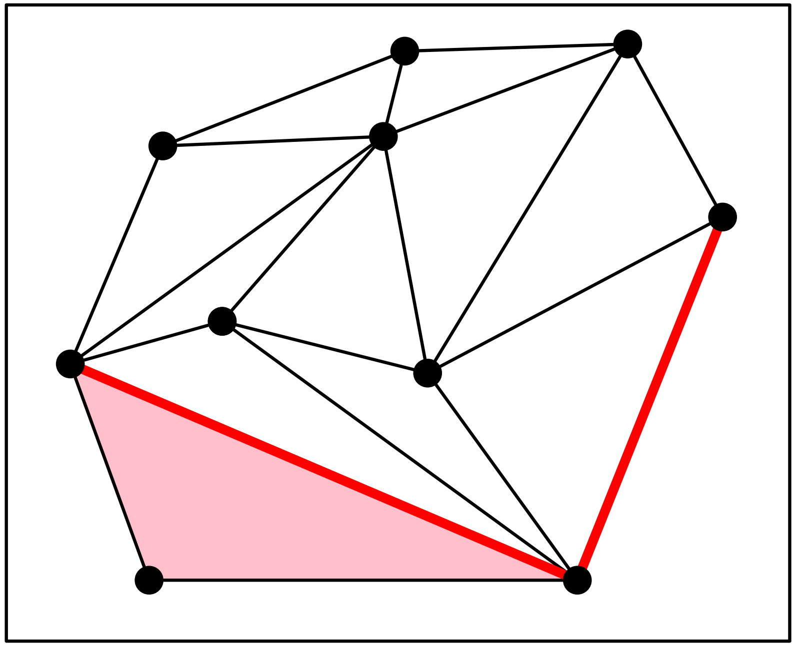

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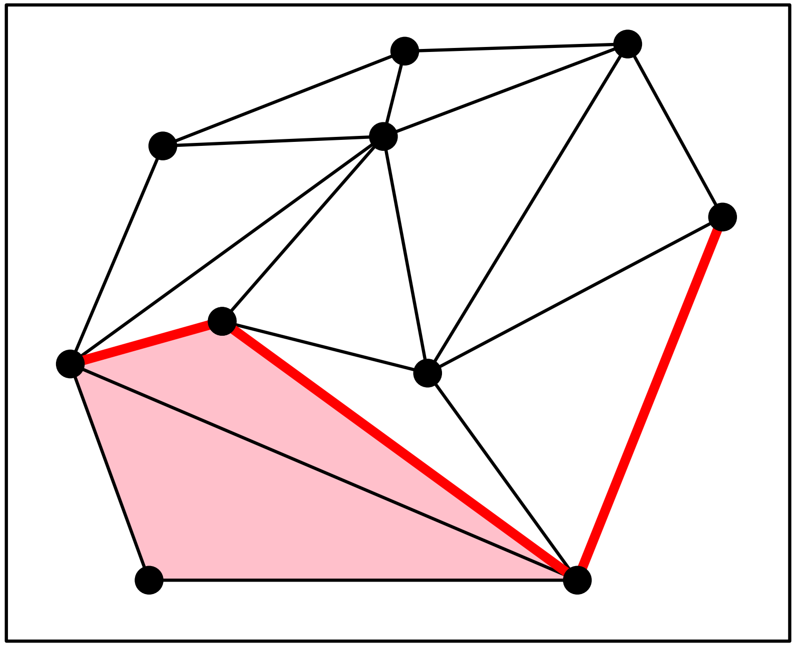

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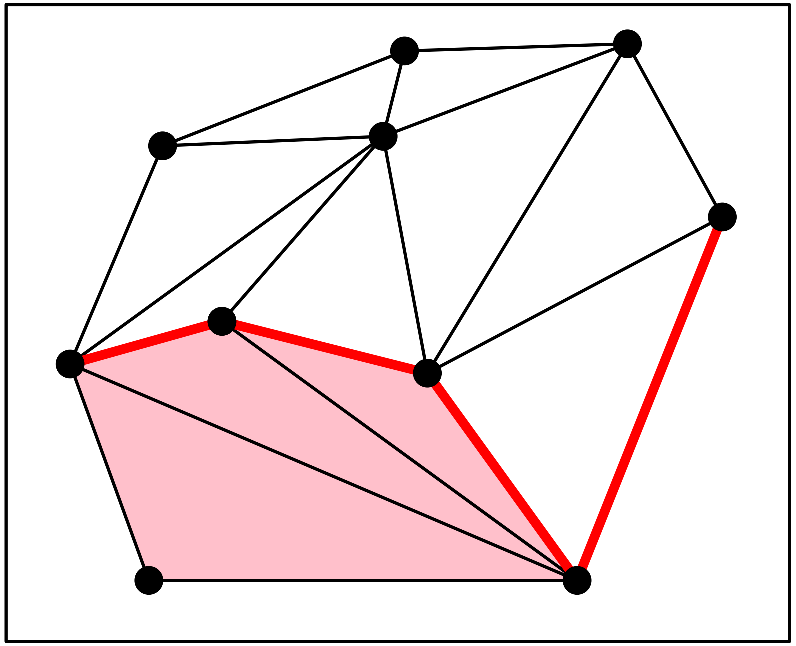

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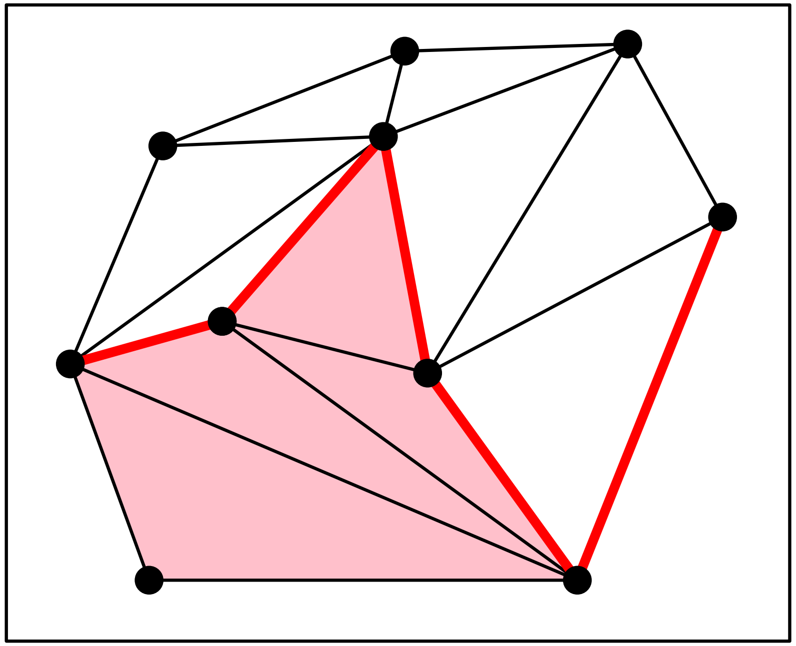

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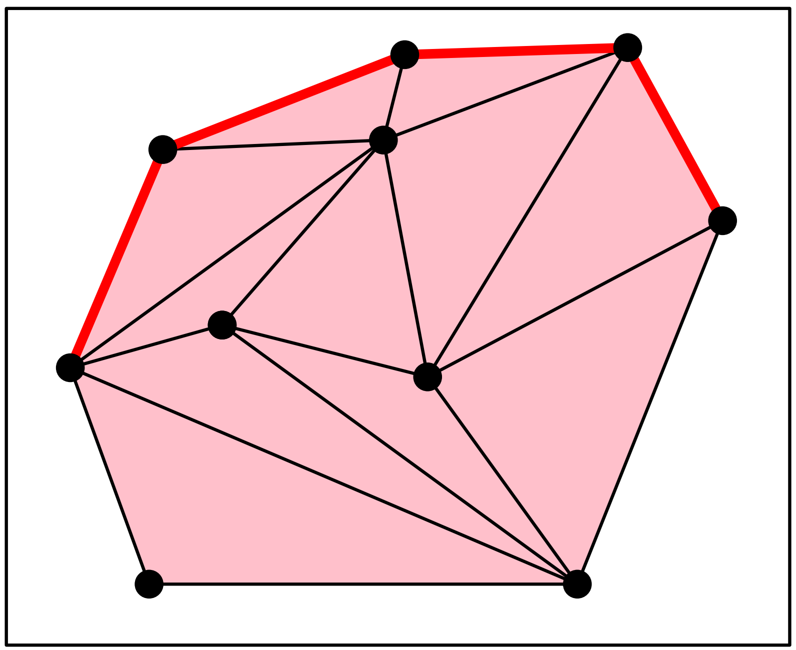

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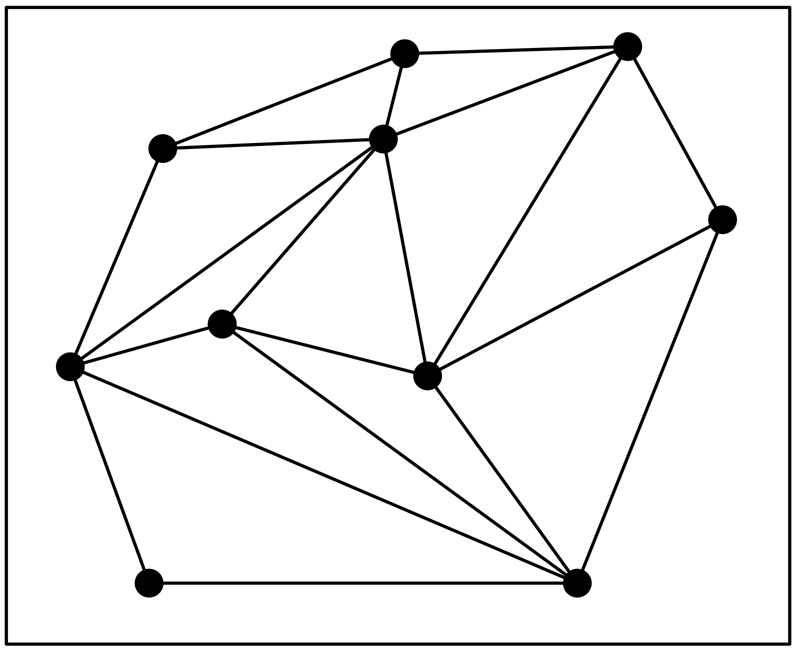

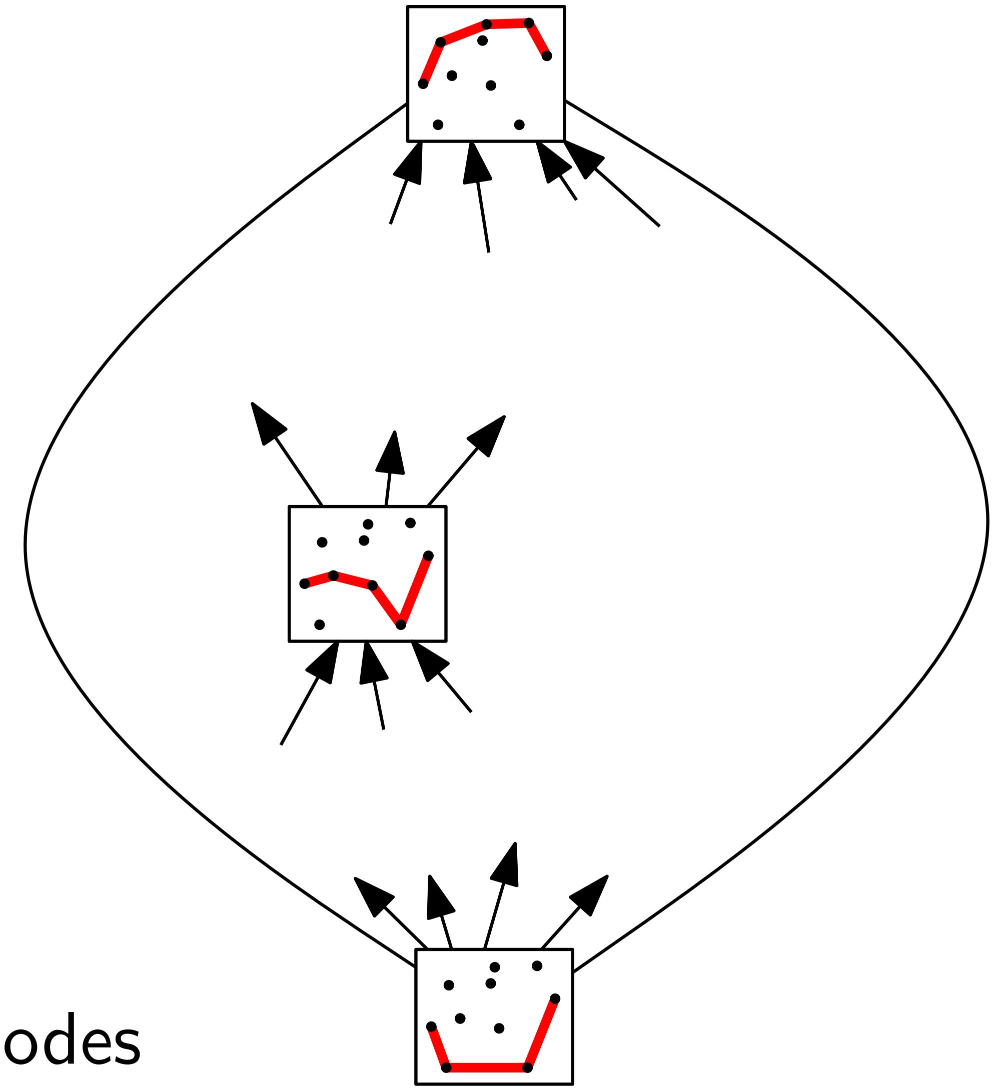

→ path in a DAG with 2^{n-2} nodes

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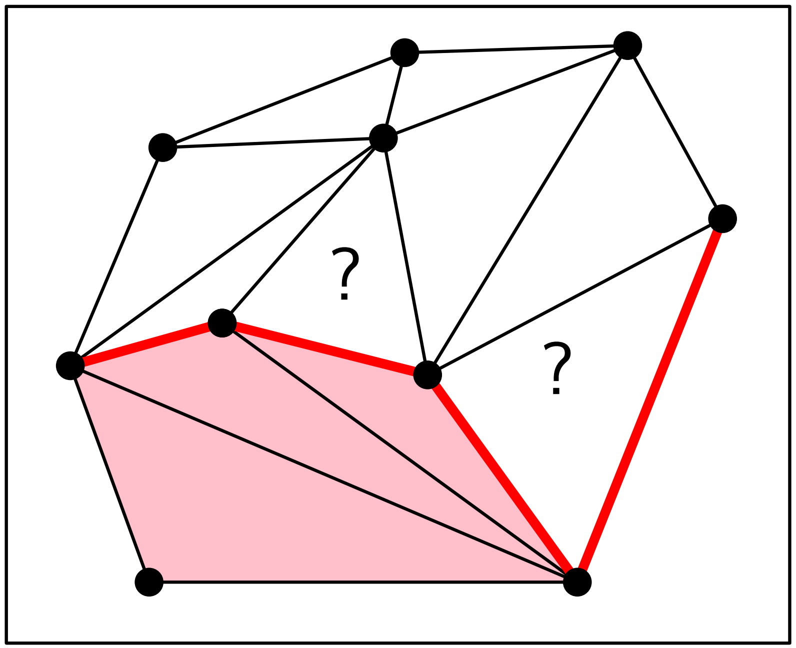

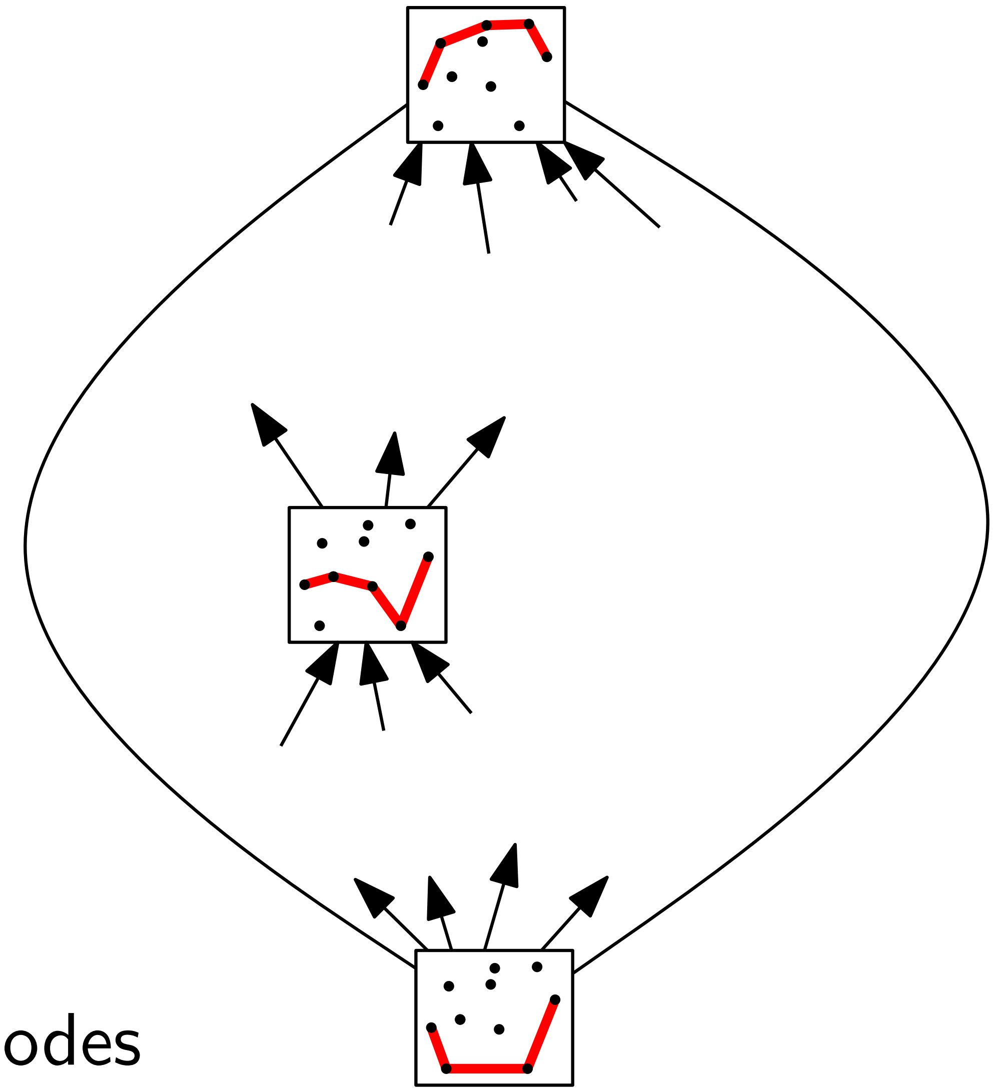

Not one-to-one!

$\rightarrow$ path in a DAG with 2^{n-2} nodes

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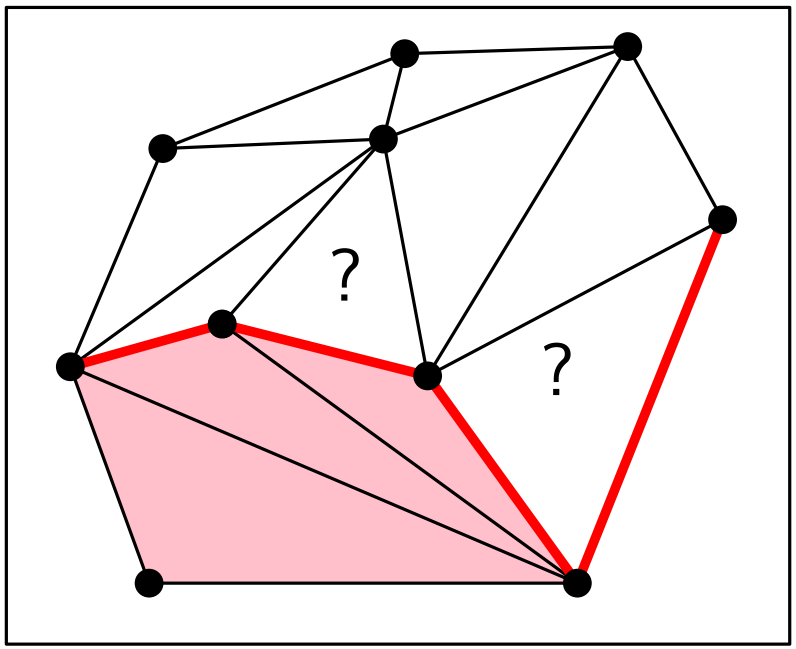

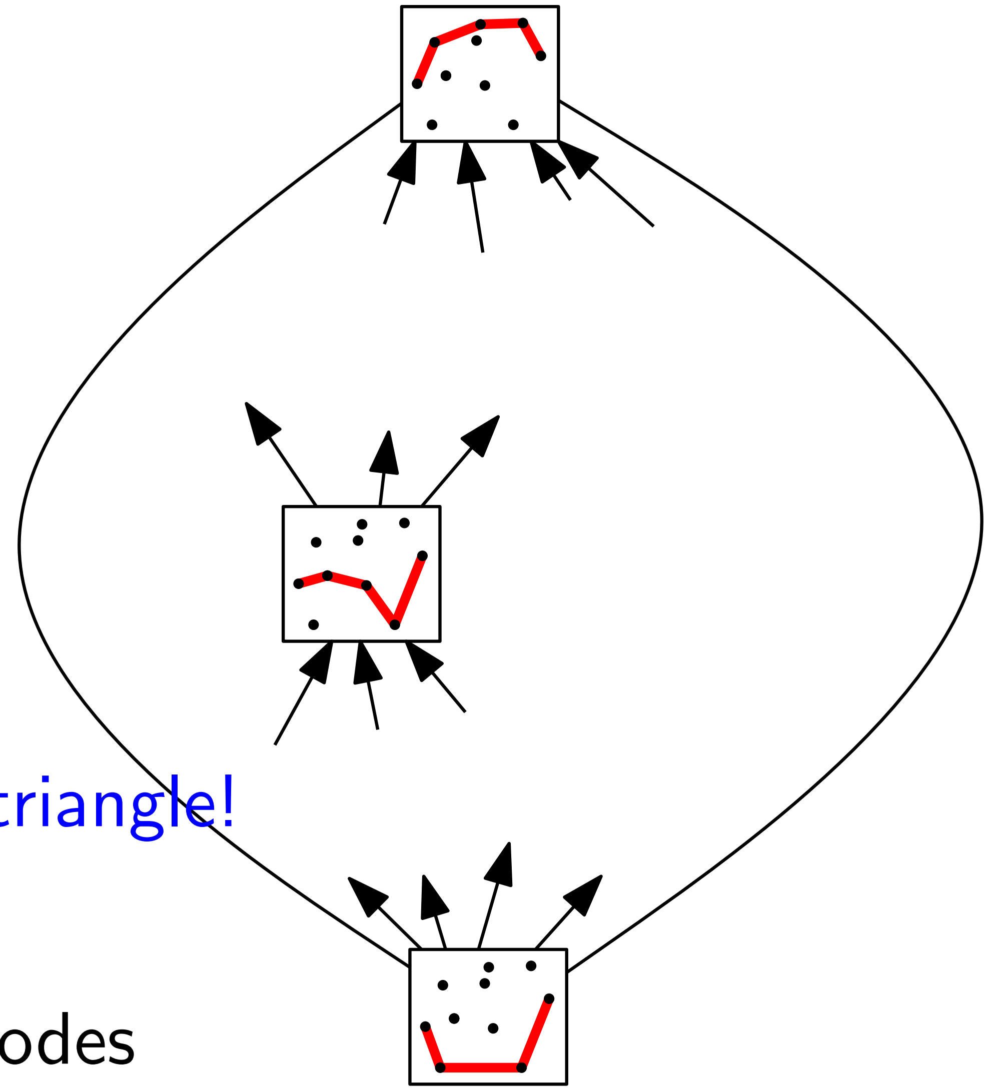

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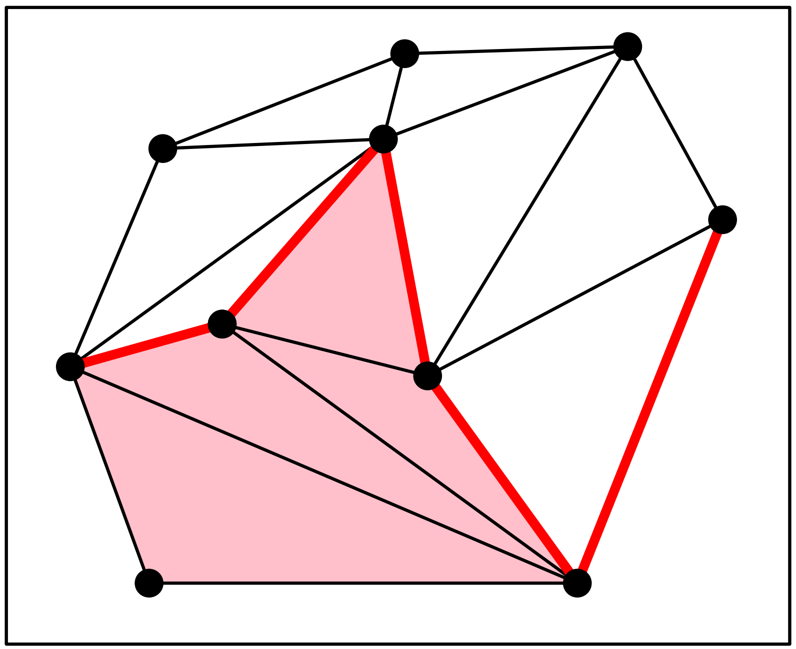

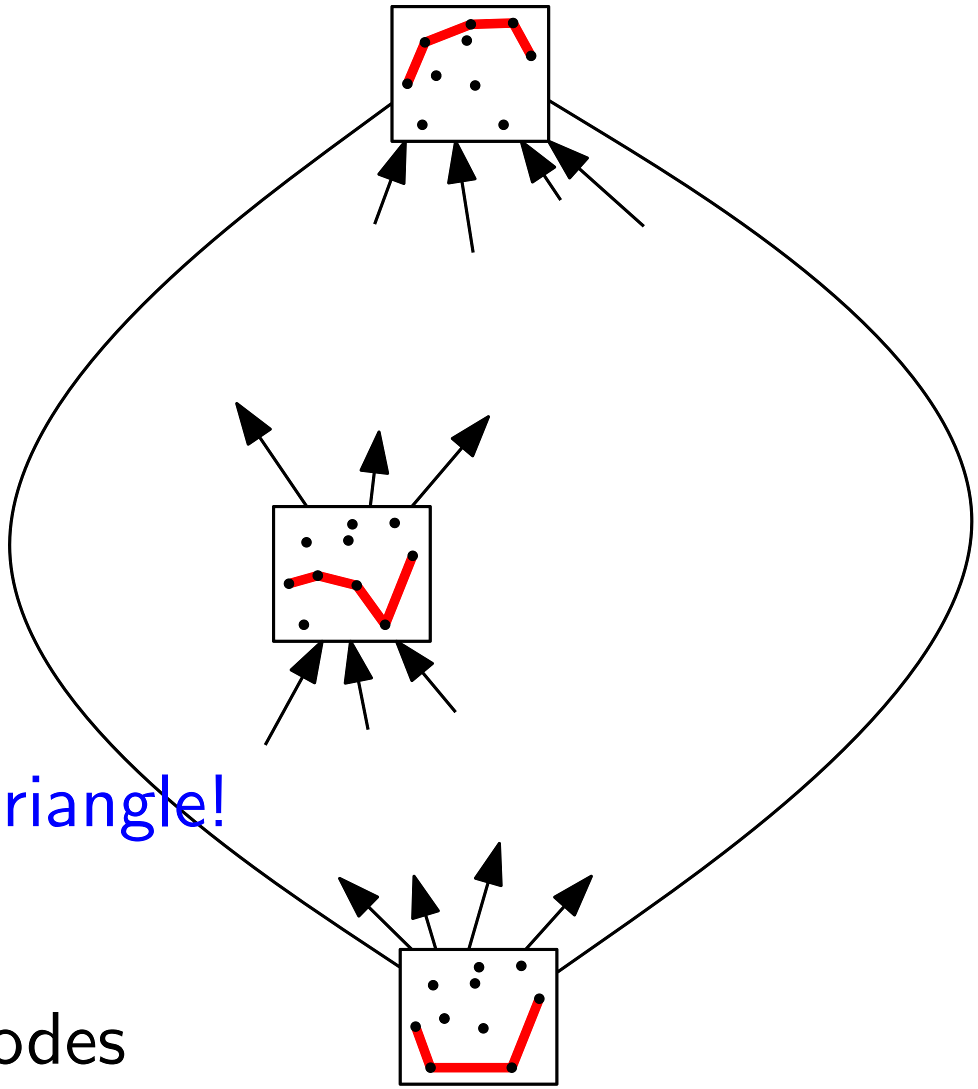

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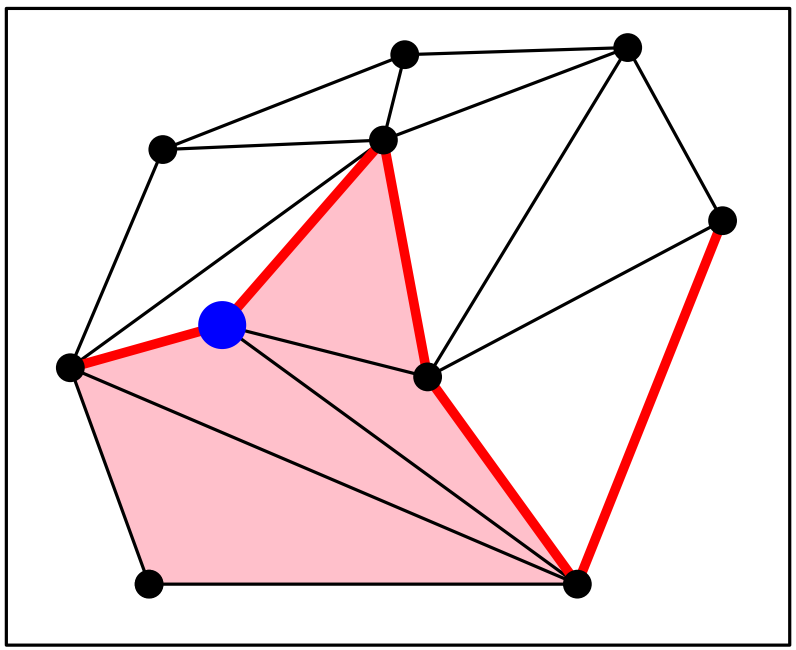

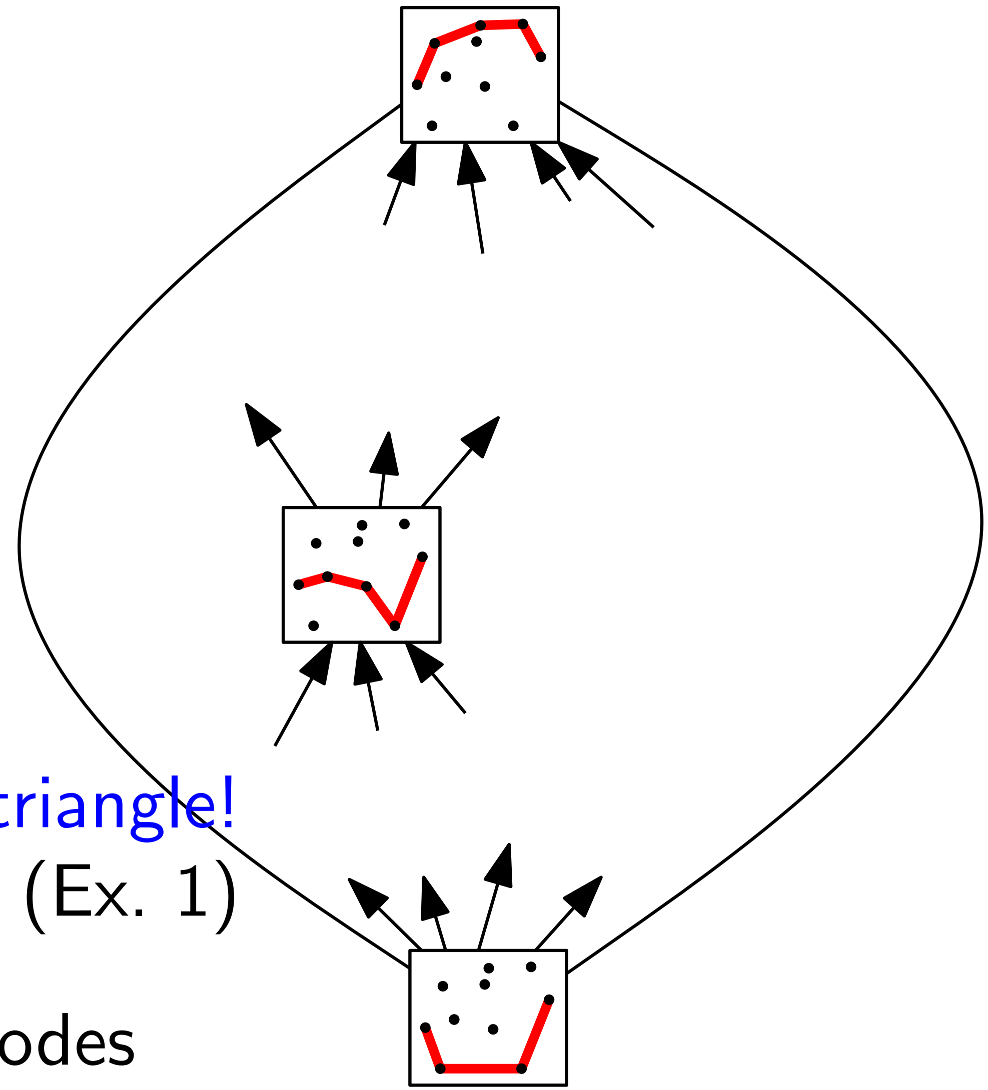

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MARK the position of change. (Ex. 1)

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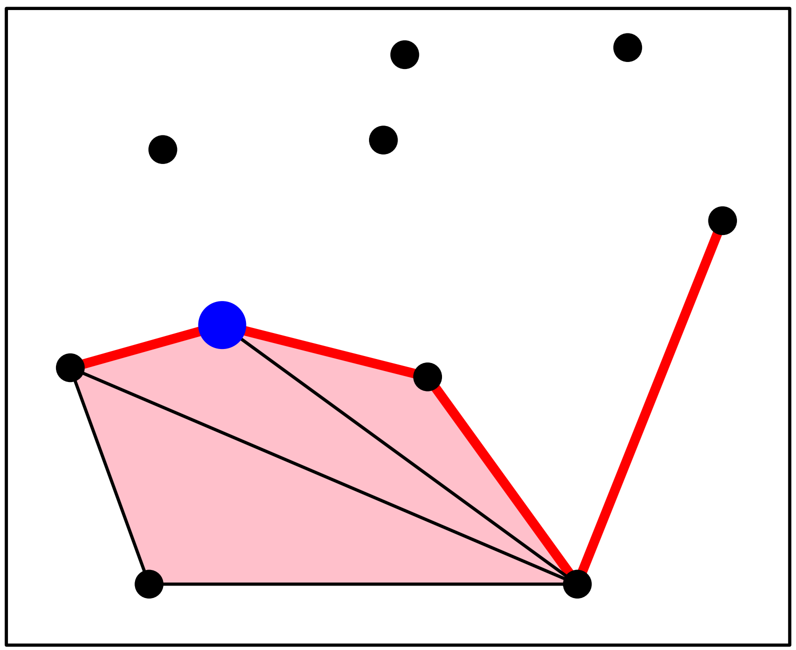

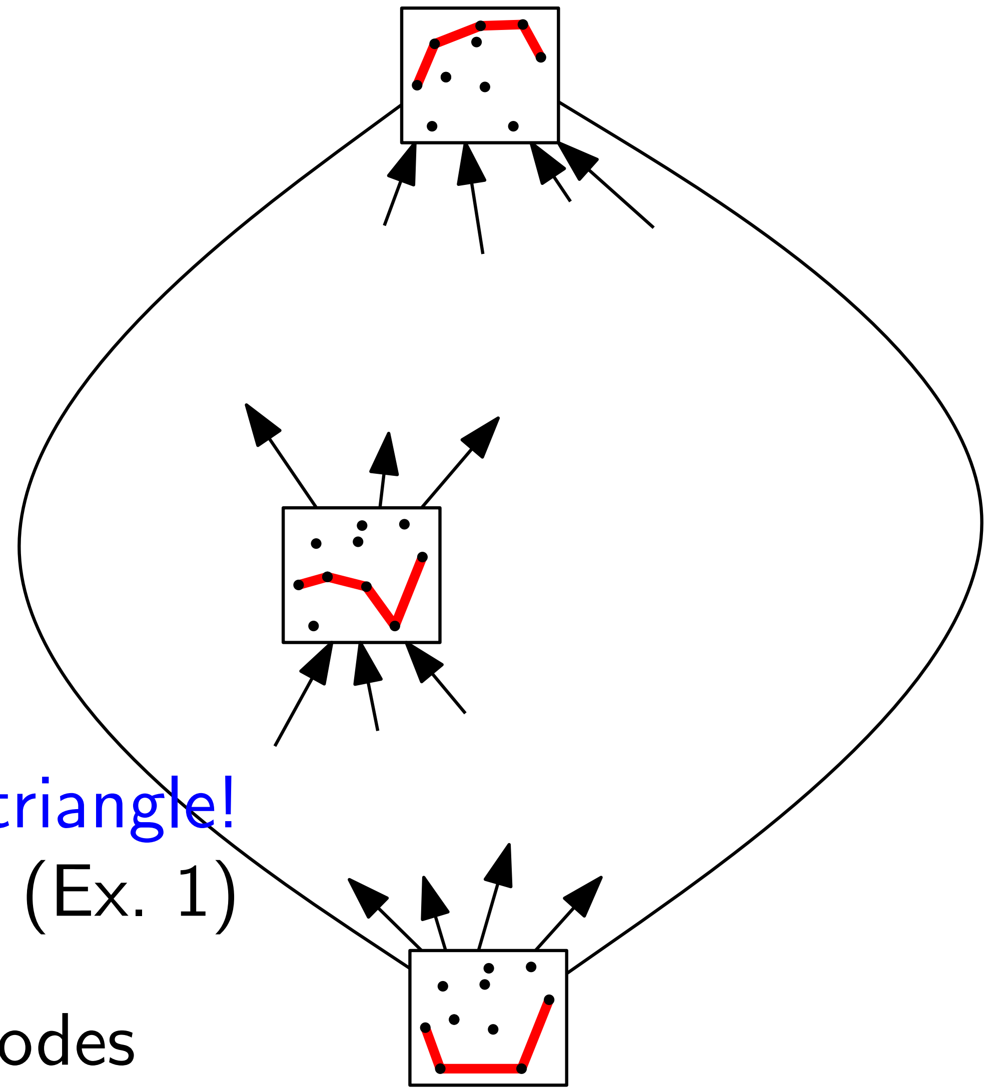

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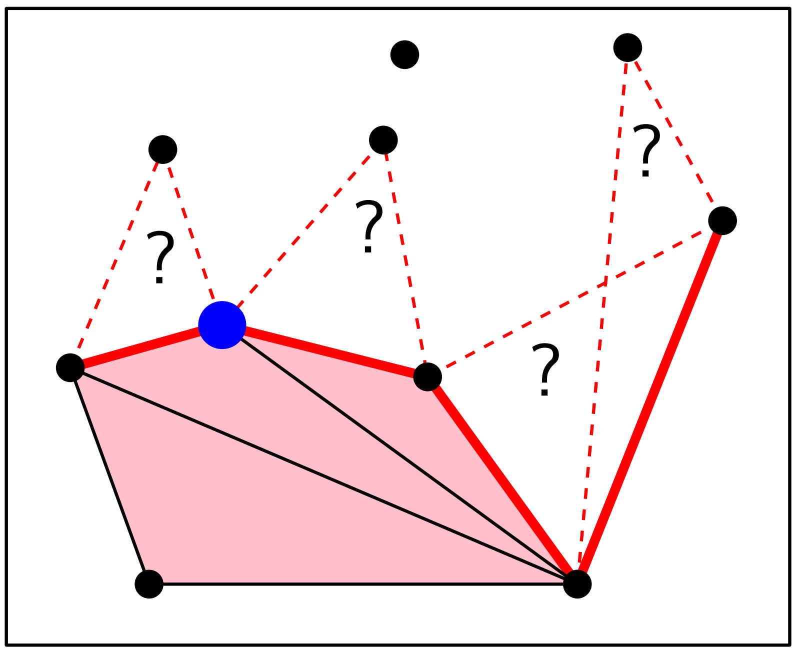

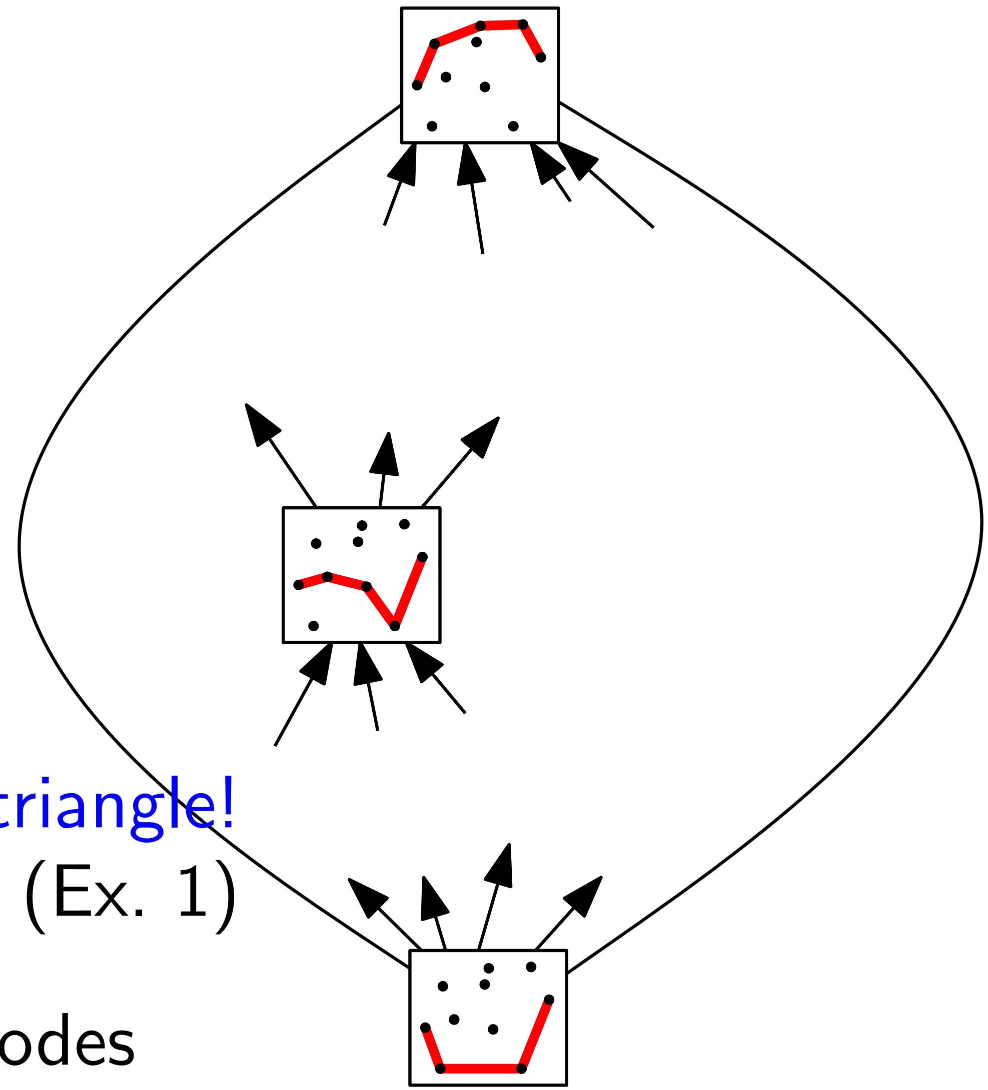

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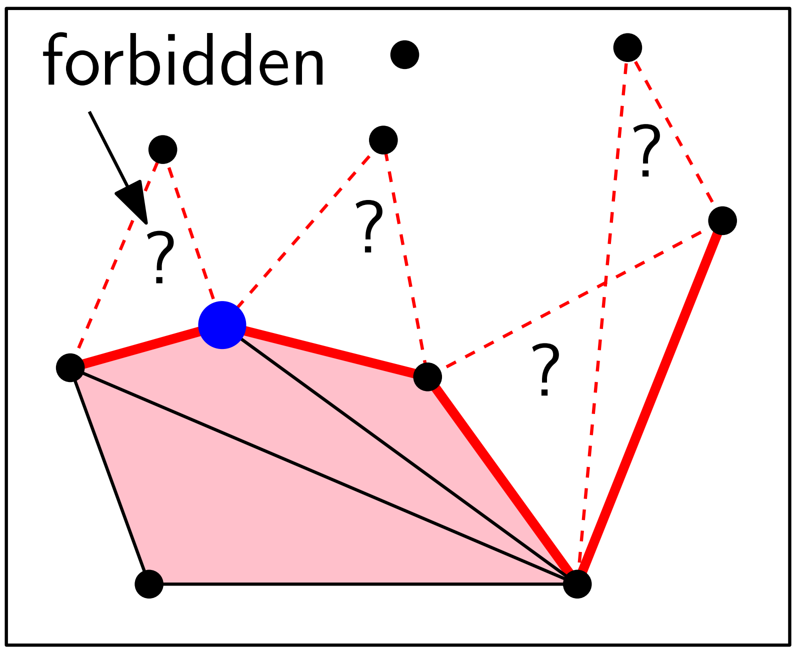

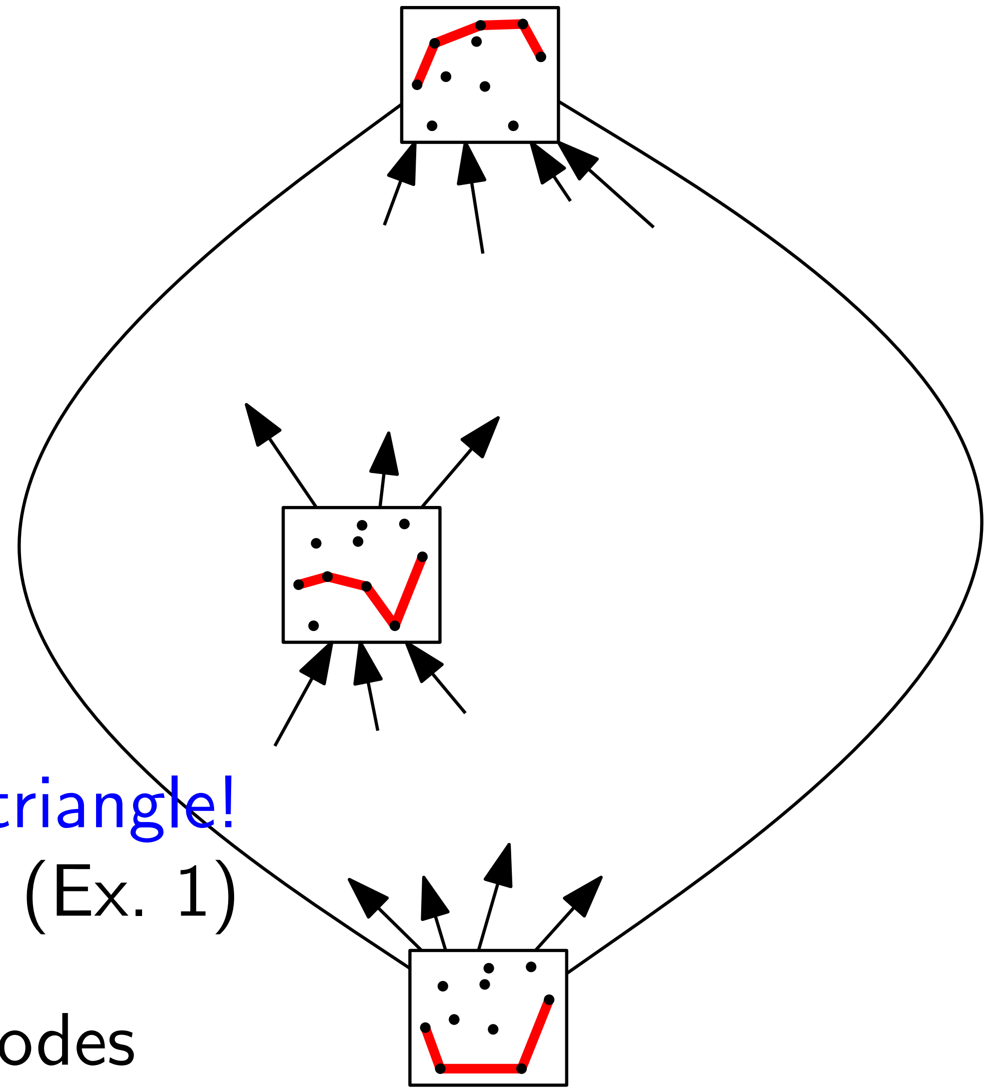

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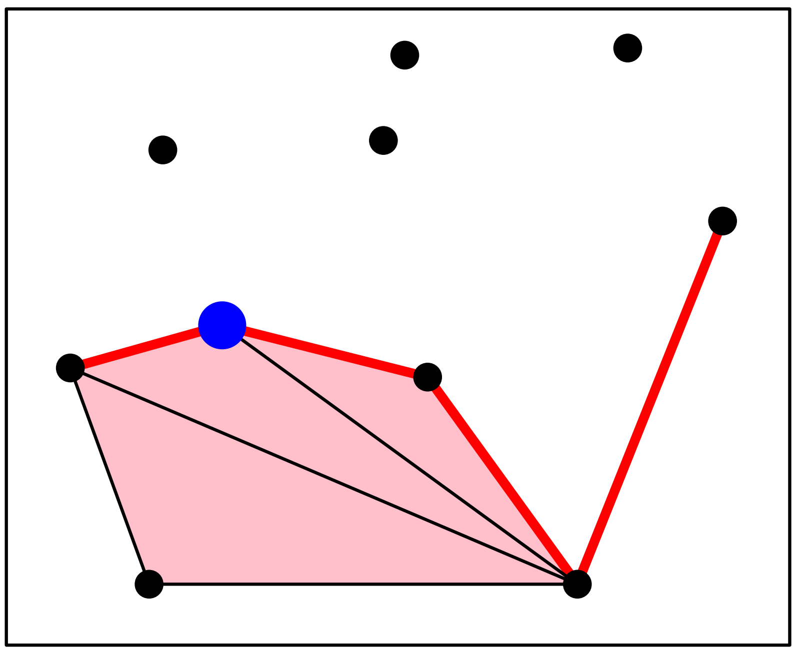

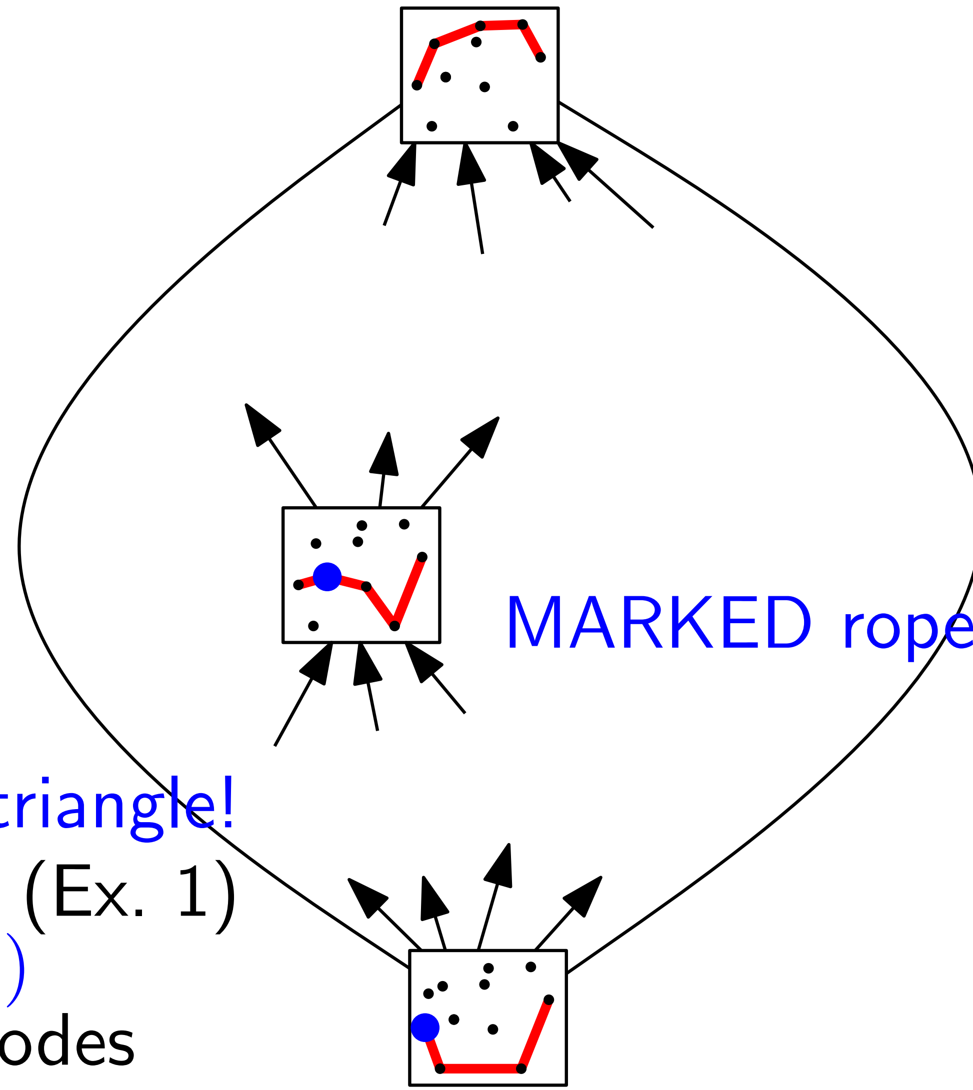

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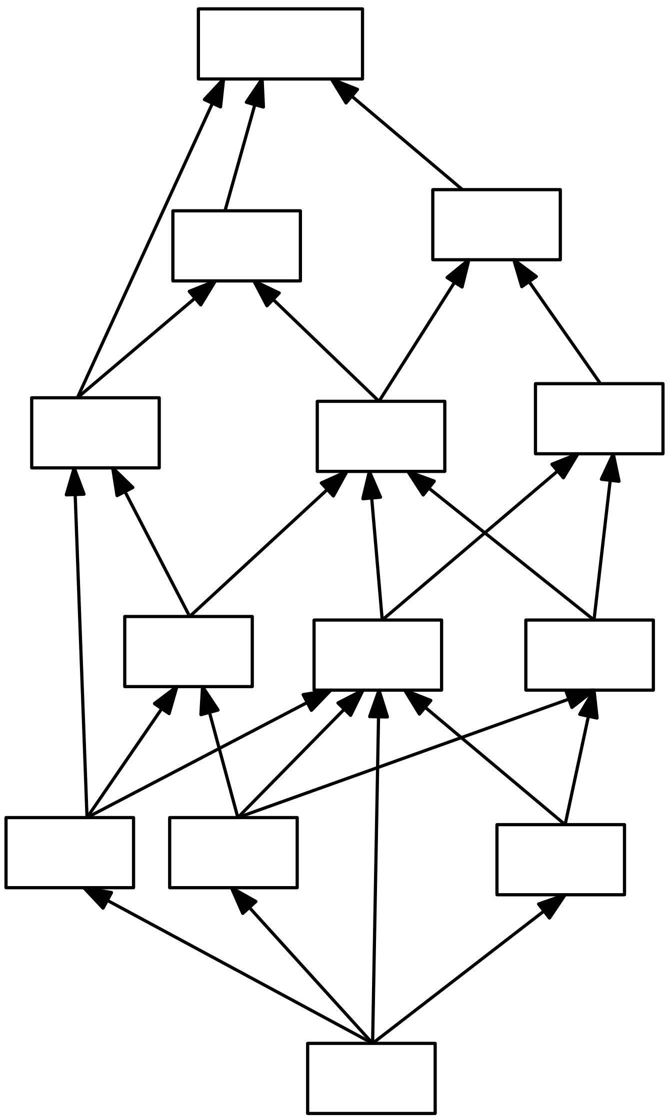
$\rightarrow$ path in a DAG with ~~2^{n-2}~~ $O(n2^n)$ nodes

Counting source-sink paths in a DAG



$N(v) :=$ #paths from source to v

Compute $N(v)$ from source to sink.

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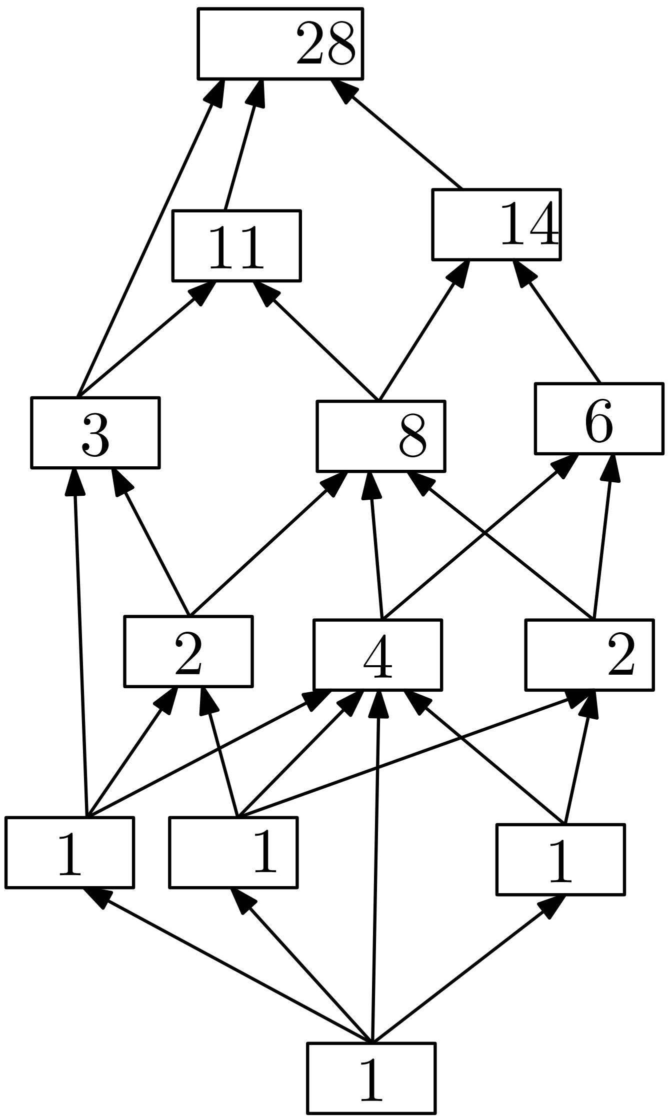

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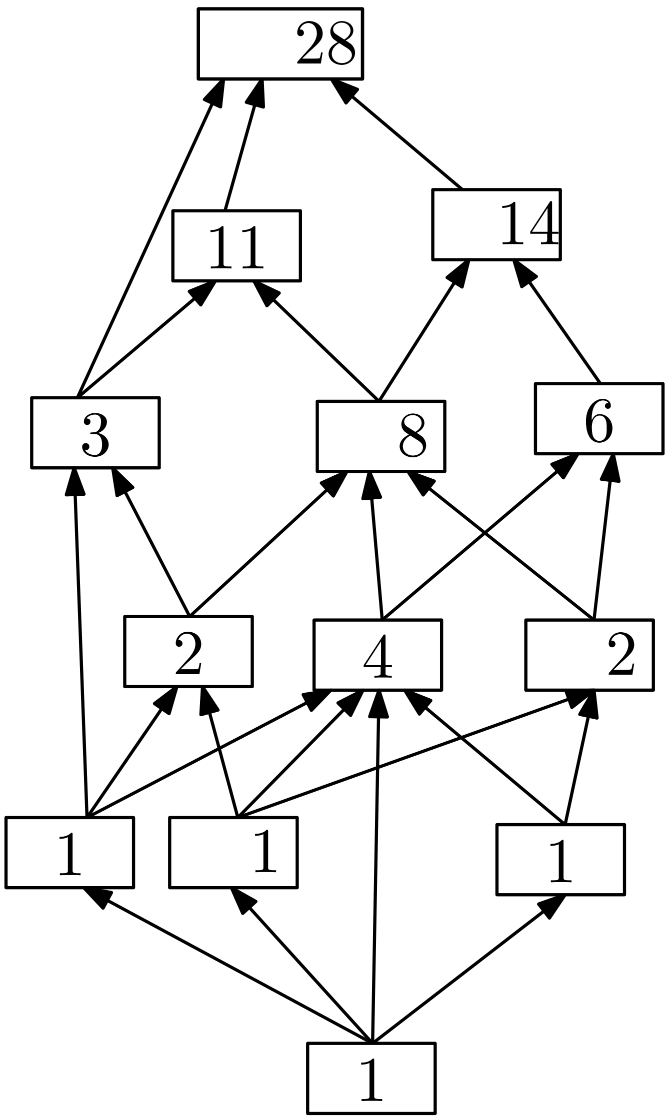

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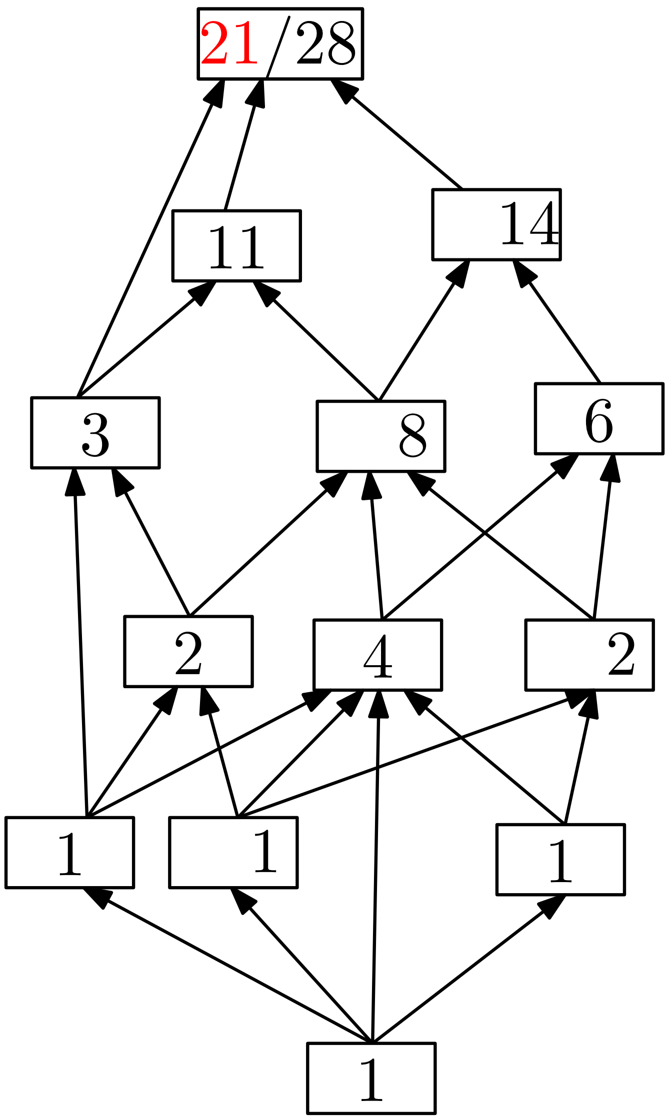
How to SAMPLE a random path:

Find a random number between 1 and 28.

...

Counting source-sink paths in a DAG



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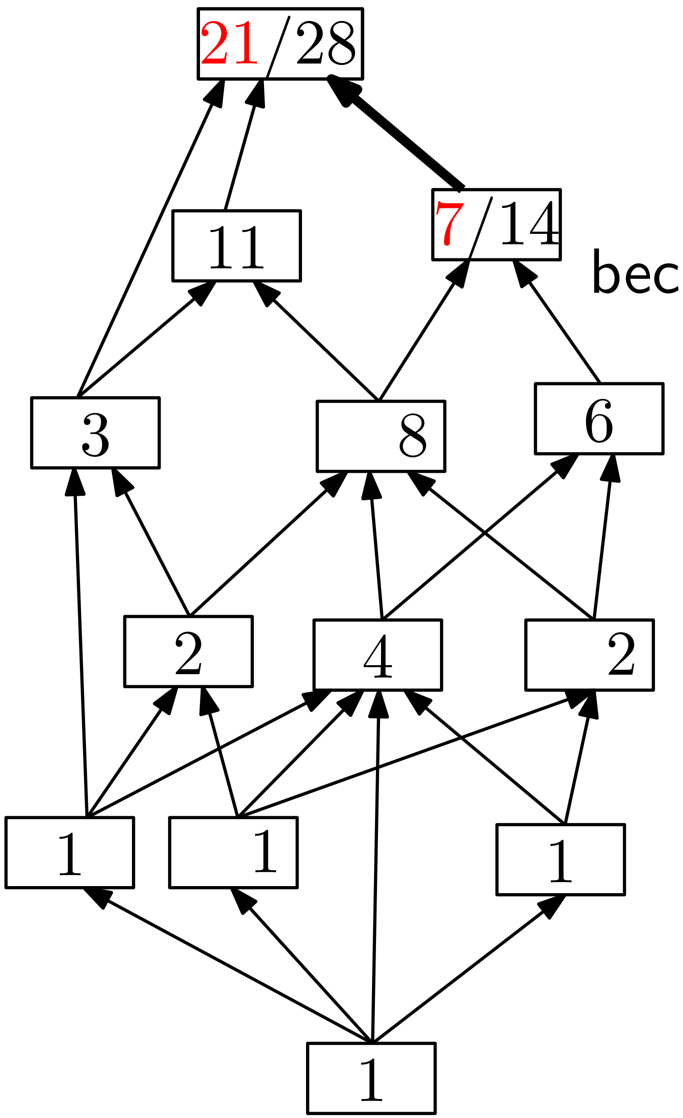

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because $21 = 3 + 11 + 7$

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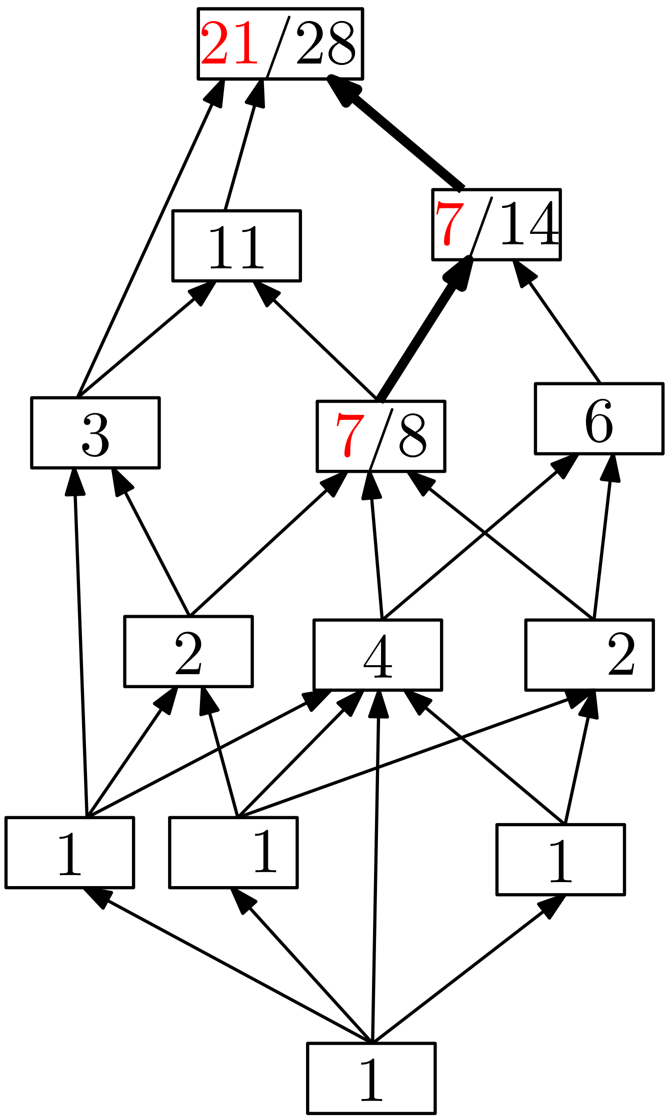

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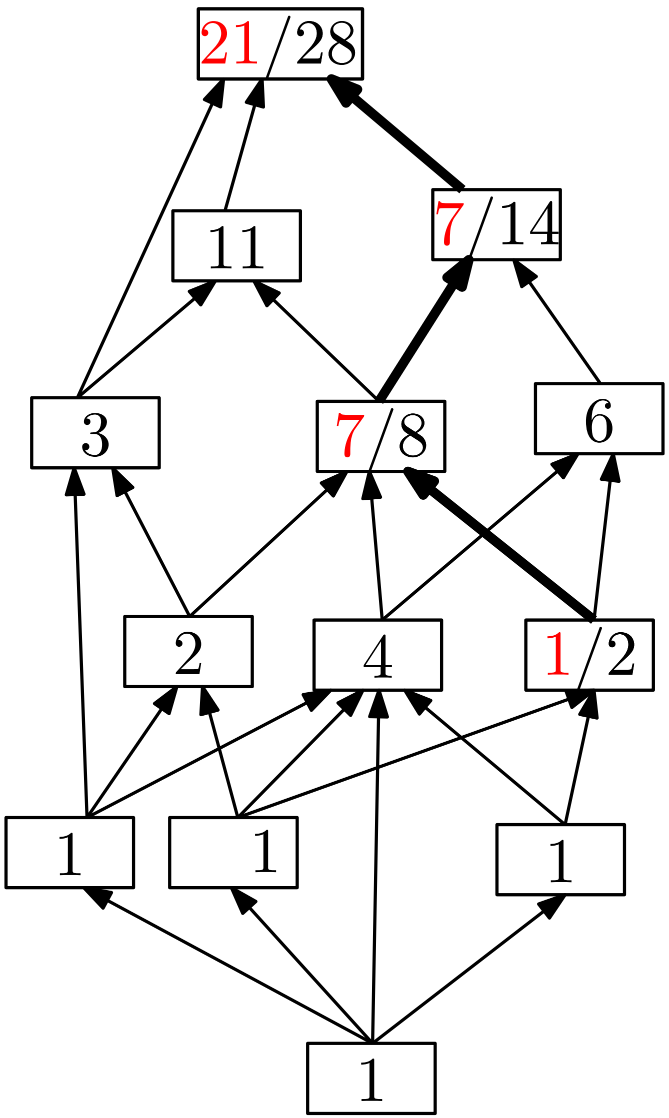

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because $7 = 2 + 4 + 1$

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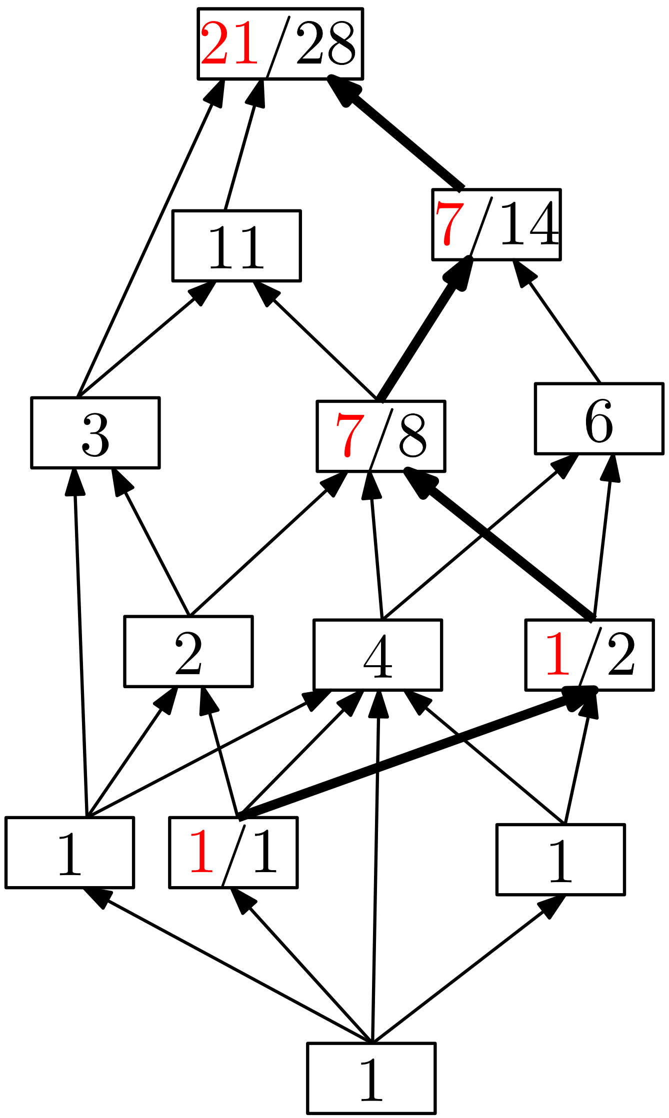

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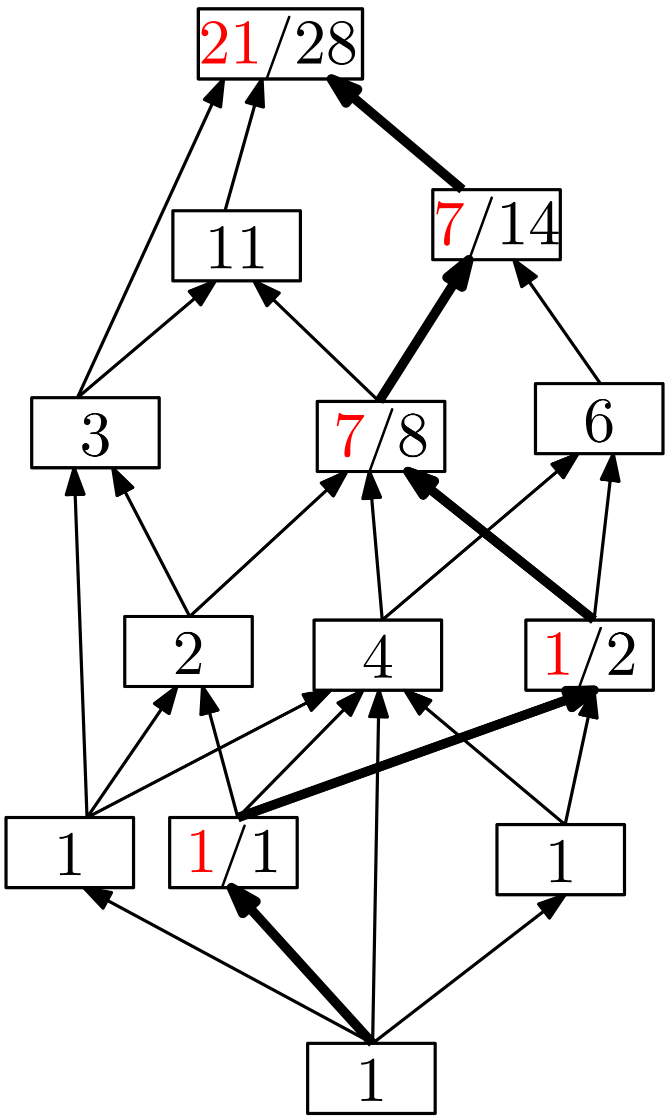

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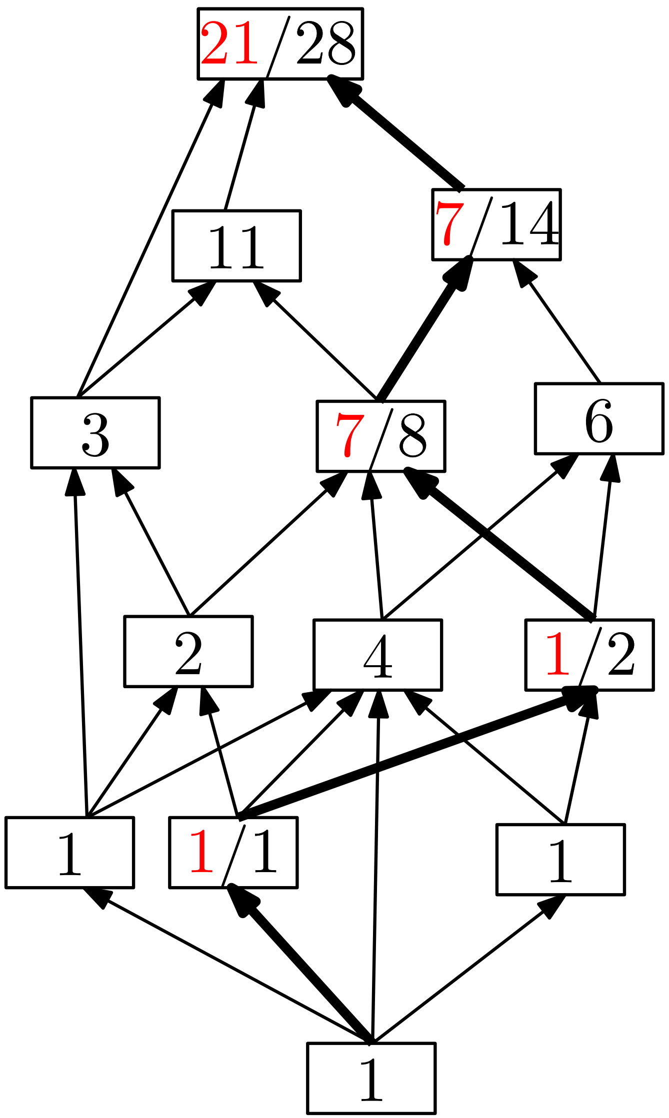

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Essentially, this is UNRANKING:
Compute a function $\{1, \ldots, N\} \to$ path

Summary

The number of triangulations can be found in $O(n^2 2^n)$ time and $O(n2^n)$ space.
With this much preprocessing and space:

- The triangulations can be enumerated with $O(n)$ delay.
- A random triangulation can be determined in $O(n \log n)$ steps.

WARNING: Have to deal with large numbers.
Counting algorithm can use modular arithmetic (Chinese remainder theorem).

Can be applied to other structures (e.g. matchings, Ex. 6)

Can be used for optimizing *decomposable* objective functions. (Nonuniqueness is not an issue.)

Other algorithms for counting

There are many other approaches (divide-and-conquer, sweep, dynamic programming).

The theoretically fastest algorithm for counting triangulations uses divide-and-conquer, based on balanced separators of size $O(\sqrt{n})$ and has supexponential runtime:

$$n^{O(\sqrt{n})}$$

Also for counting other structures.

[“cactus layers”, Marx and Miltzow, 2016]

Overview

0. Introduction

1. Count triangulations [Alvarez and Seidel, 2013]
   - and perfect matchings [Wettstein 2014]
   - Optimal triangulations

2. Coordinated primal-dual sweep
   [Biedl, Chambers, Kostitsyna, Rote, Felsner 2020]

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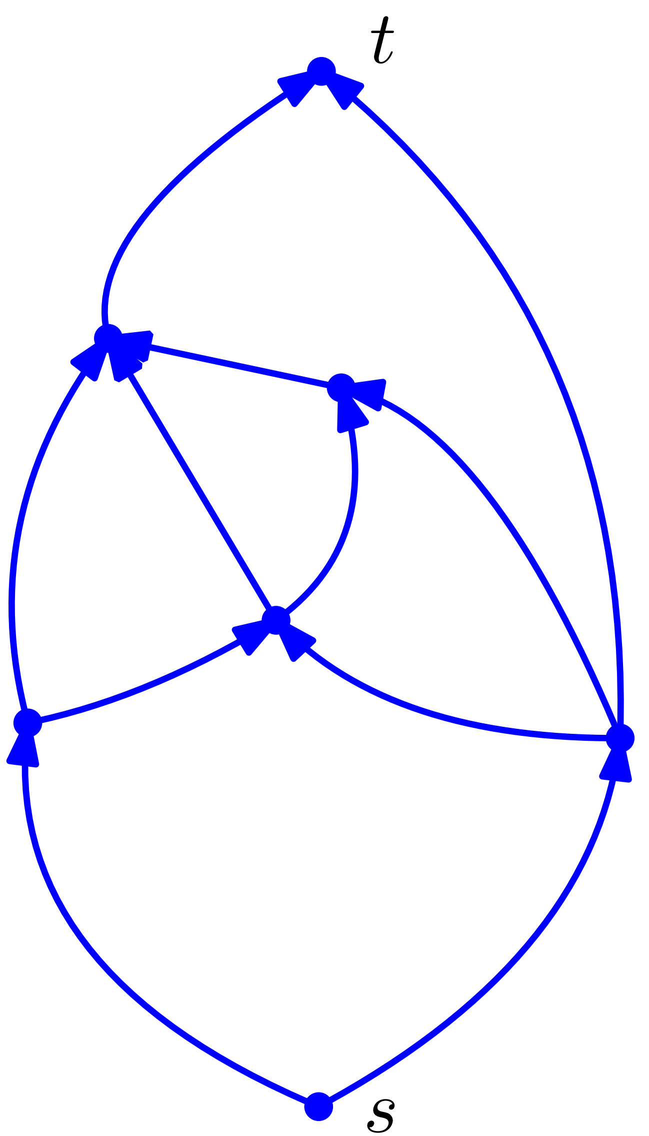
2. Bipolar orientations (s-t-planar graphs)



- plane directed acyclic graph
- a single source s and
  a single sink t

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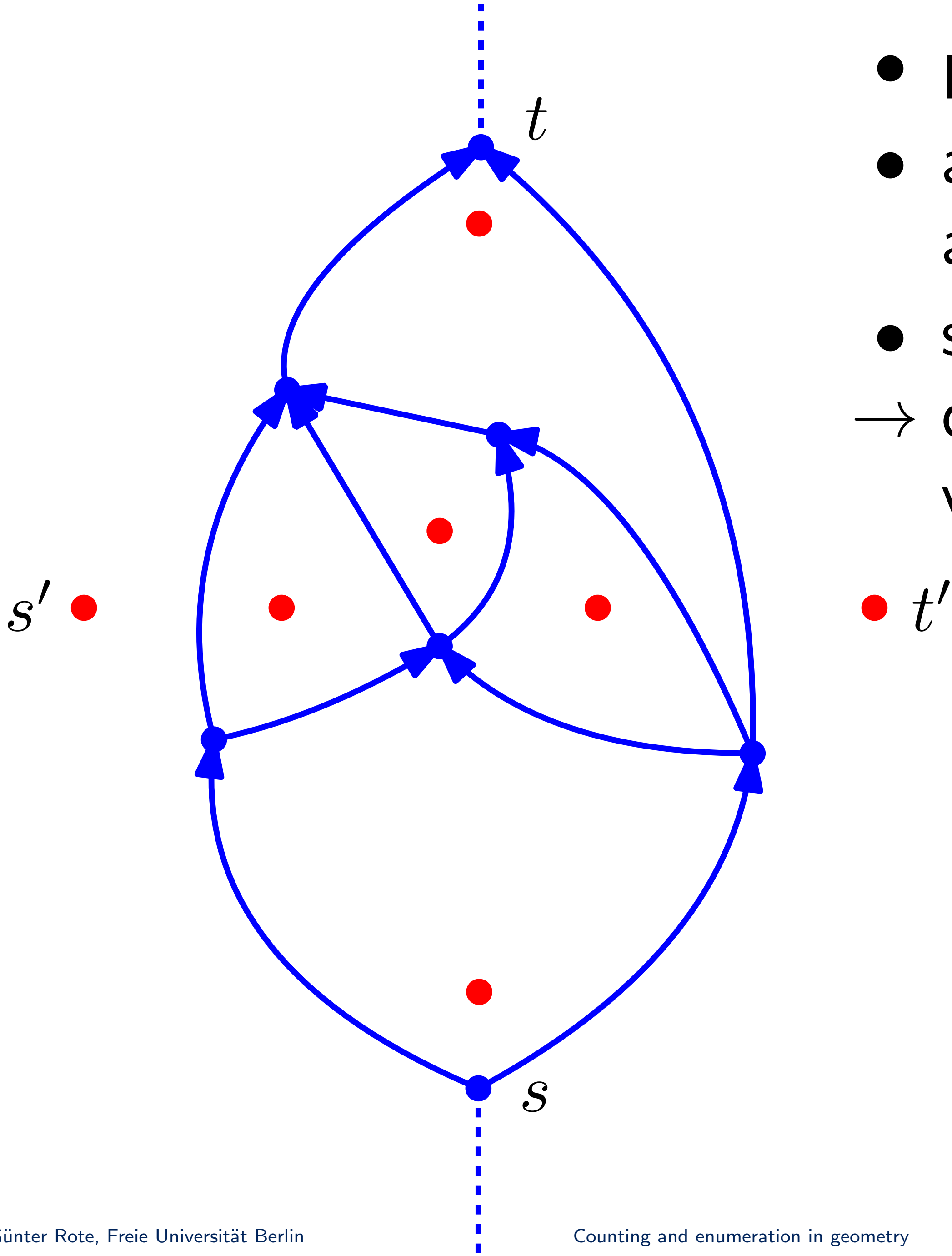

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$\rightarrow$ dual graph with a *left* outer vertex s' and a *right* vertex t'

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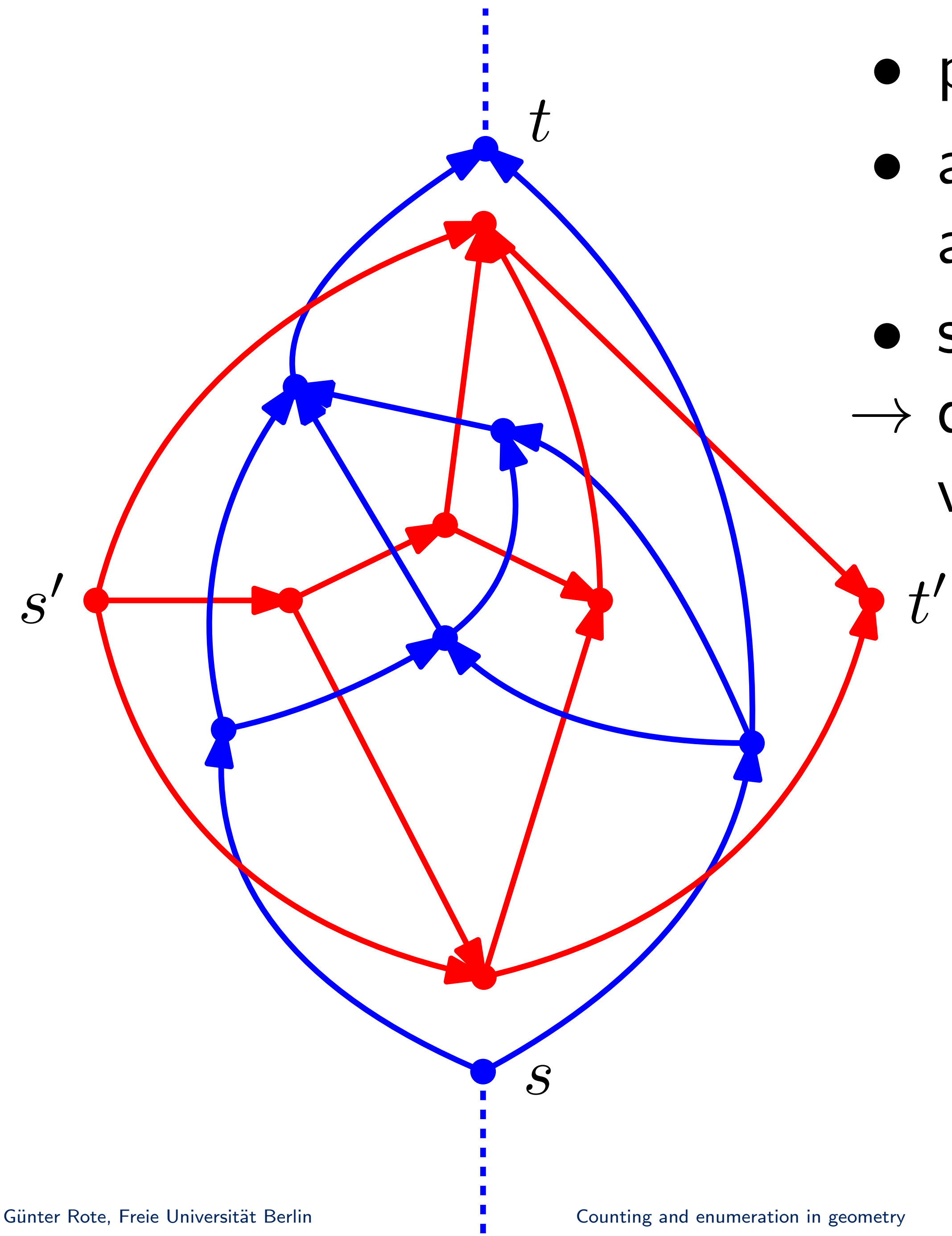

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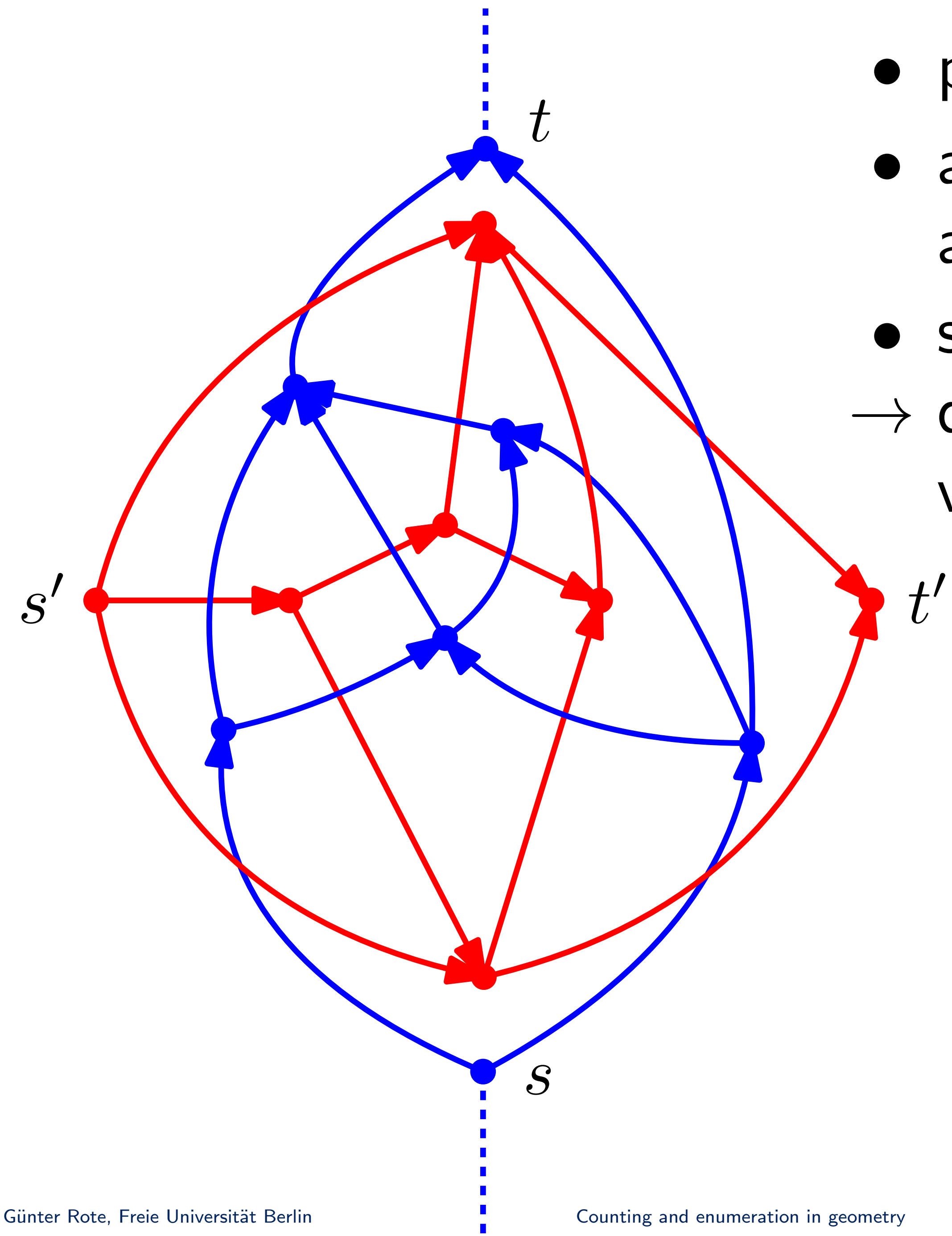

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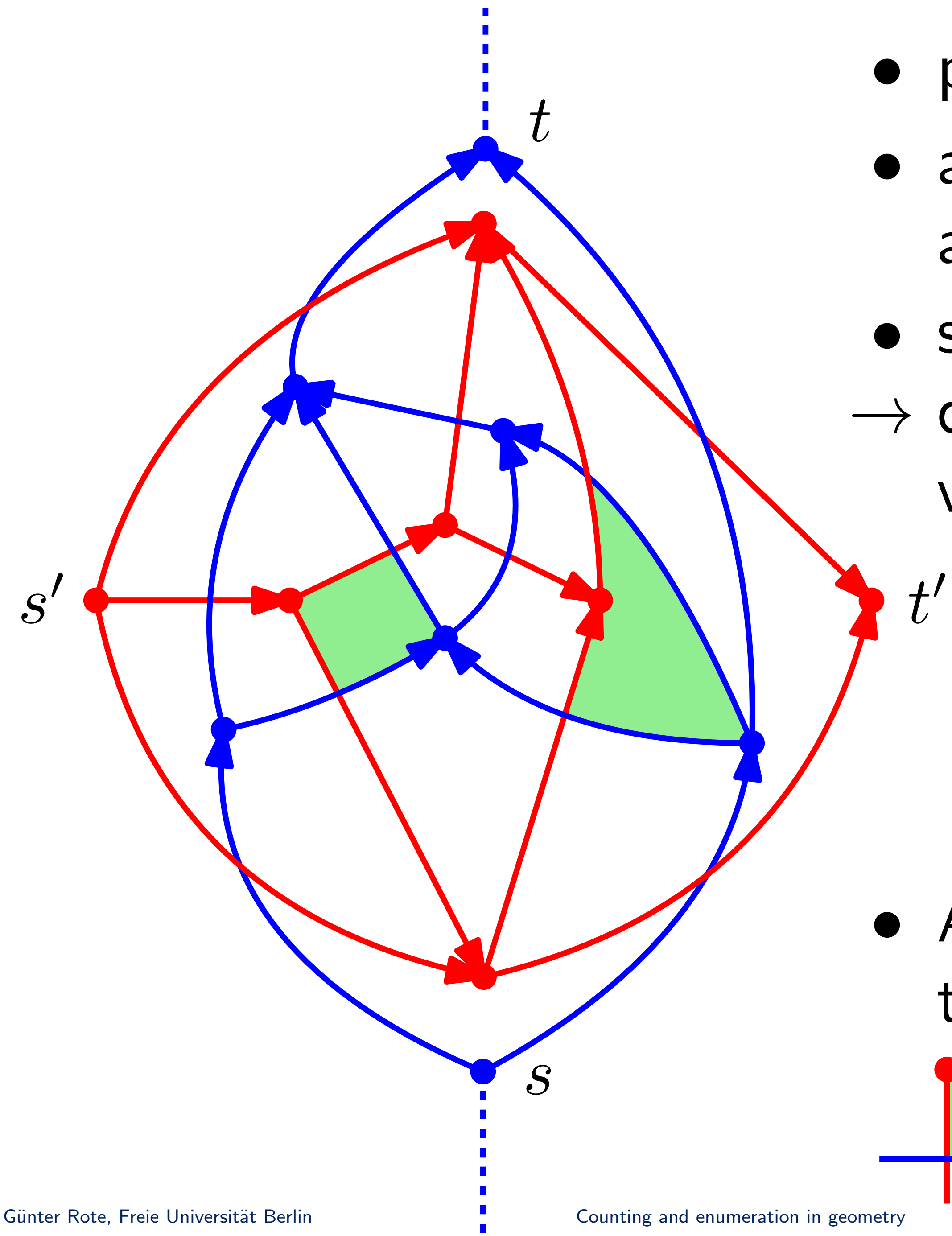

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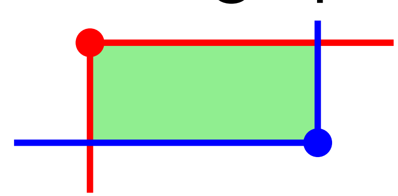
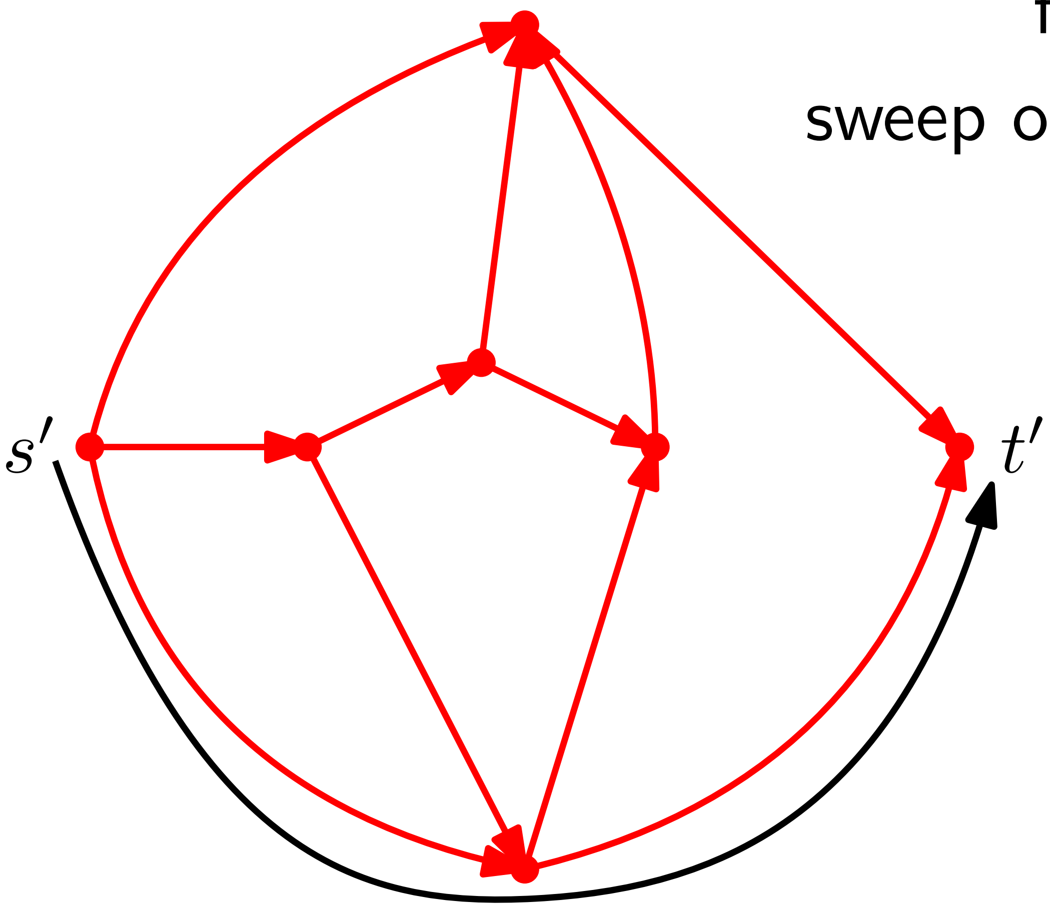
- All faces in the overlay of the two graphs are quadrilaterals:



Coordinated primal-dual sweep

- sweep the dual graph with an s'–t' rope from bottom to top

sweep over the *leftmost* possible face



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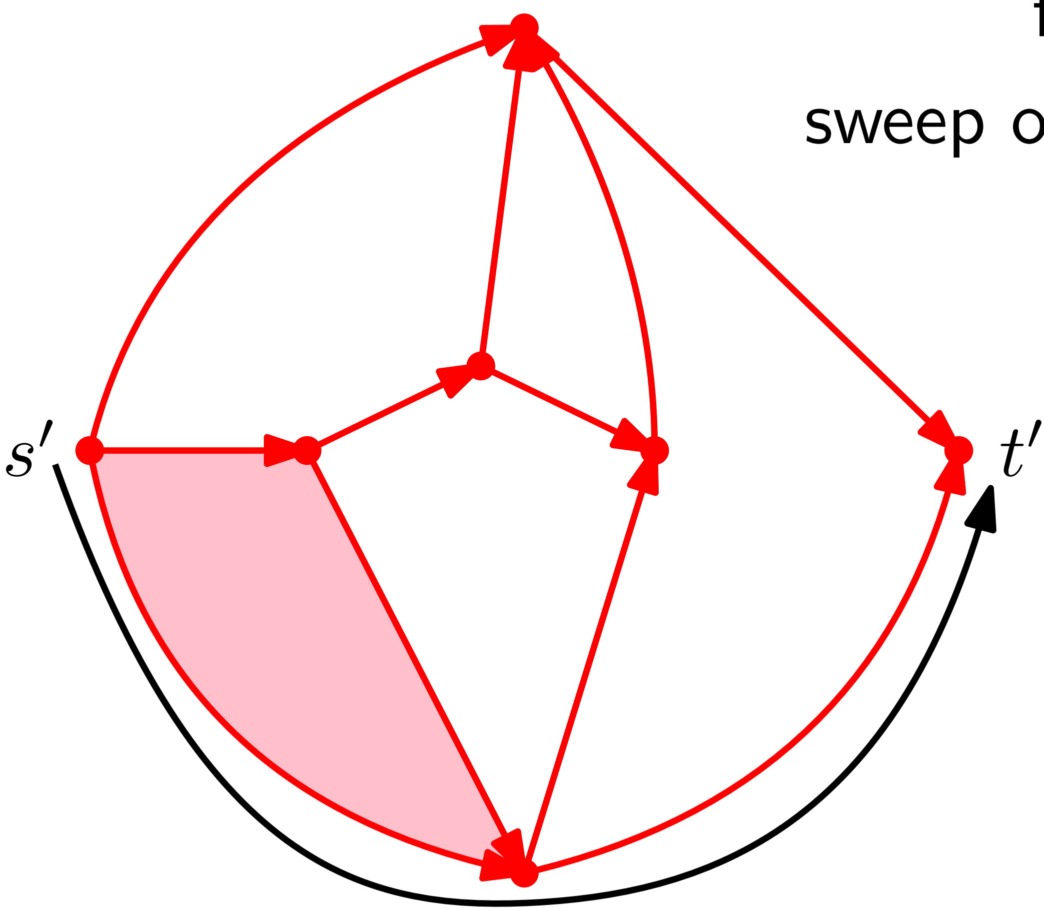

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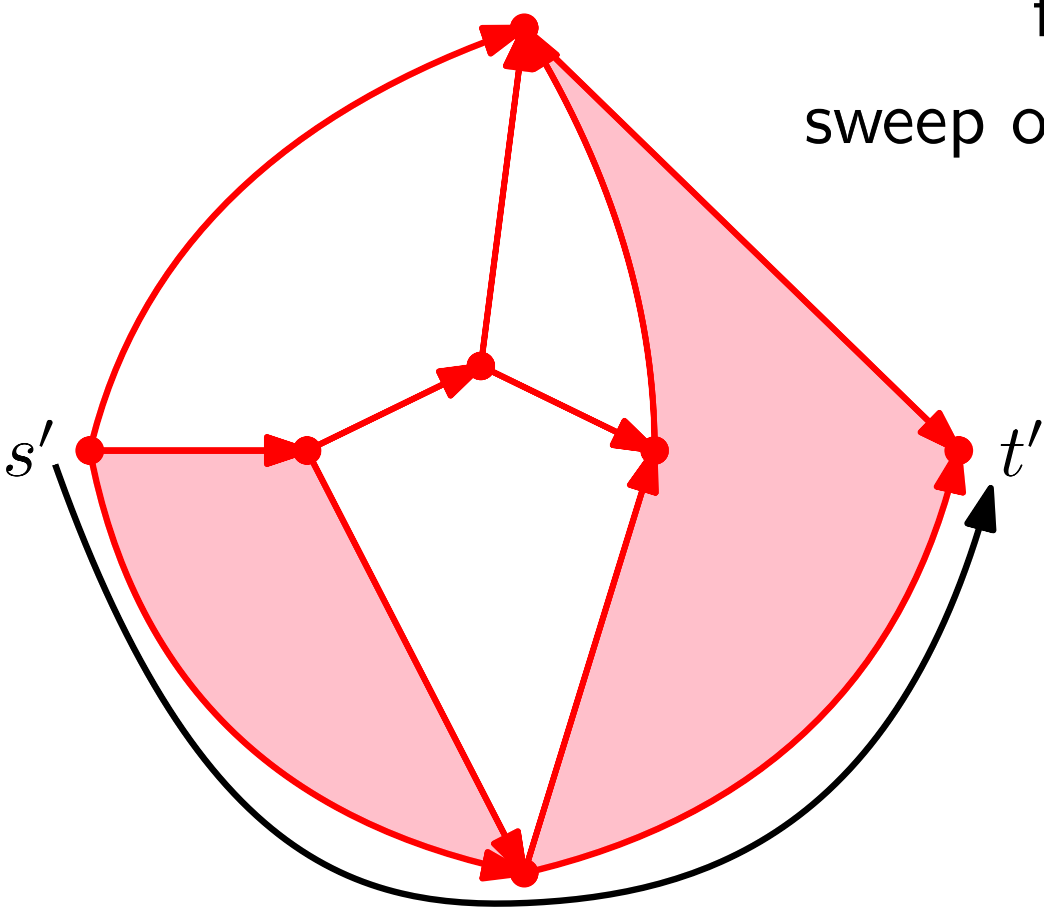

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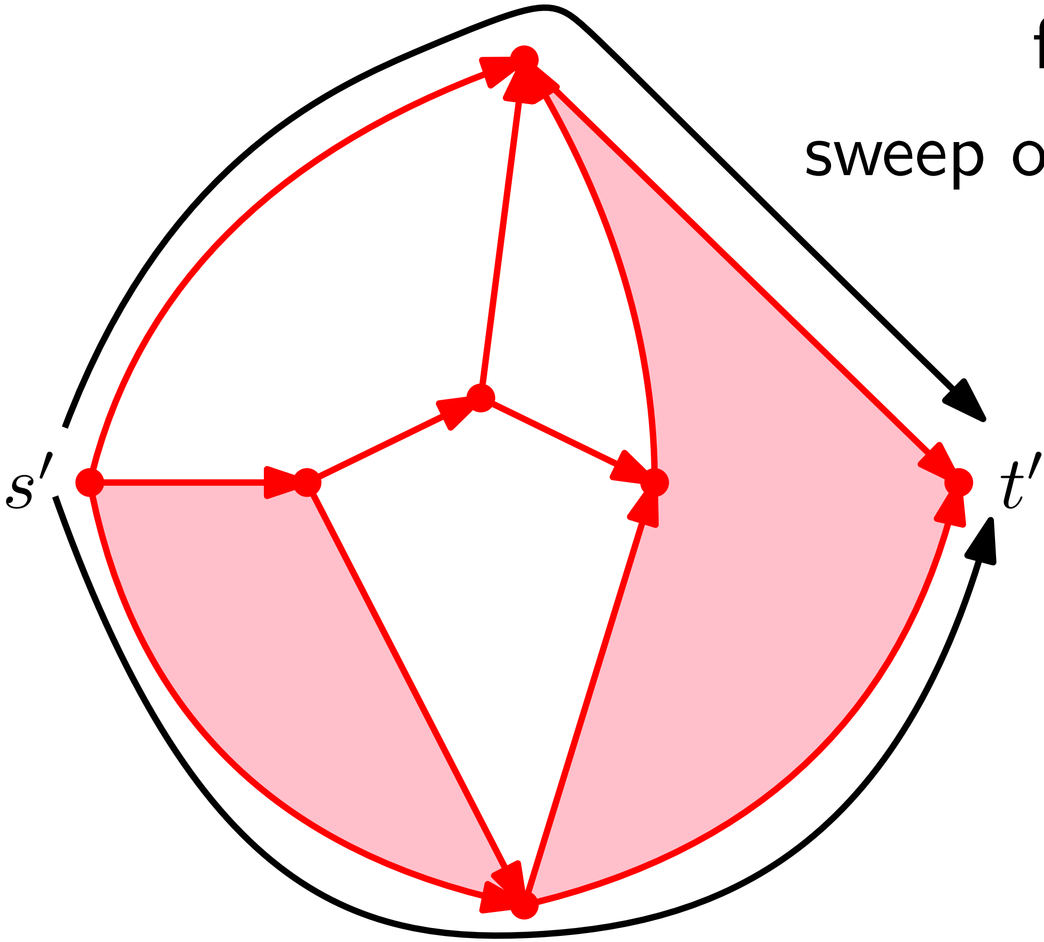

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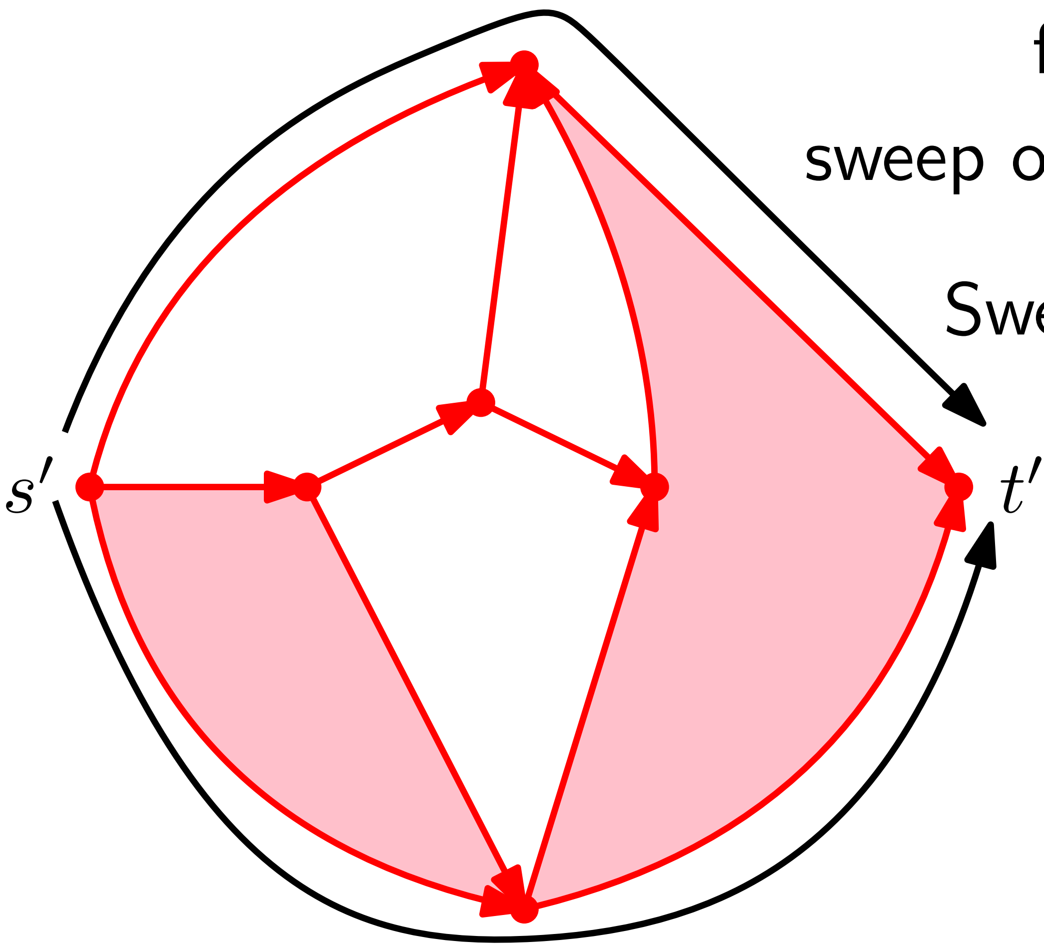

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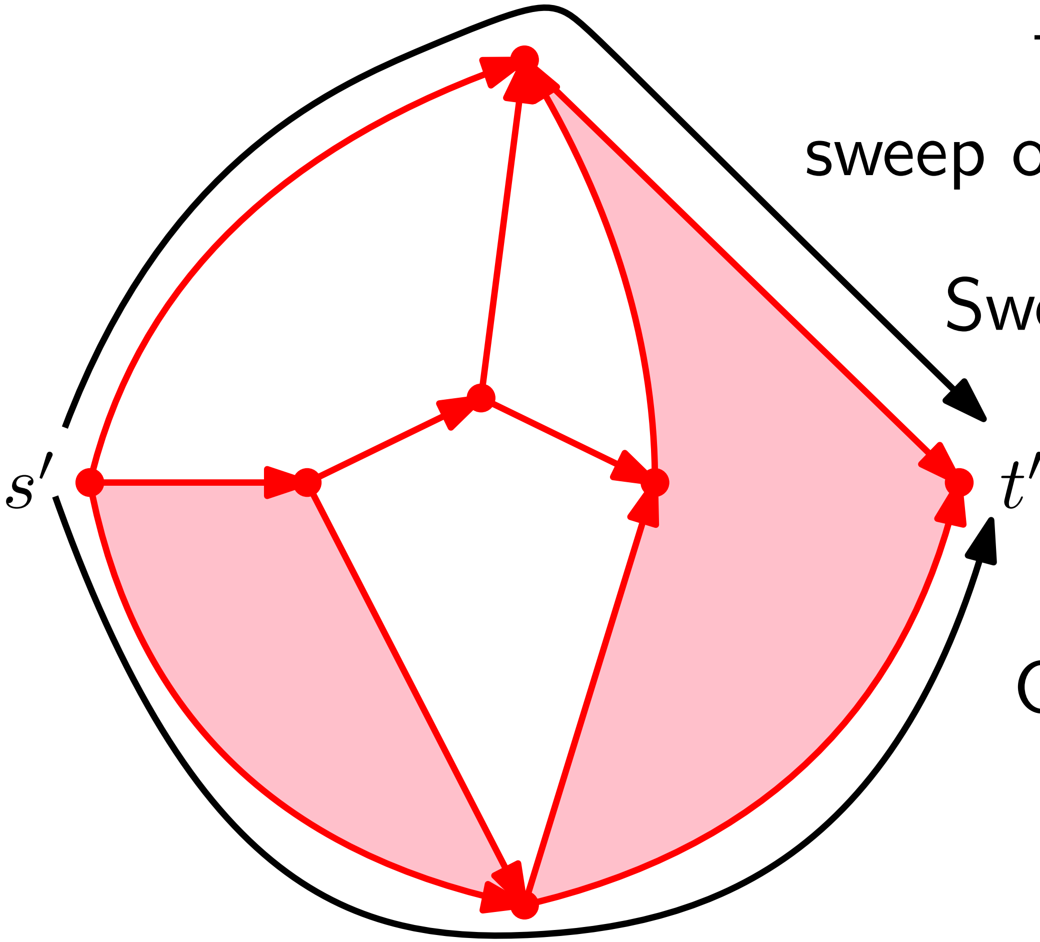

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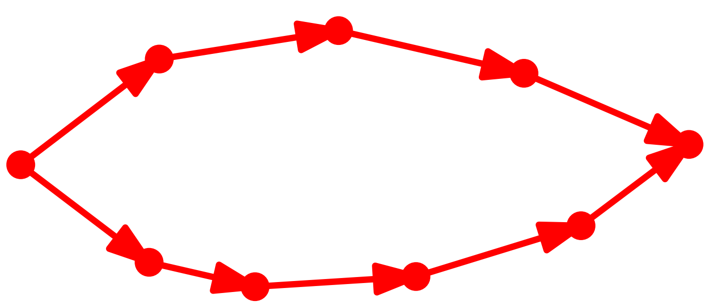
General form of a face (Ex. 2b)



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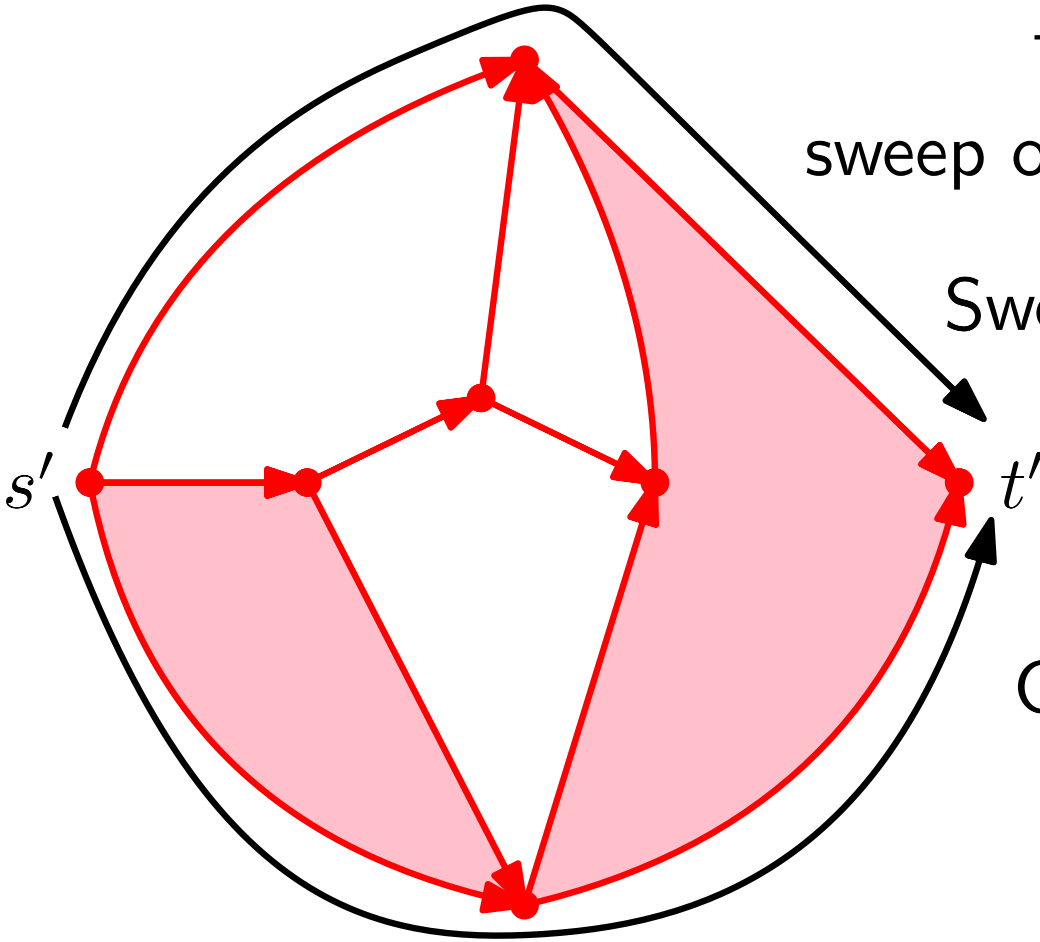

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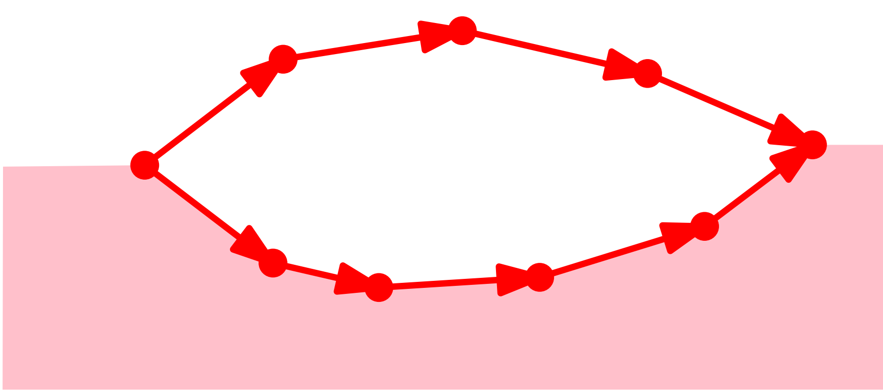
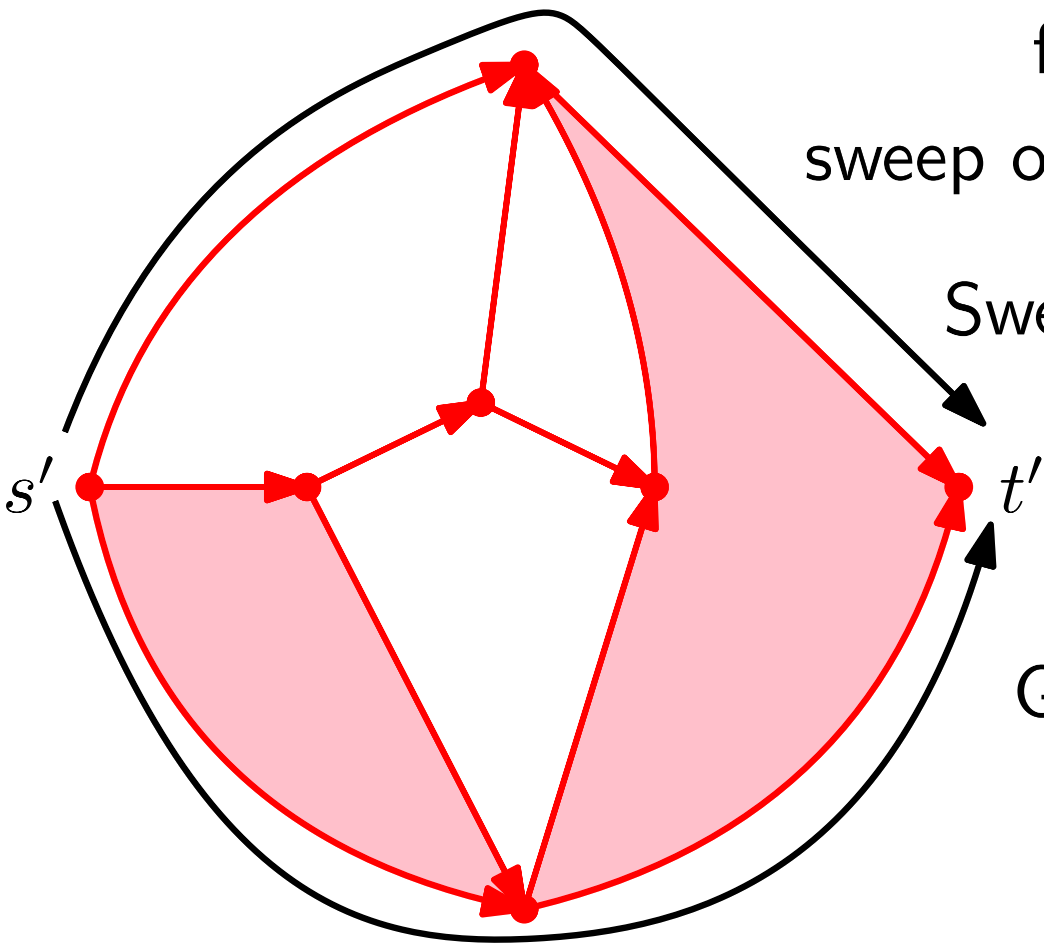


Coordinated primal-dual sweep



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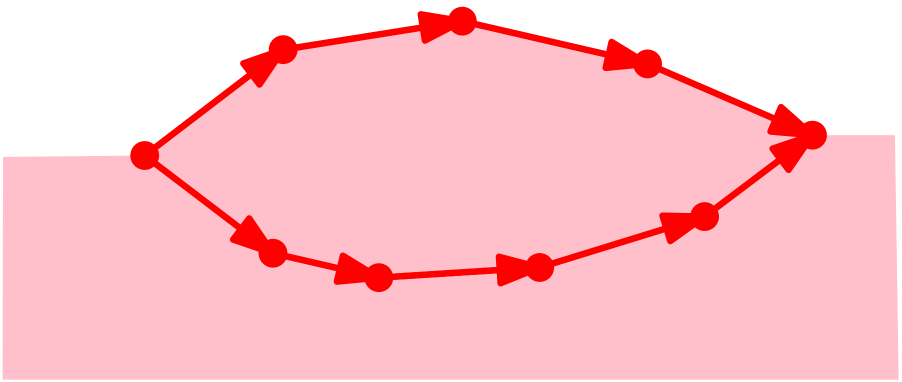

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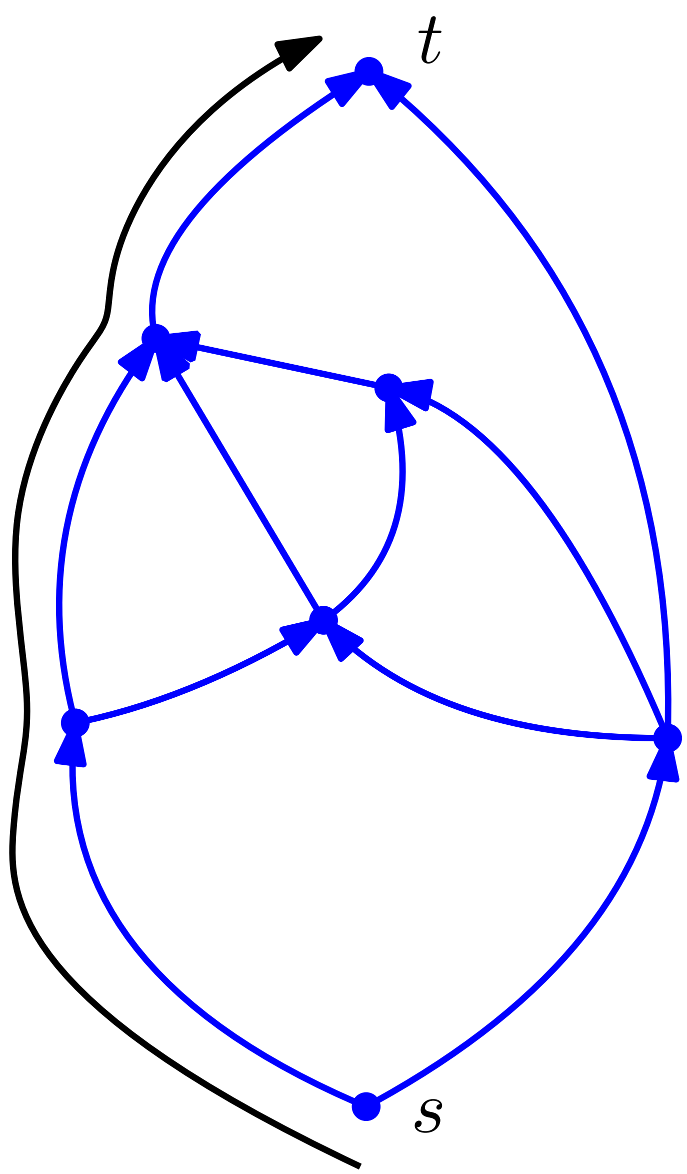
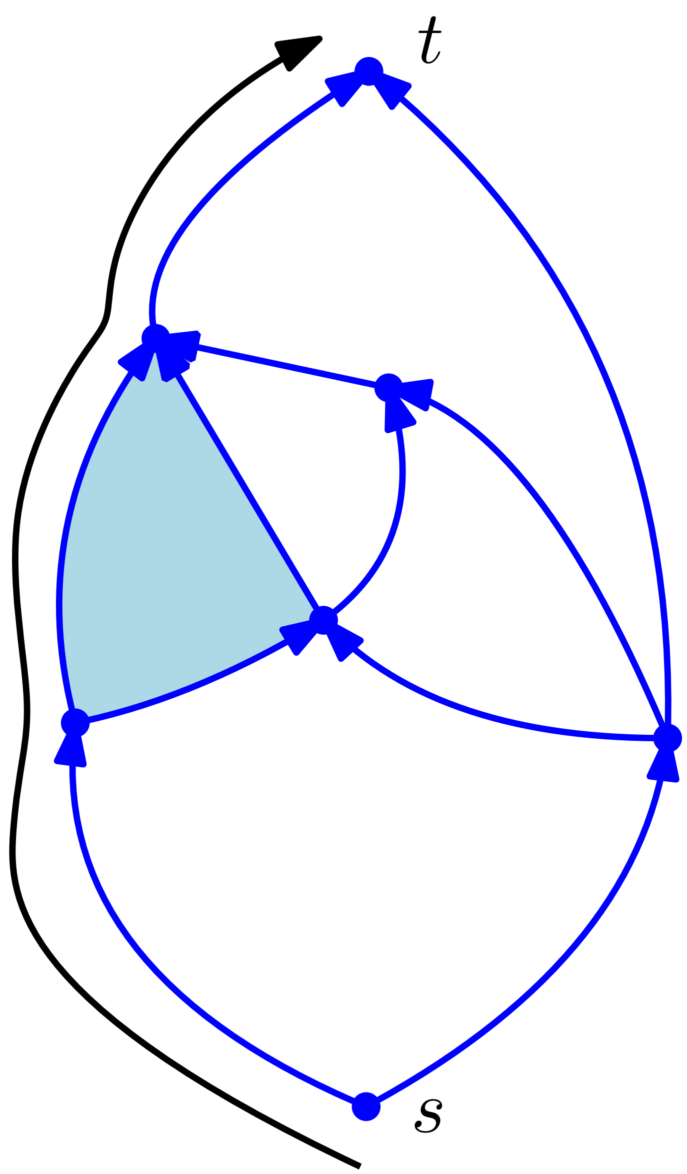


- sweep the primal graph with an s–t rope from left to right

Coordinated primal-dual sweep



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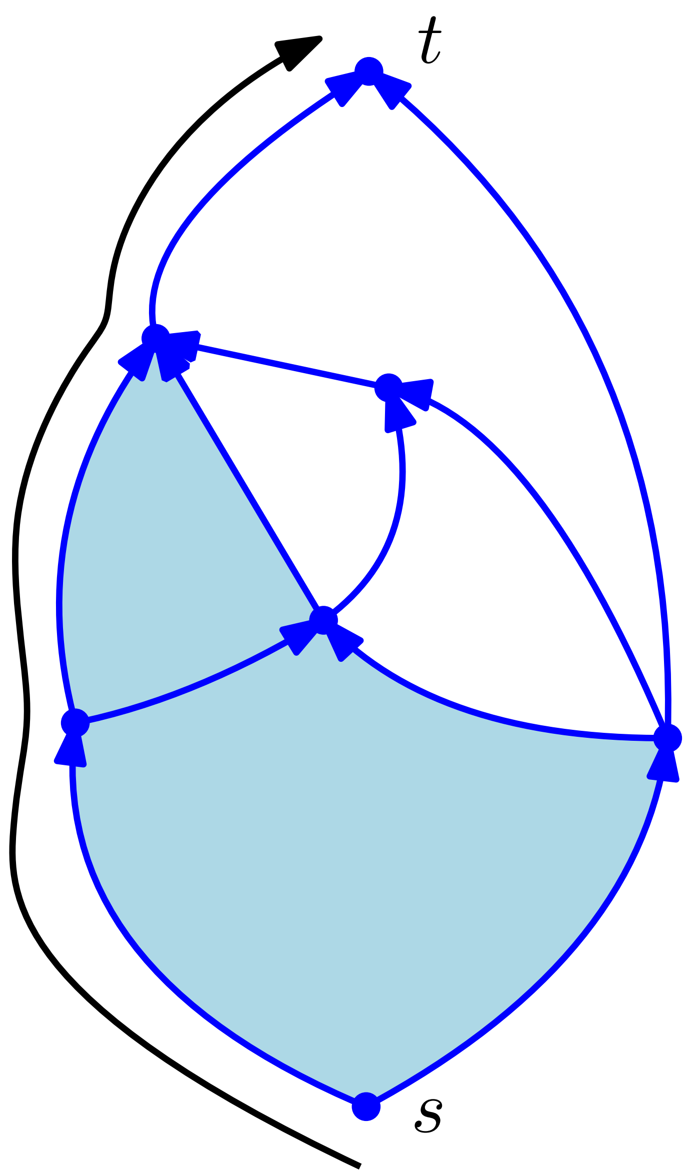

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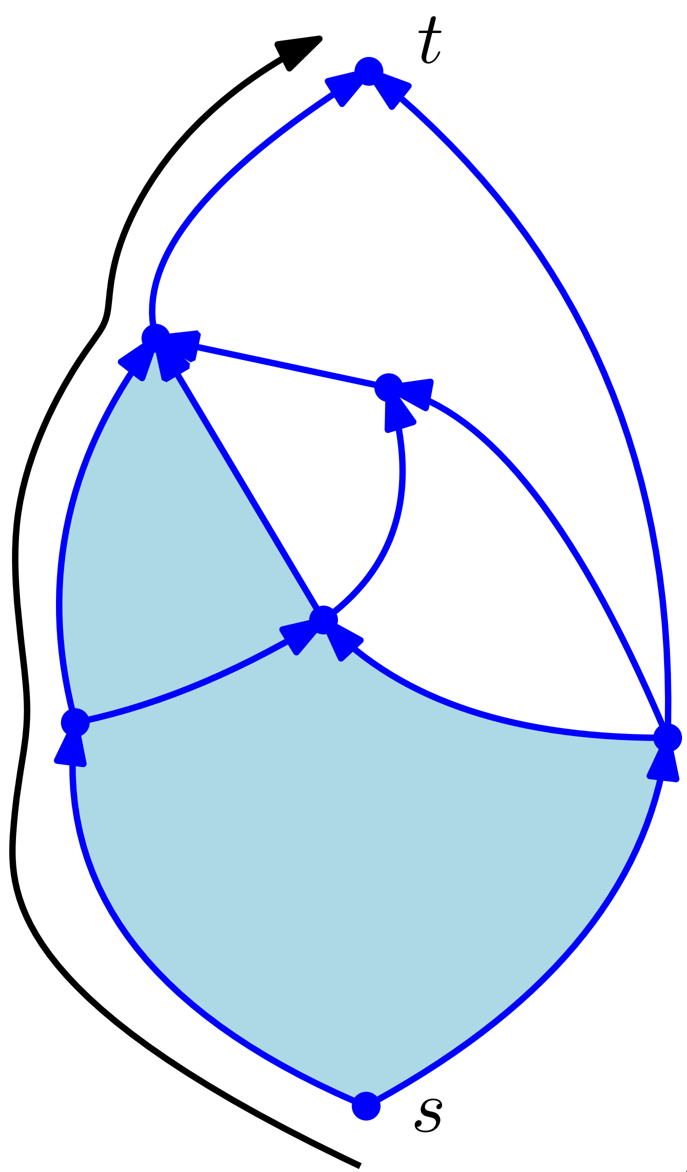

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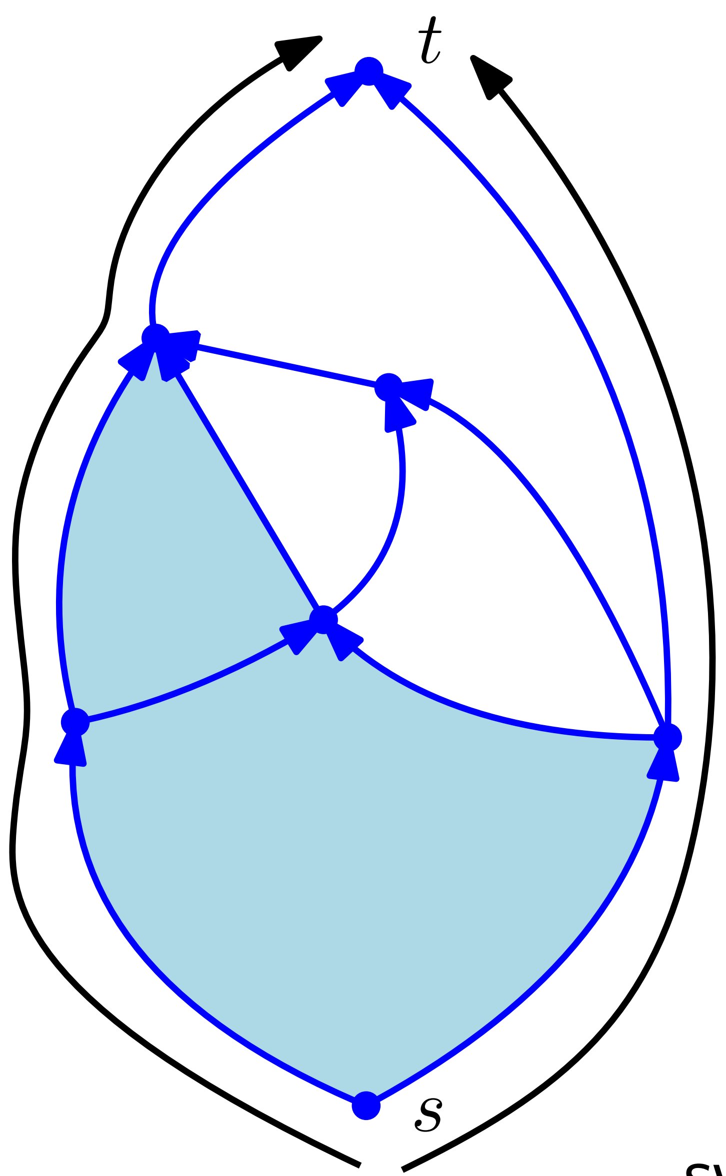
sweep over the *lowest* possible face

Coordinated primal-dual sweep



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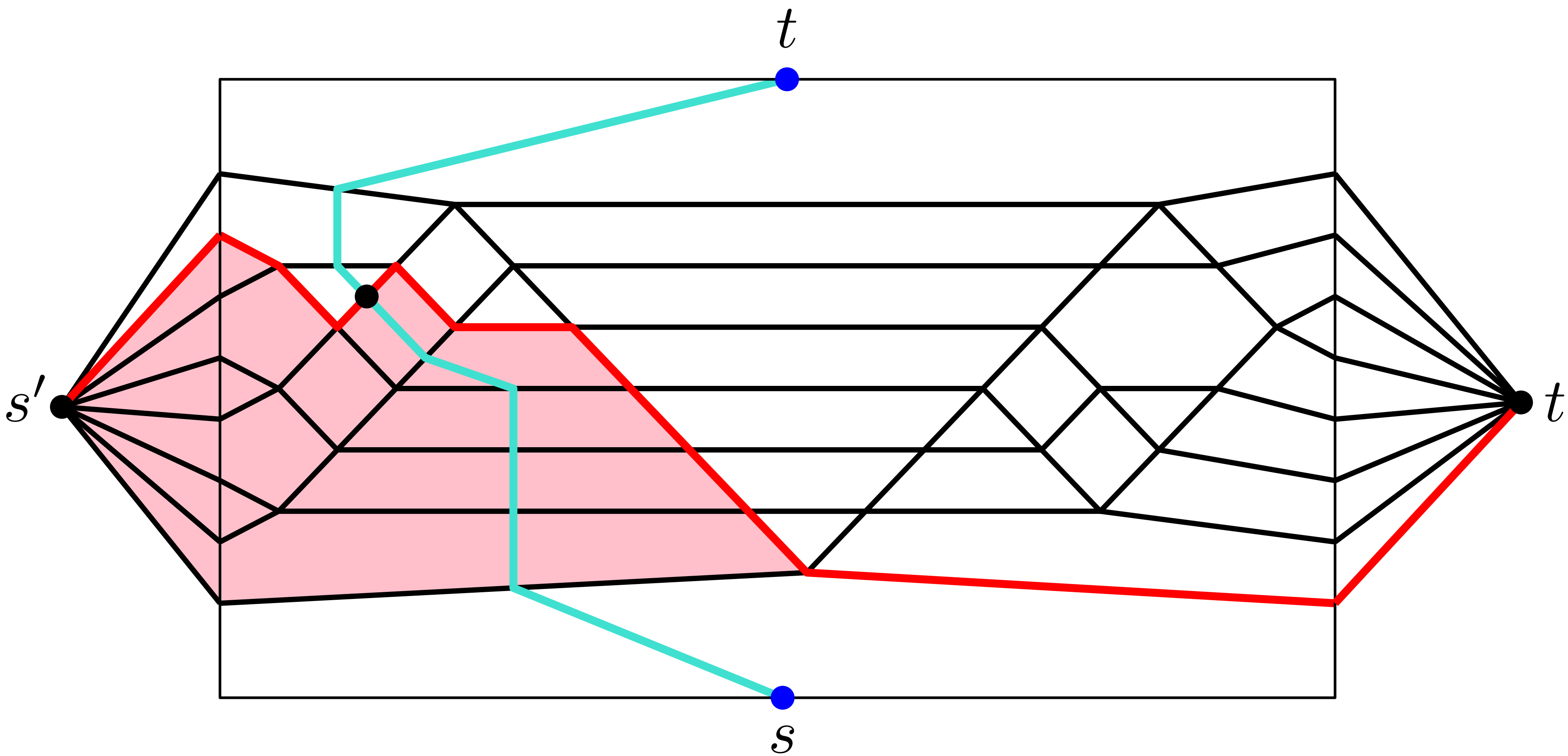
Animation



dual rope in the dual (multi-)graph

primal rope (The primal graph is not shown.)

page.mi.fu-berlin.de/rote/Papers/slides/Wuerzburg-2020-Simultaneous-sweep-Animation.pdf

Coordinated sweep

Freie Universität Berlin

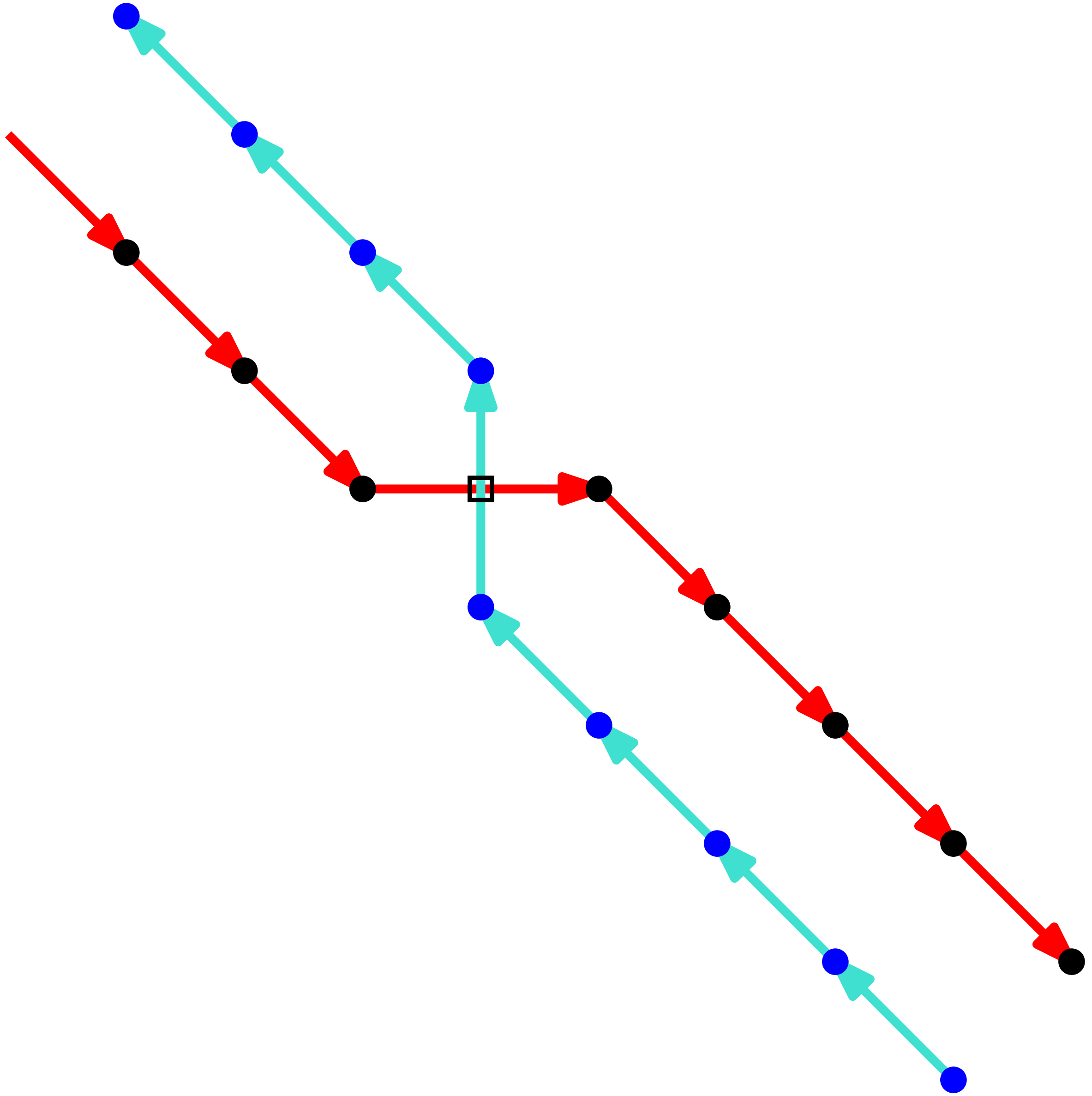
There is a (unique) coordinated primal-dual sweep with the following properties:

- The primal rope always crosses the dual rope exactly once.
- The primal and the dual rope stay "close" to each other.
- Exactly one rope can advance, depending on the situation at the crossing.
- Every primal-dual edge pair is visited exactly once.
- Each individual sweep is a leftmost/bottommost sweep.

[Biedl, Chambers, Kostitsyna, Rote, Felsner 2020]
in connection with sweeping over a pseudoline arrangement, see Ex. 4.

Coordinated sweep

general situation:



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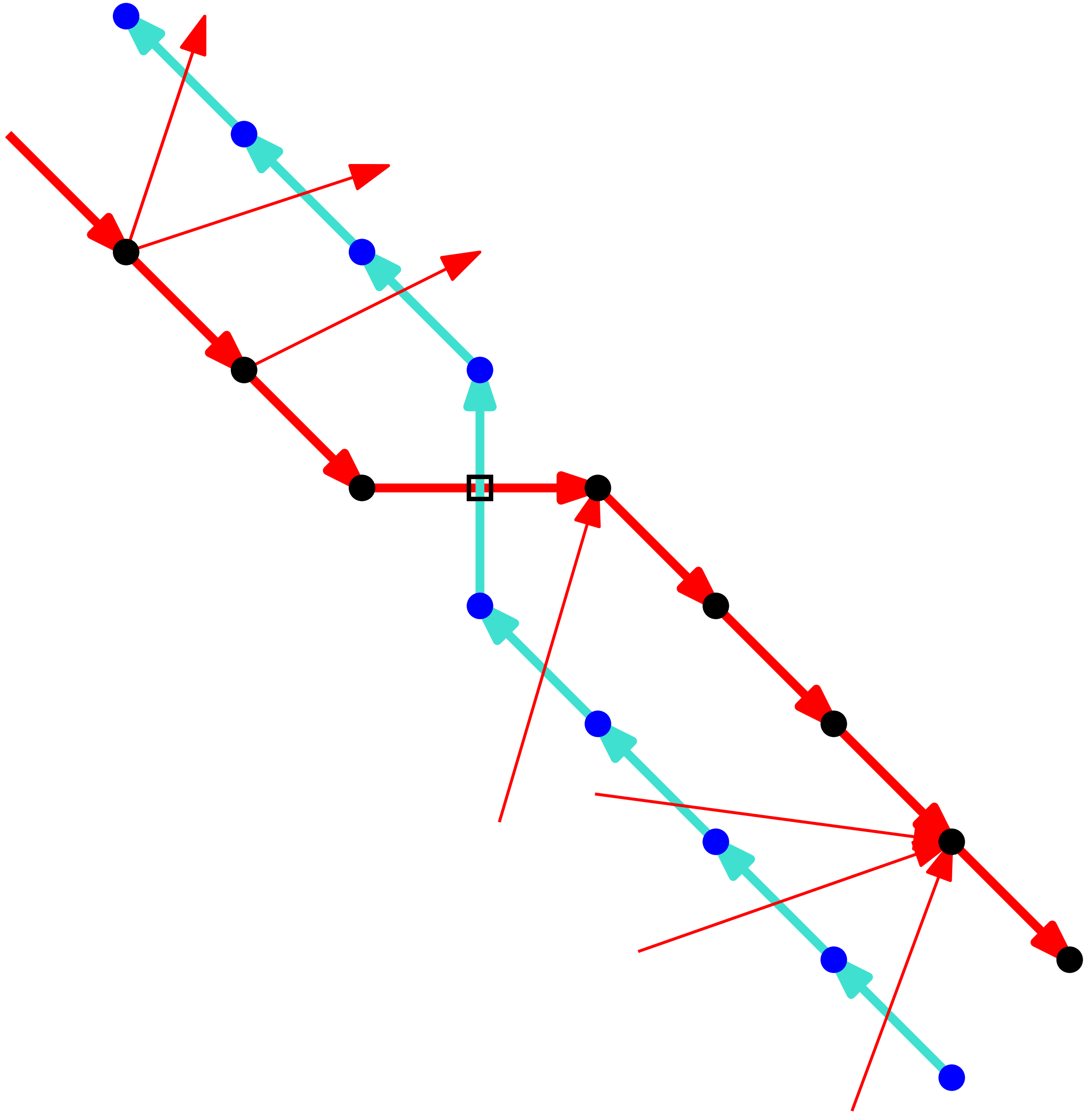

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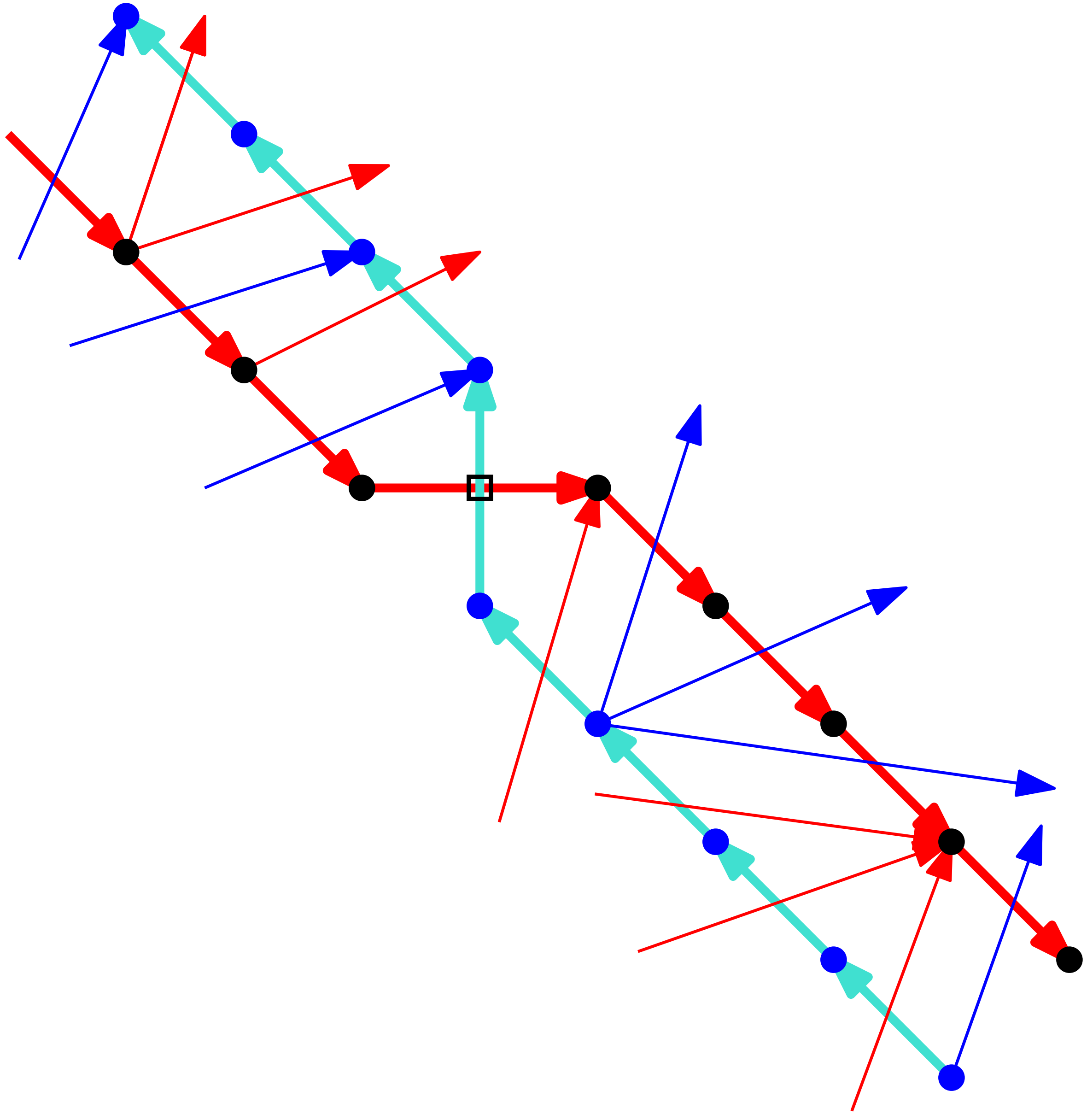

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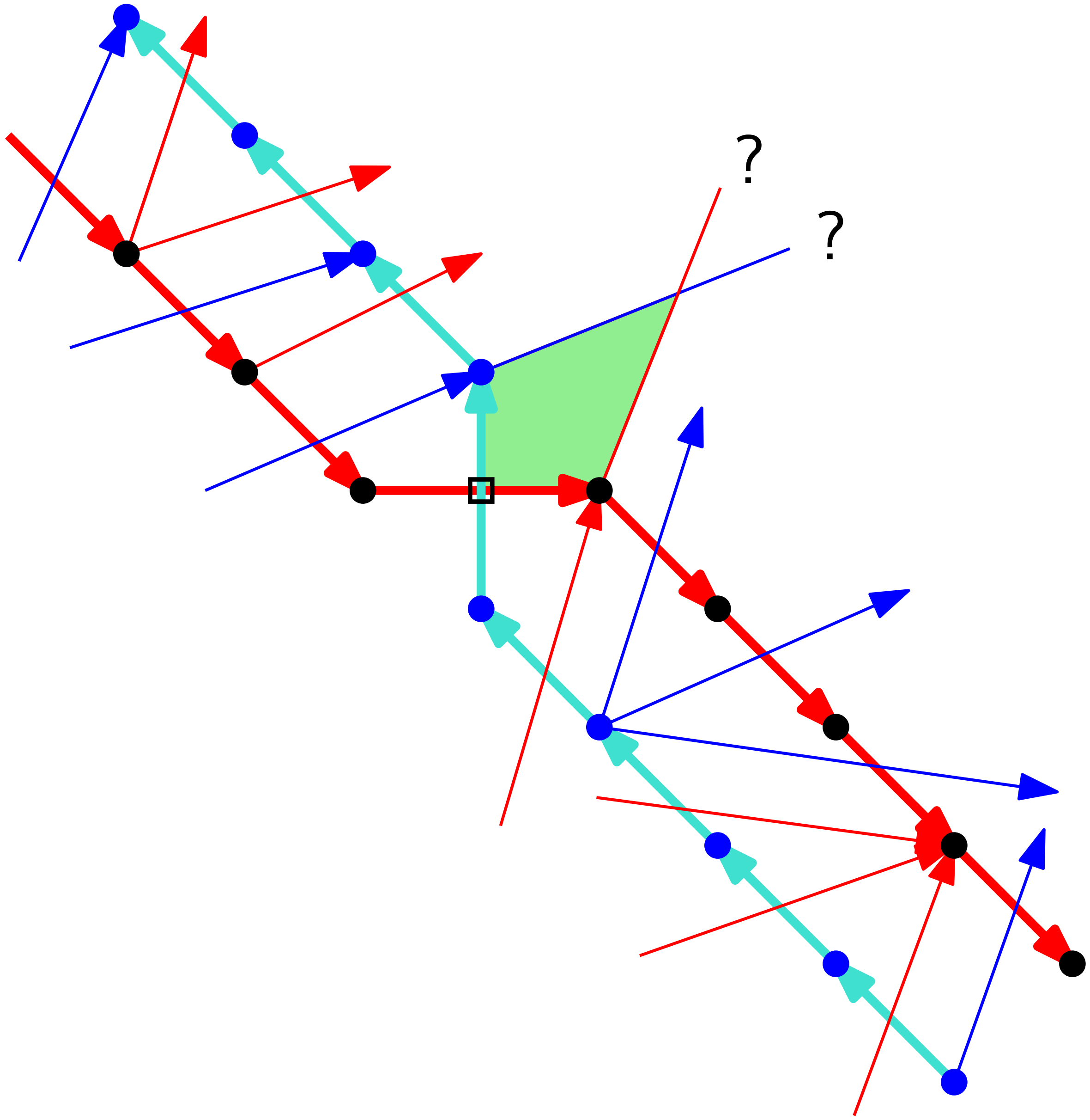

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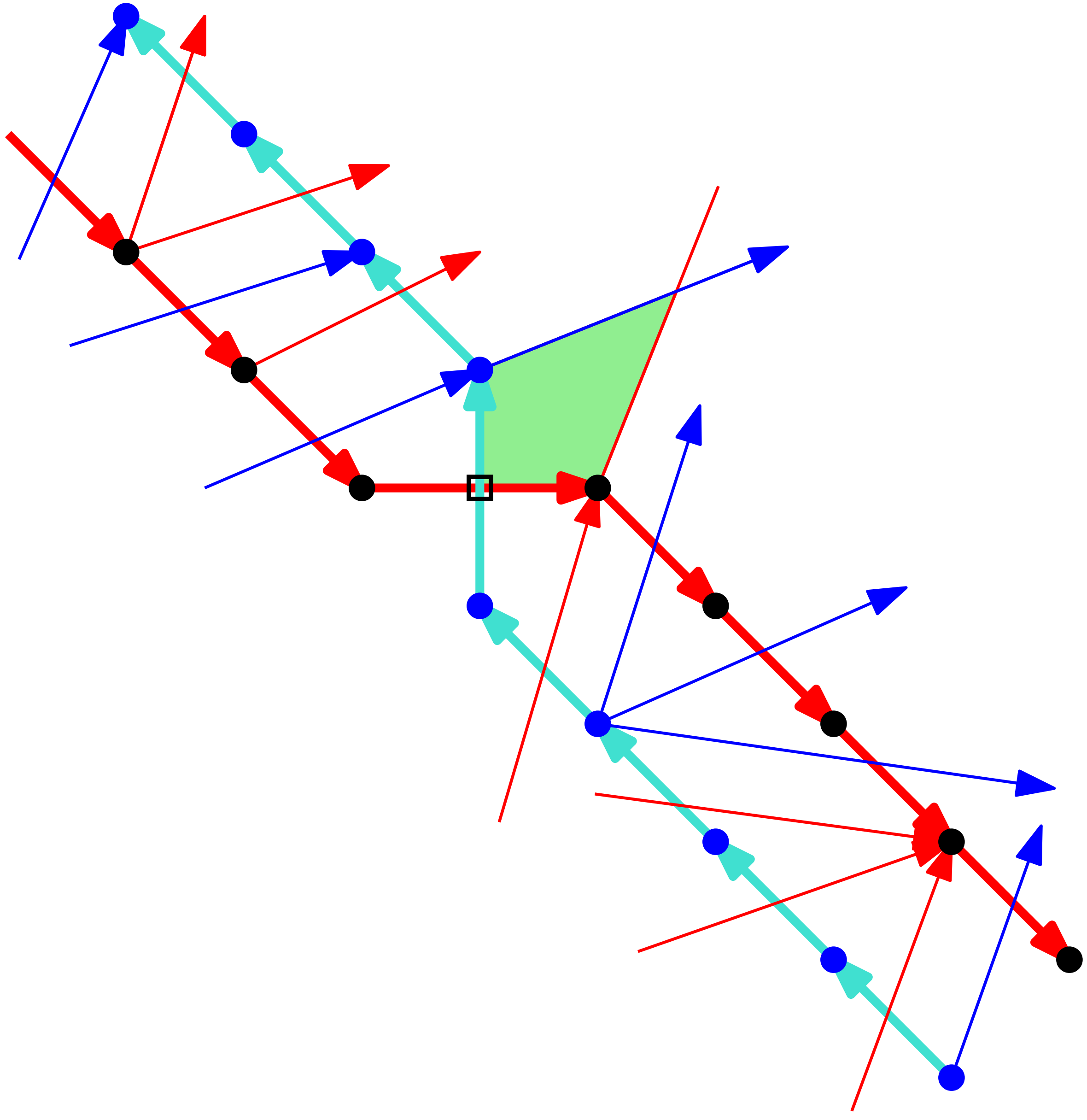

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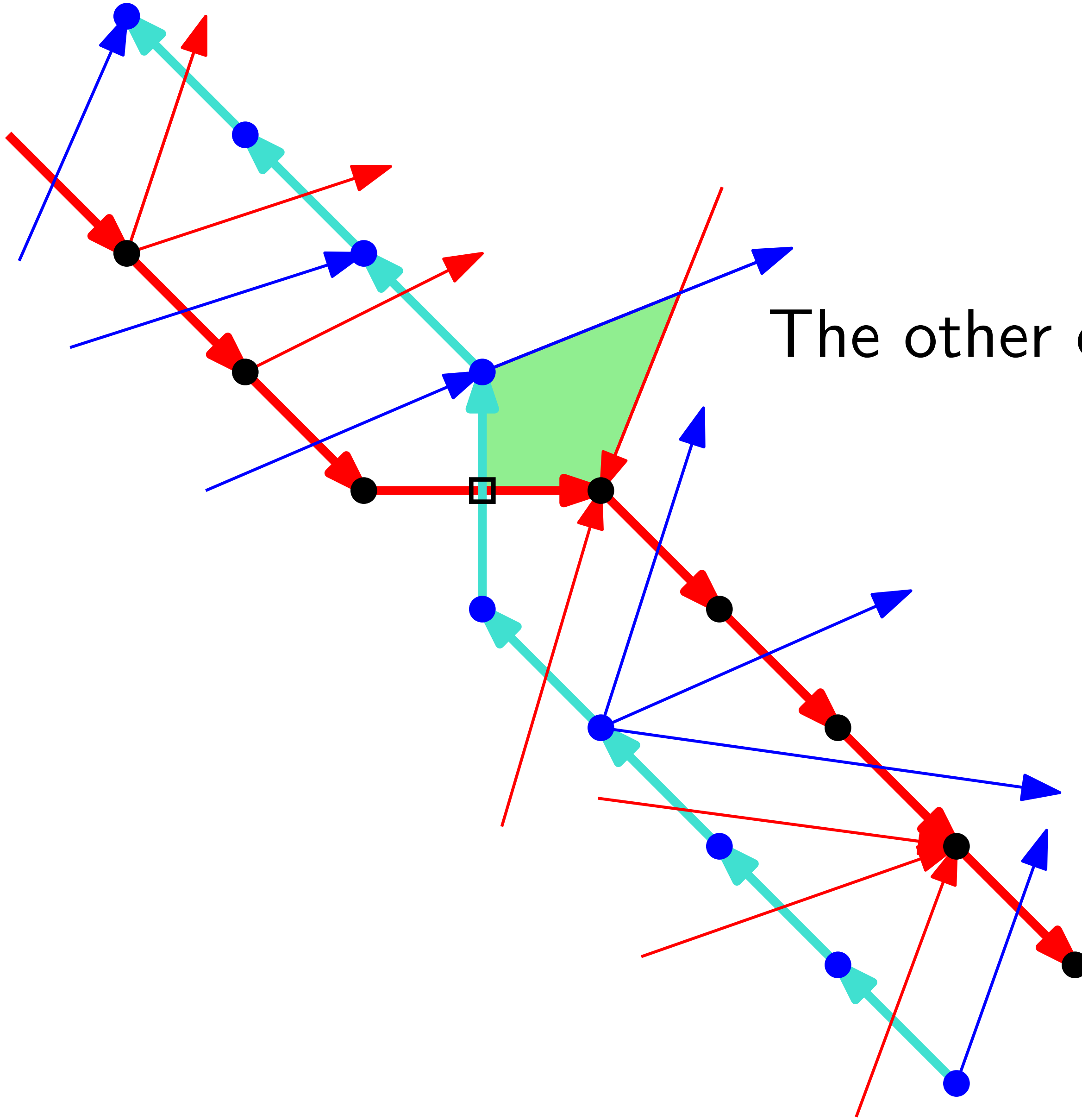

The other case is symmetric.

Coordinated sweep

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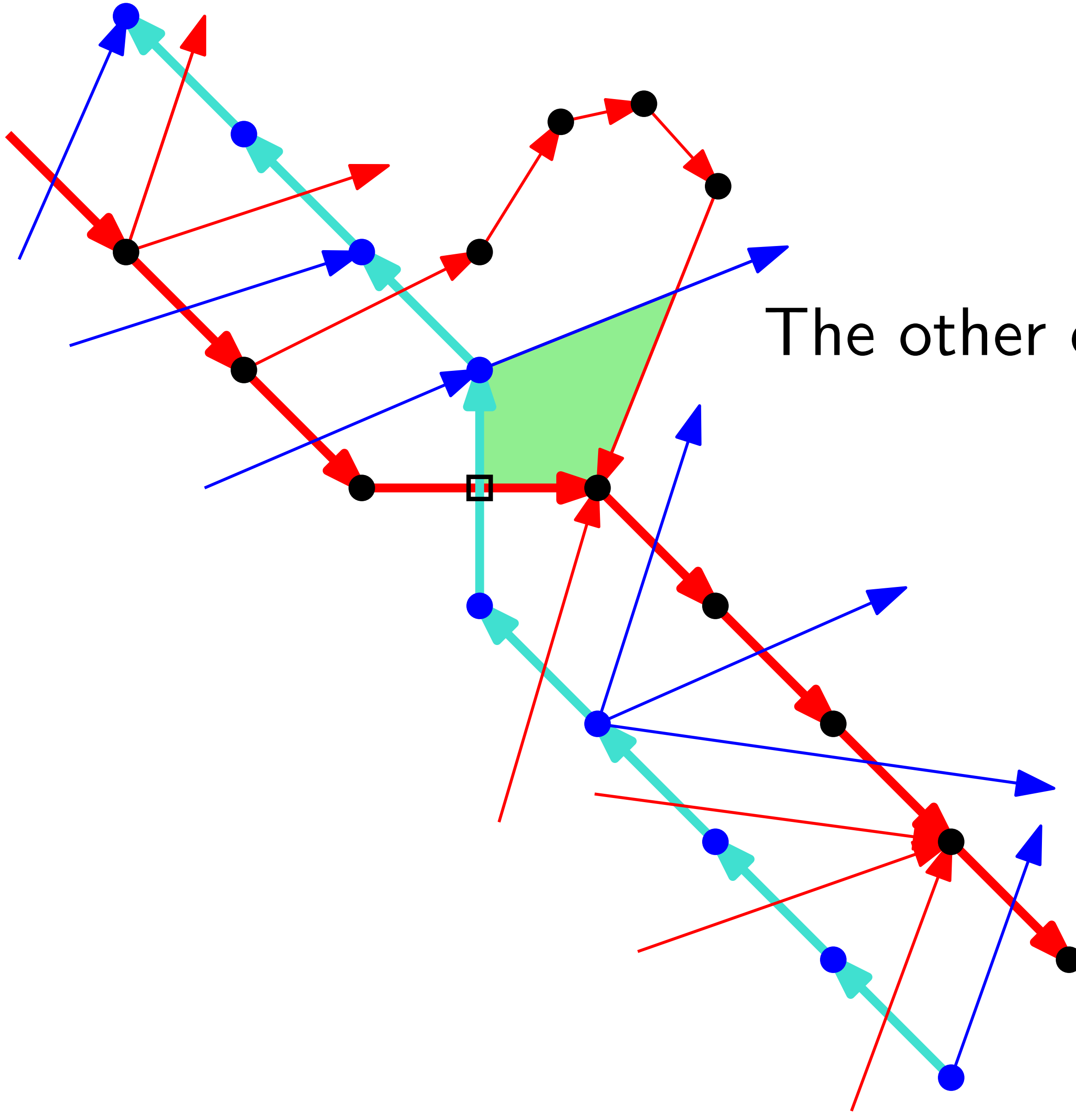

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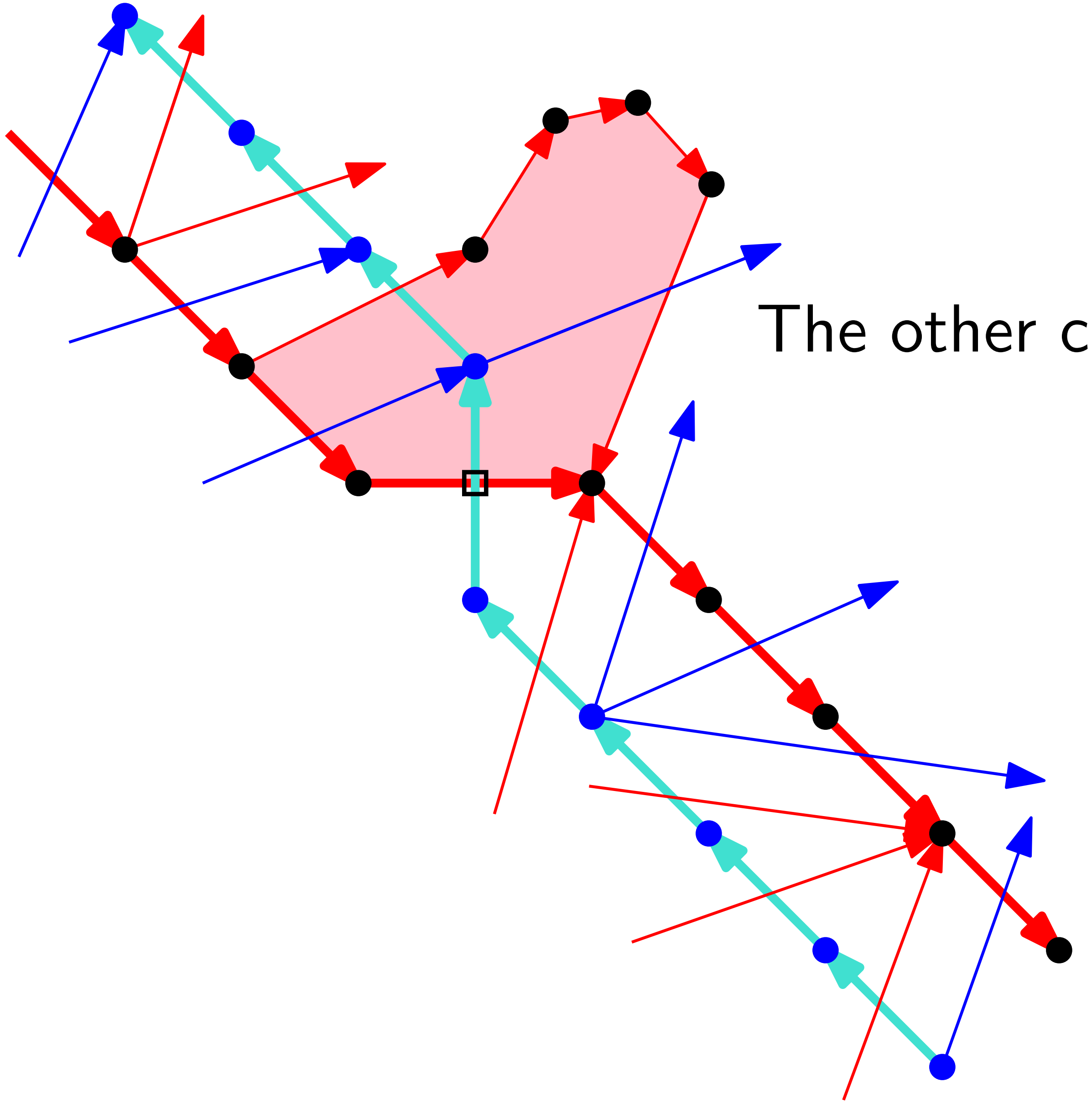

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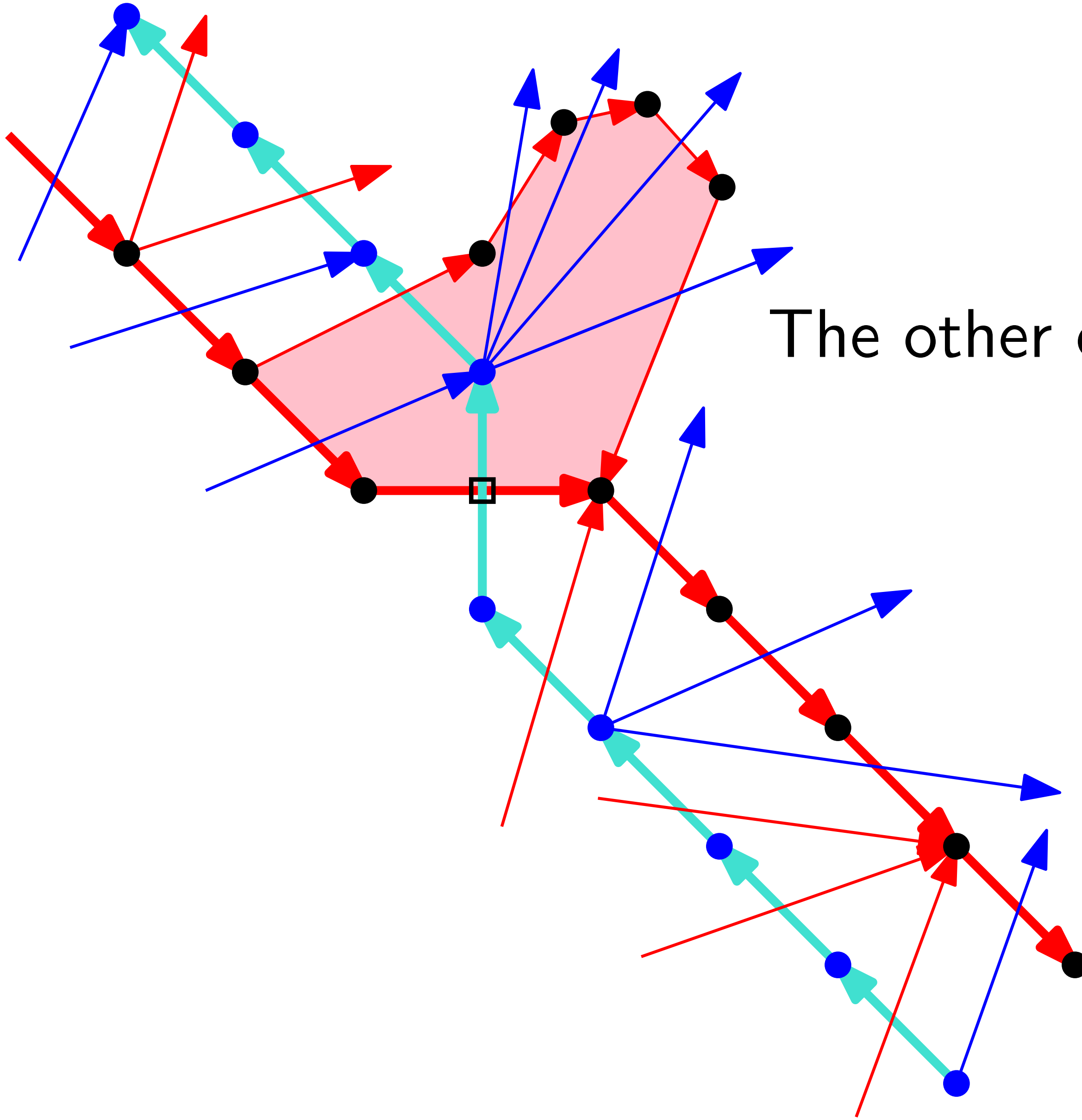

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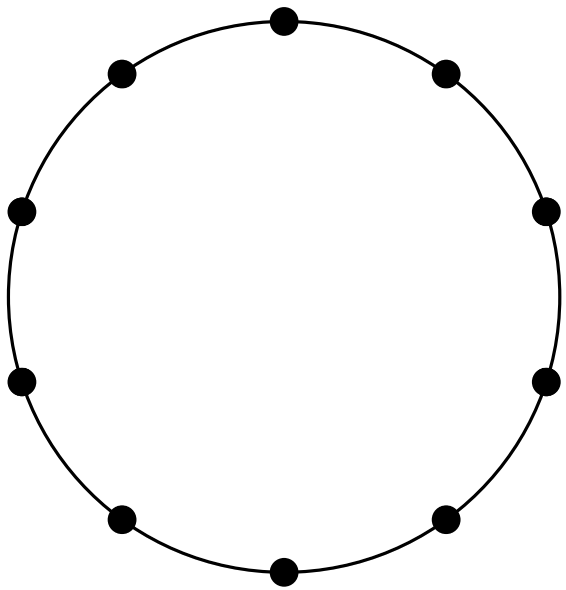
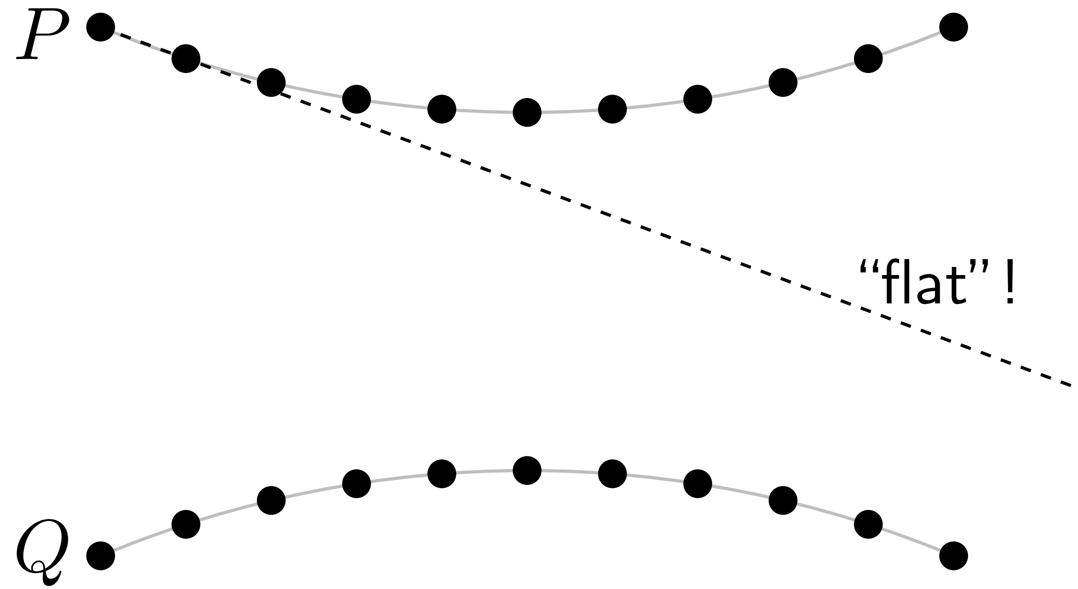
3. Perf. matchings in structured points



convex position



double-chain

smallest possible number of perfect matchings: $\Theta^*(2^n)$

$\Theta^*(3^n)$

[García, Noy, Tejel 2000]

Upper bound: $O^*(10.06^n)$ [Sharir, Welzl 2006]

$*$ $=$ up to a polynomial factor

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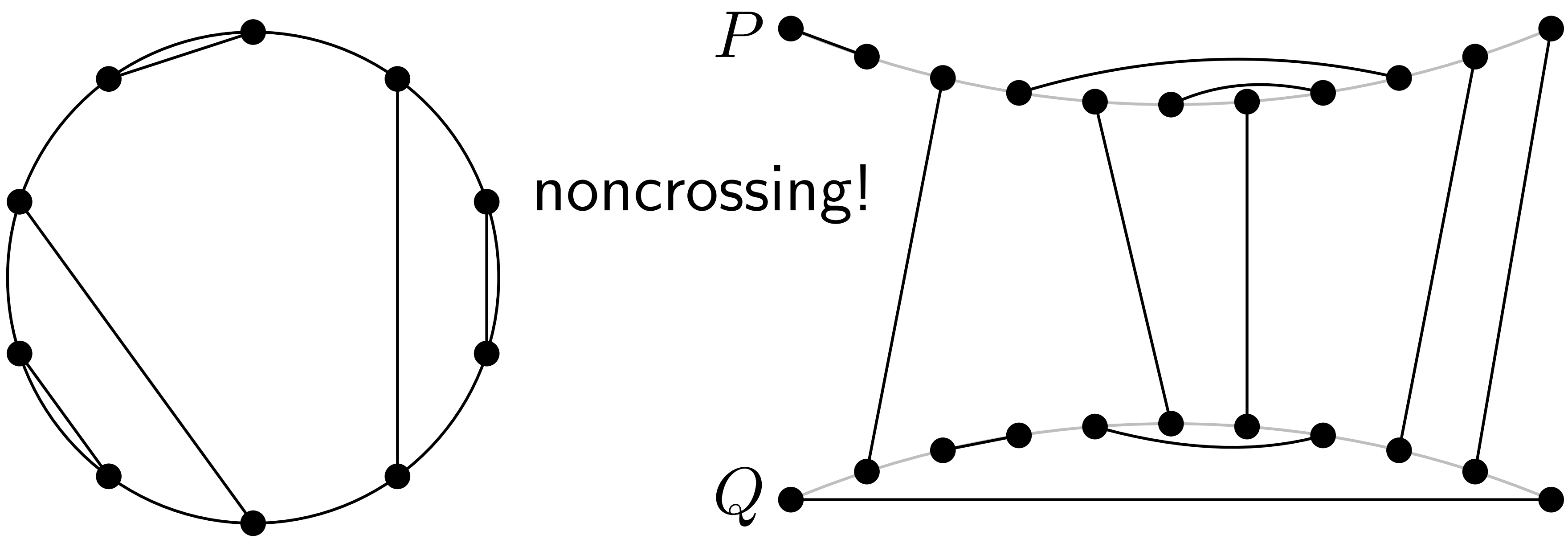

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Current lower bound record:

The generalized double-zigzag chain

$|P| = rn + 1$

$r = 8$: $\Theta^*(3.0930^n)$ [Asinowski and Rote 2018]

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Here: $r = 3$ *without corners*: $\Theta^*(3.037^n)$

3. Perf. matchings in structured points

Current lower bound record:

The generalized double-zigzag chain

~~$|P| = rn + 1$~~ $|P| = 3n$

$r = 8$: $\Theta^*(3.0930^n)$ [Asinowski and Rote 2018]

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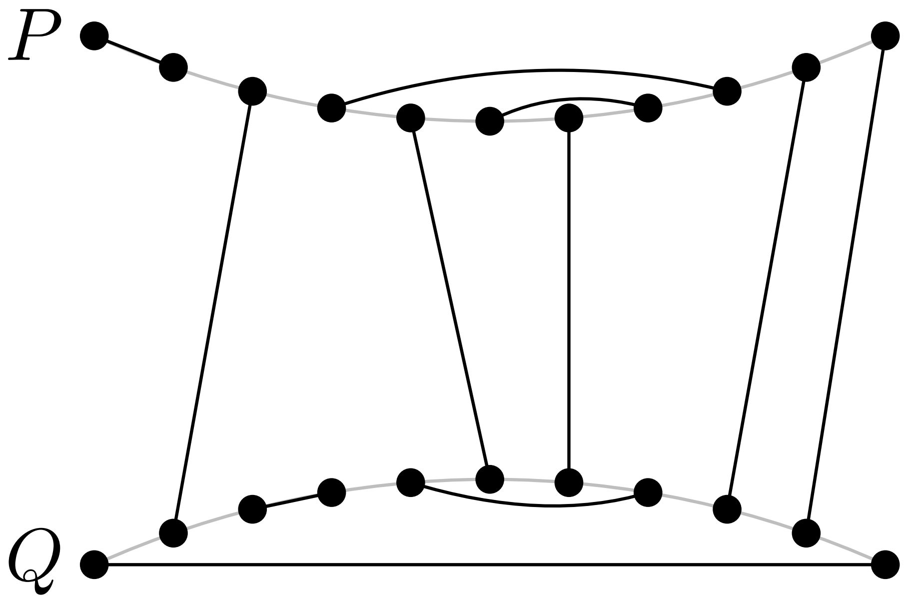
Perfect matchings in double-X



$$|P| = |Q| = n$$

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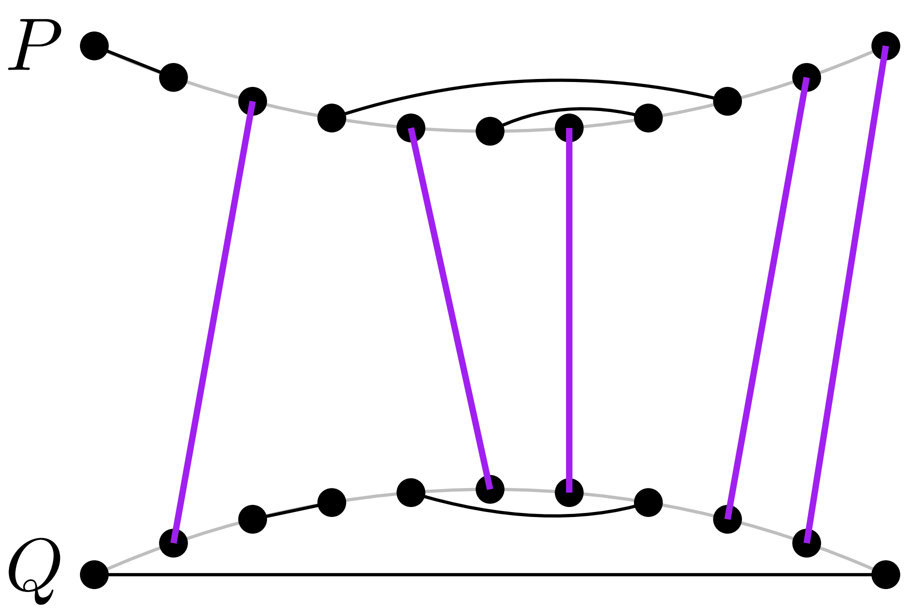

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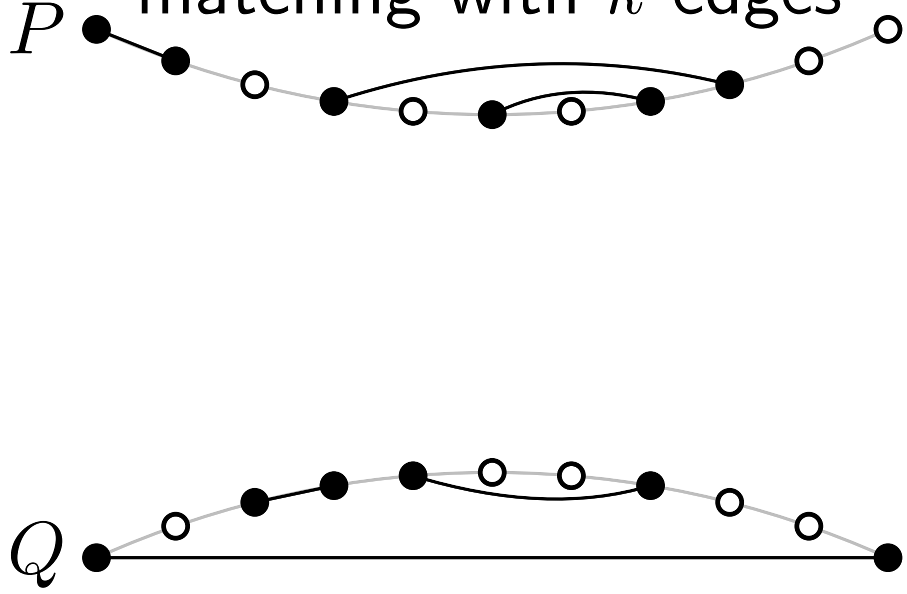

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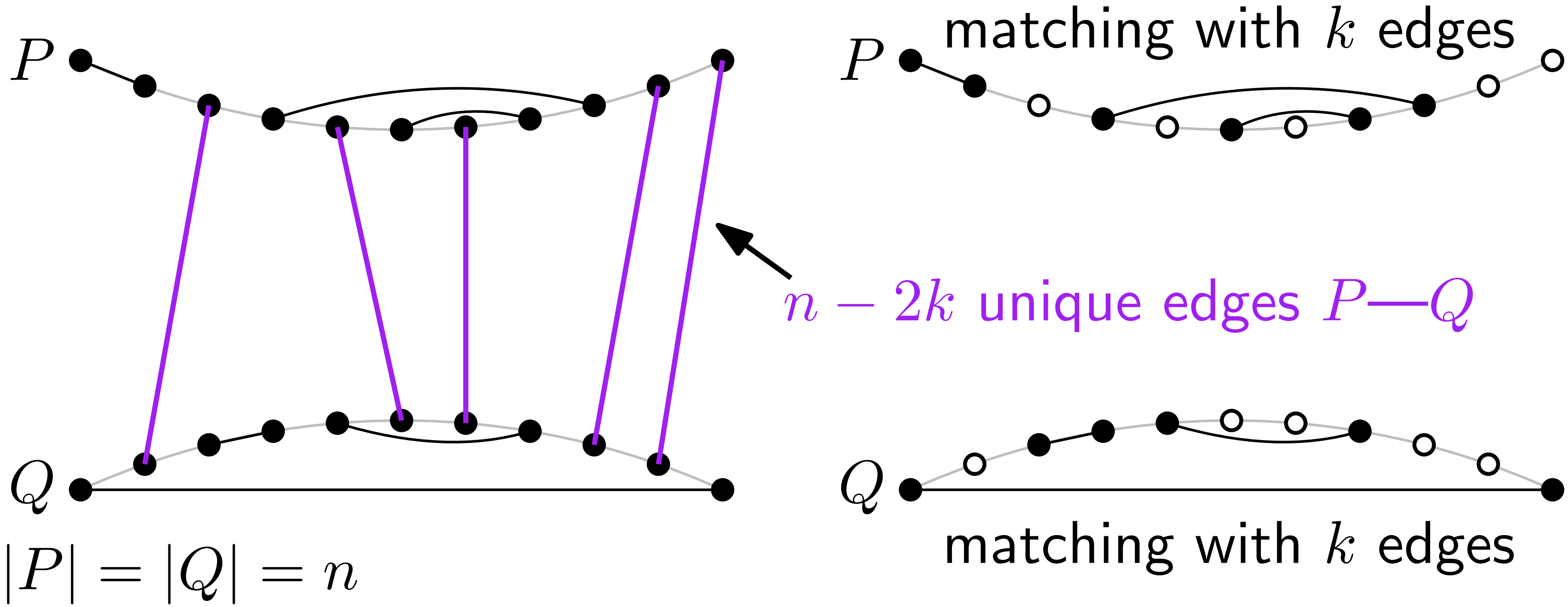

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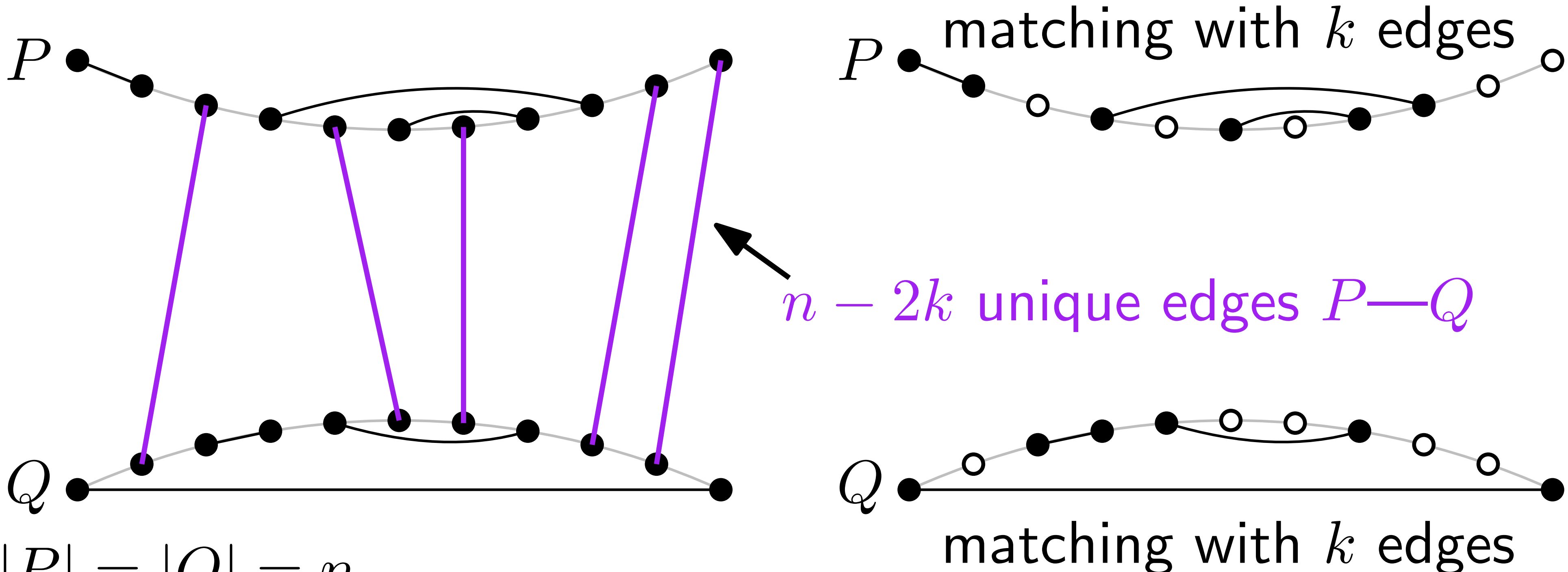


$|P| = |Q| = n$

$$\text{PM(double-}X) = \sum_{k=0}^{n/2} M_k(X)^2 \qquad M(X) = \sum_{k=0}^{n/2} M_k(X)$$

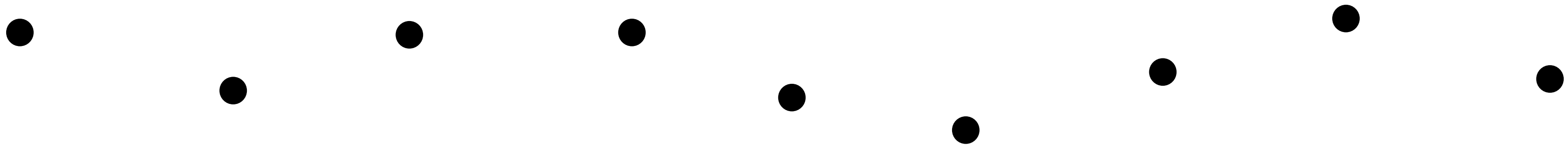
$M(X), M_k(X) = \#$ matchings of X (with k edges)

$\implies M(X)^2/\frac{n}{2} \le \text{PM(double-}X) \le M(X)^2$

$\implies \Theta^*(3^{2n})$ (Ex. 5)

More general “flat” X

P



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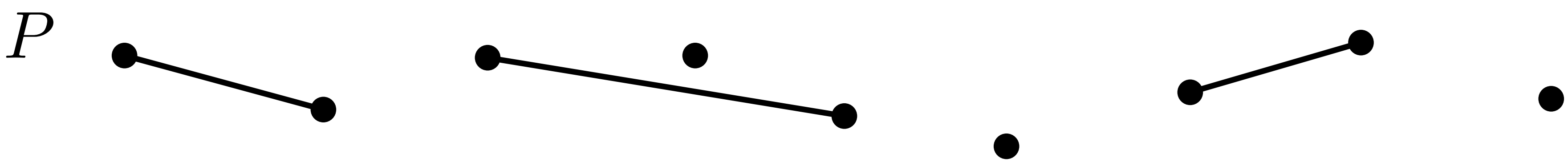

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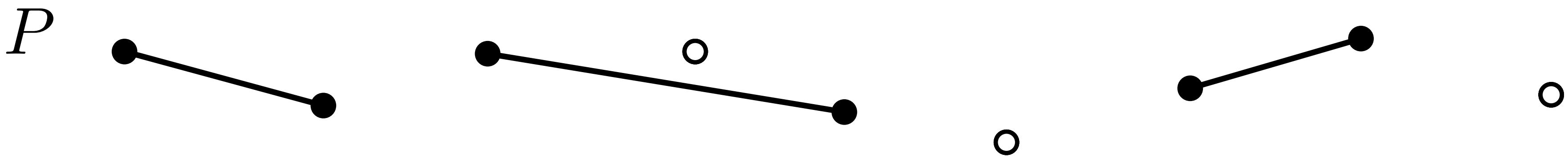

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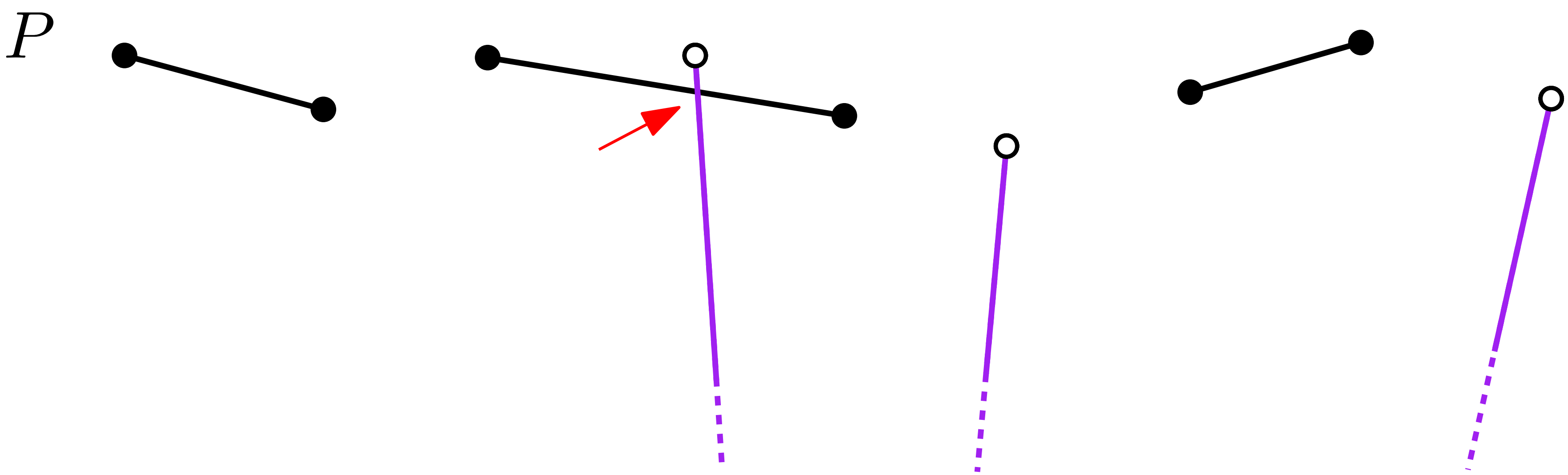
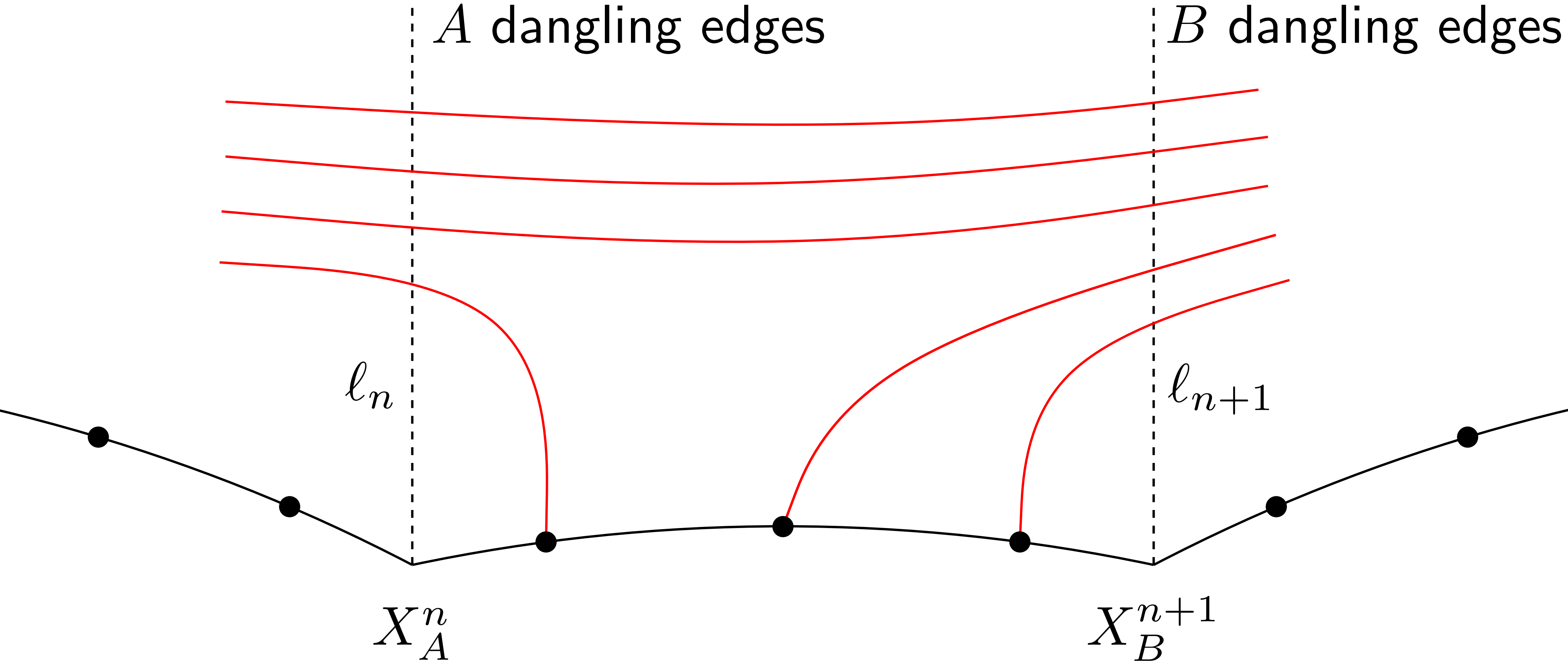


Must count only *down-free* matchings of P:

The unmatched points must be visible from below!

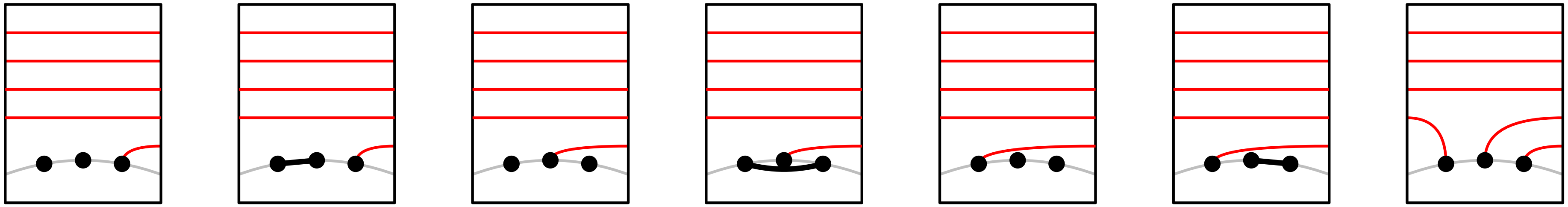
Dynamic Programming Recursion



$X_A^n = \#$ possibilities after n arcs with A dangling edges

Dynamic Programming Recursion

$$X_5^{n+1} = X_2^n + 3X_3^n + 7X_4^n + 6X_5^n + 7X_6^n + 3X_7^n + X_8^n$$



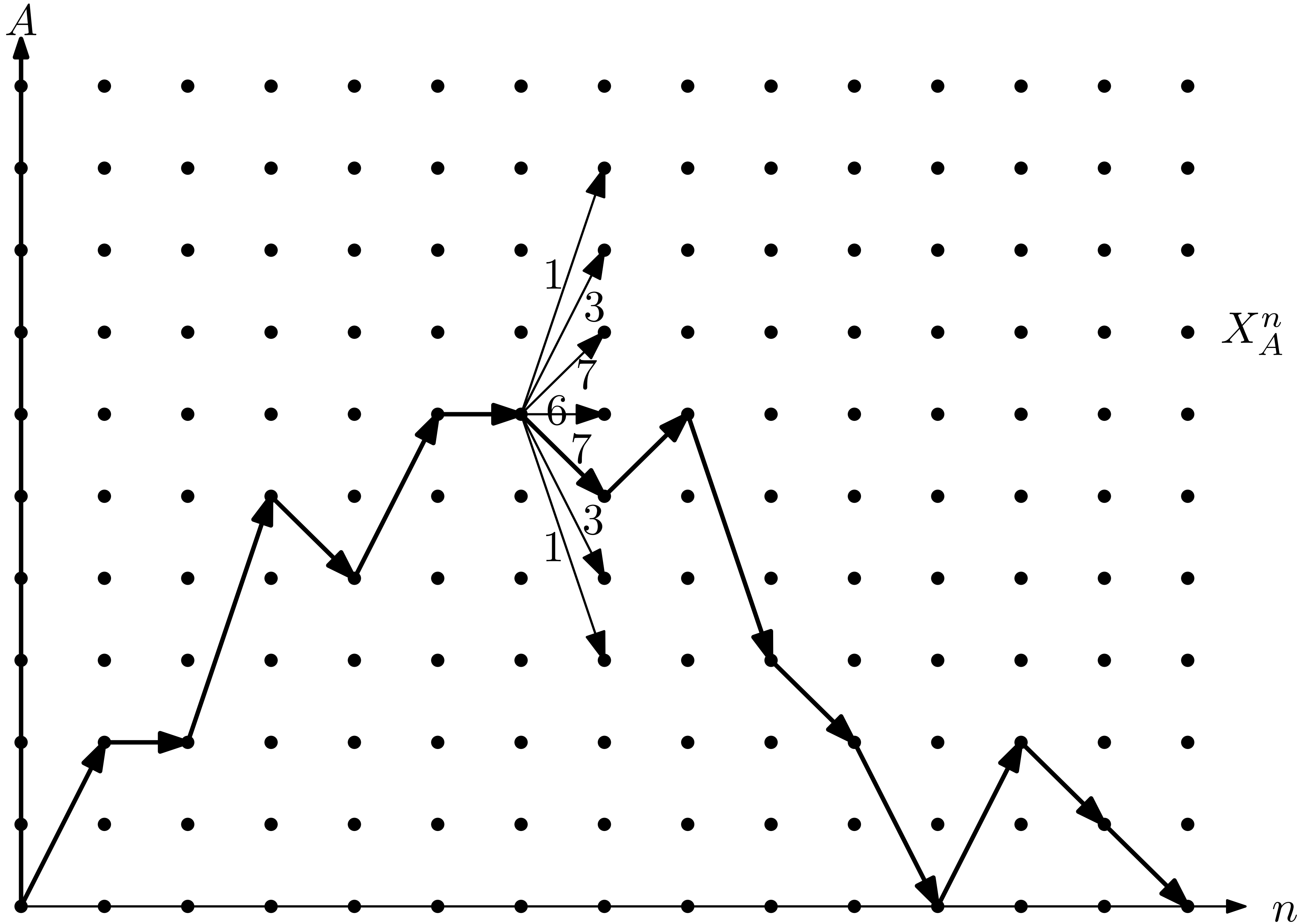
$$\begin{pmatrix} X_0^{n+1} \\ X_1^{n+1} \\ X_2^{n+1} \\ X_3^{n+1} \\ X_5^{n+1} \\ X_6^{n+1} \\ X_7^{n+1} \\ X_8^{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 3 & 6 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 6 & 6 & 7 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 7 & 6 & 7 & 3 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 7 & 6 & 7 & 3 & 1 & 0 & \cdots \\ 0 & 1 & 3 & 7 & 6 & 7 & 3 & 1 & \cdots \\ 0 & 0 & 1 & 3 & 7 & 6 & 7 & 3 & \cdots \\ 0 & 0 & 0 & 1 & 3 & 7 & 6 & 7 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 3 & 7 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} X_0^n \\ X_1^n \\ X_2^n \\ X_3^n \\ X_5^n \\ X_6^n \\ X_7^n \\ X_8^n \\ \vdots \end{pmatrix}$$

total #points

row sum $28 \implies$ vectors grow like $28^n/n^{3/2} \implies \Theta^*(3.037^N)$

[Banderier and Flajolet, 2002]

Weighted lattice paths



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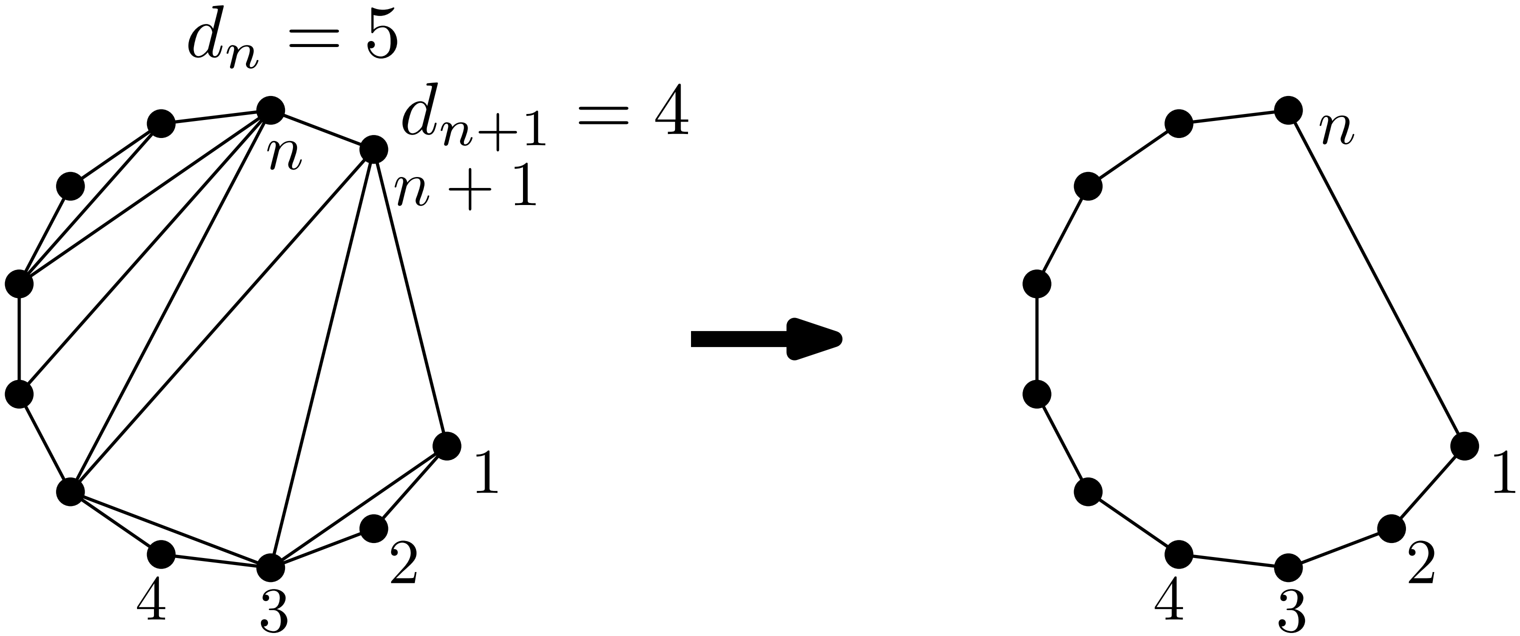
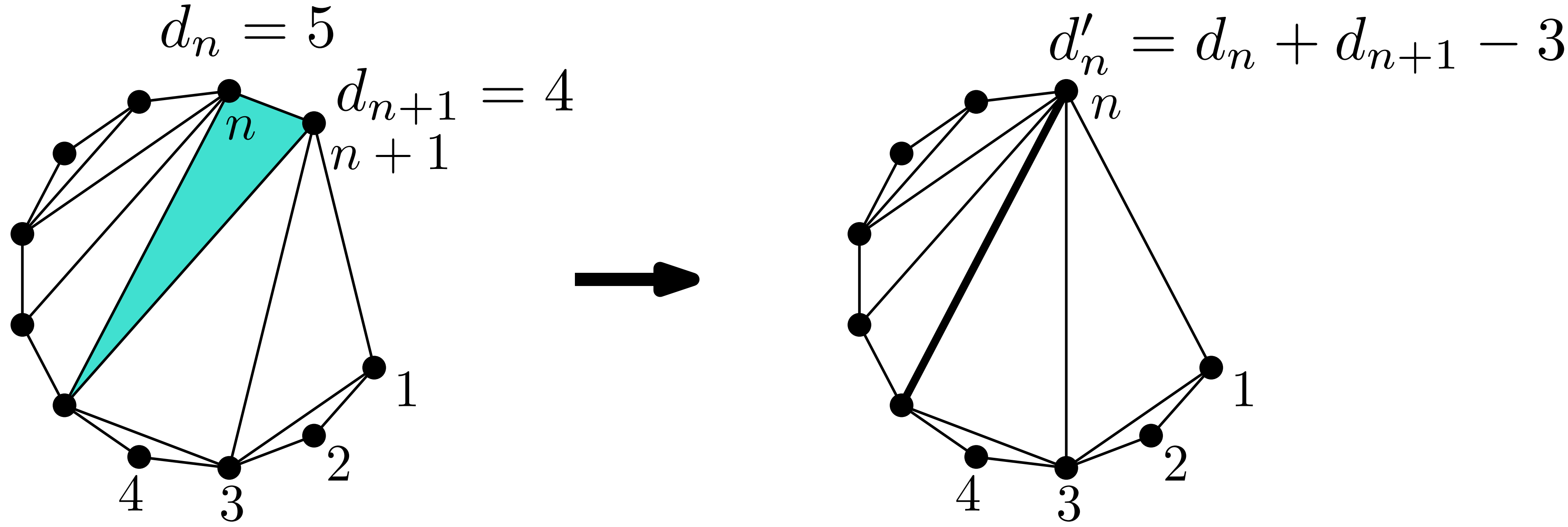
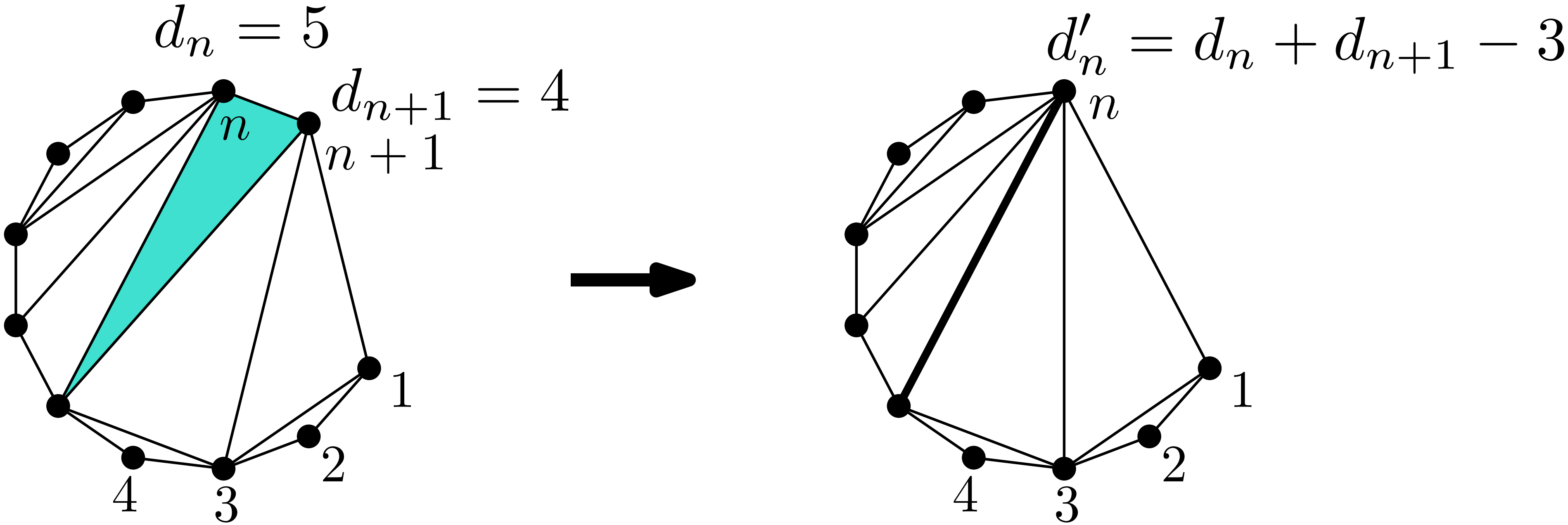
4. Triangulations of a convex n-gon

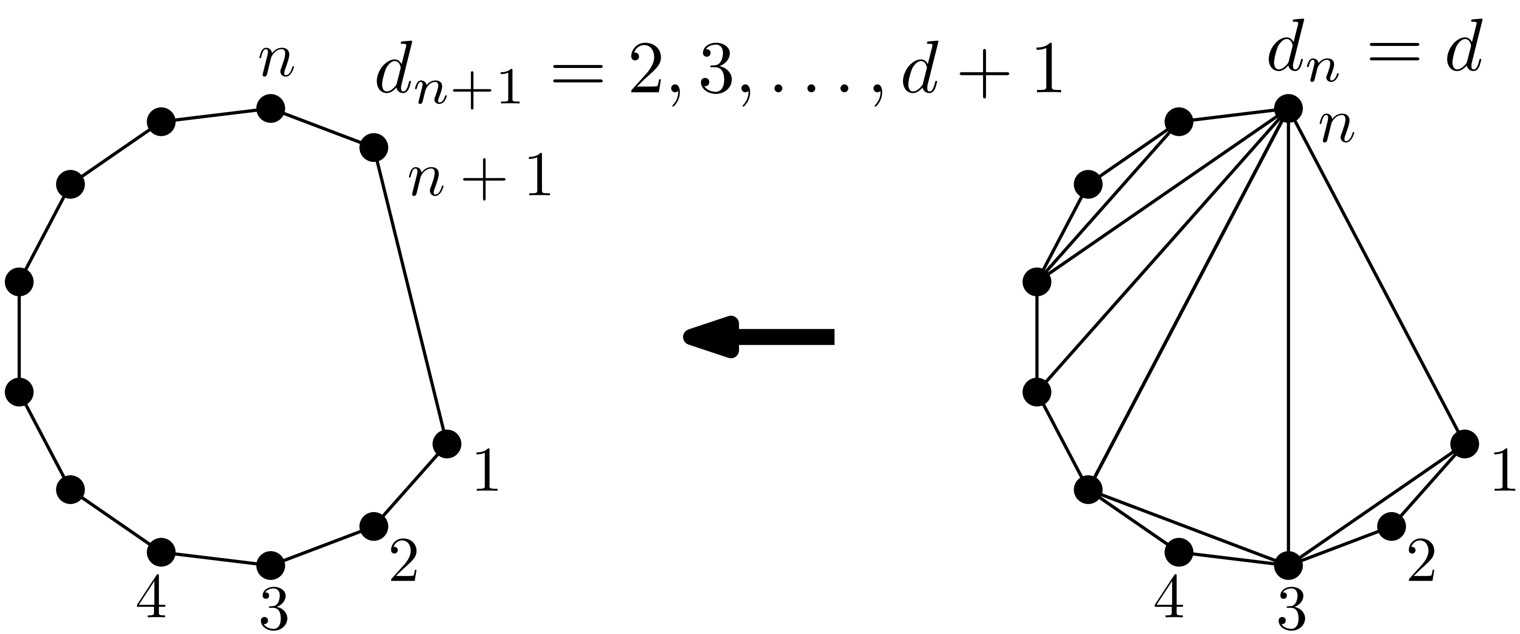


4. Triangulations of a convex n-gon

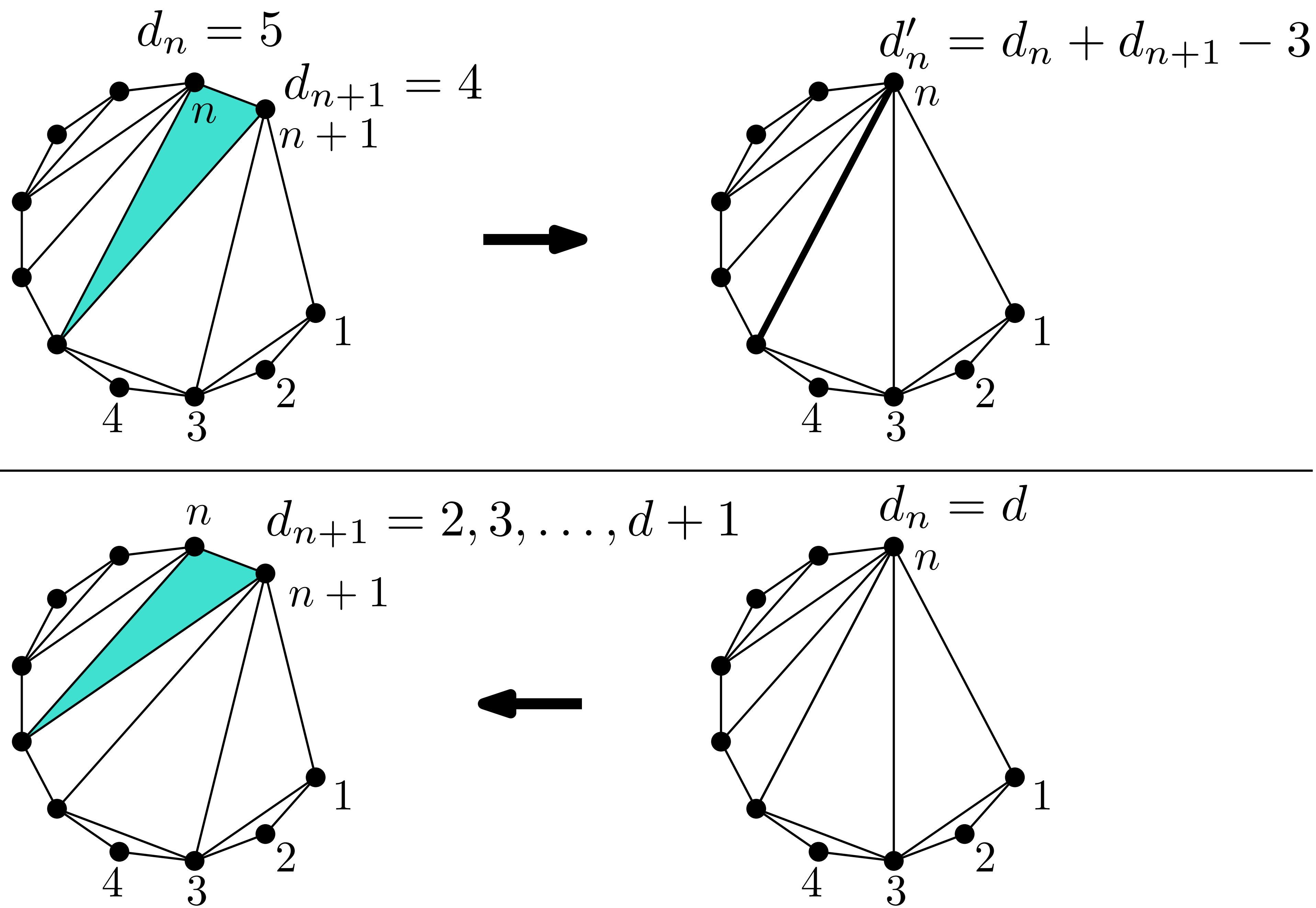


4. Triangulations of a convex n-gon





4. Triangulations of a convex n-gon



4. Triangulations of a convex n-gon

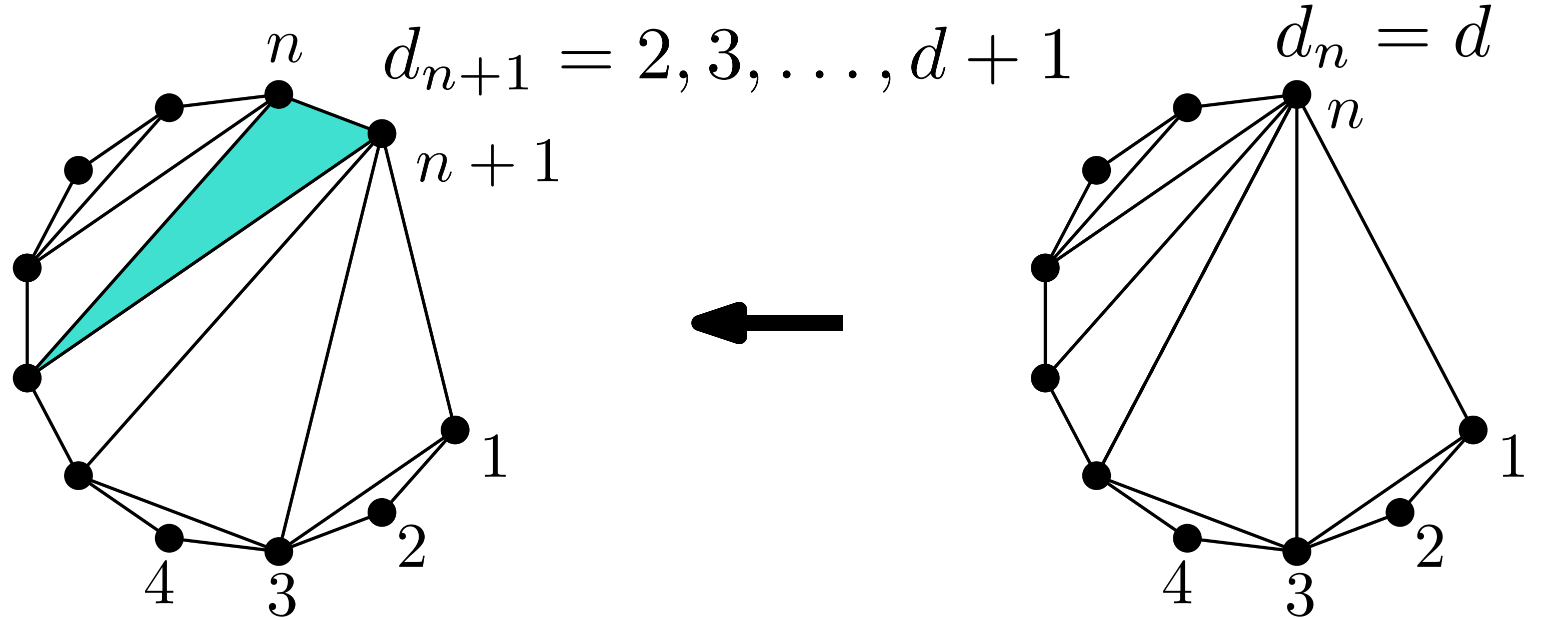
Triangulation of n-gon with last vertex of degree $d_n = d$

$\rightarrow$

Triangulation of $(n+1)$-gon with last vertex of degree

$$d_{n+1} = 2 \text{ or } 3 \text{ or } 4 \text{ or } \dots \text{ or } d, \text{ or } d+1$$

[Hurtado & Noy 1999]
"tree of triangulations"



4. Triangulations of a convex n-gon

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$\rightarrow$

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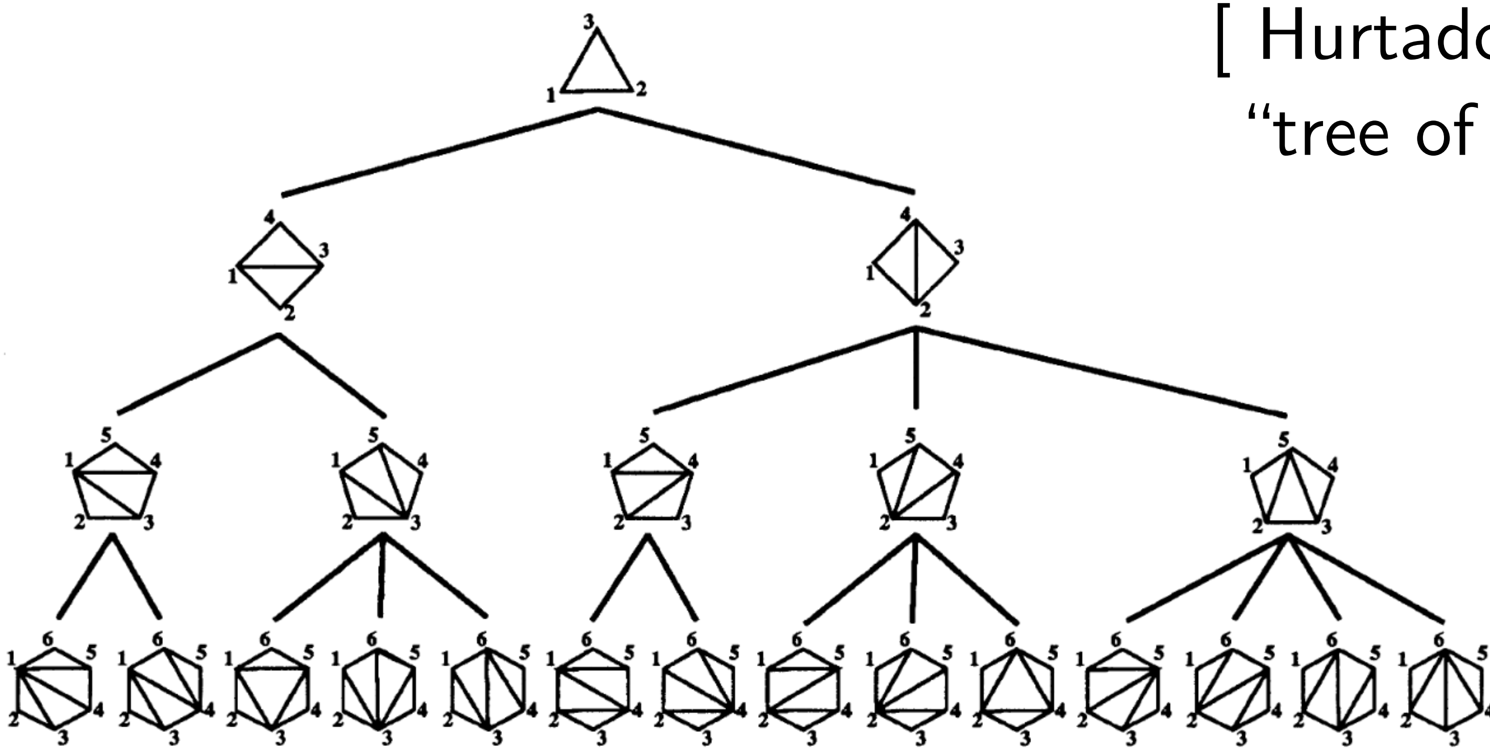

Fig. 4. Levels three to six of the tree of triangulations.

Triangulation of n-gon with last vertex of degree $d_n = d$

$\rightarrow$

Triangulation of $(n+1)$-gon with last vertex of degree

$$d_{n+1} = 2 \text{ or } 3 \text{ or } 4 \text{ or } \ldots \text{ or } d, \text{ or } d+1$$

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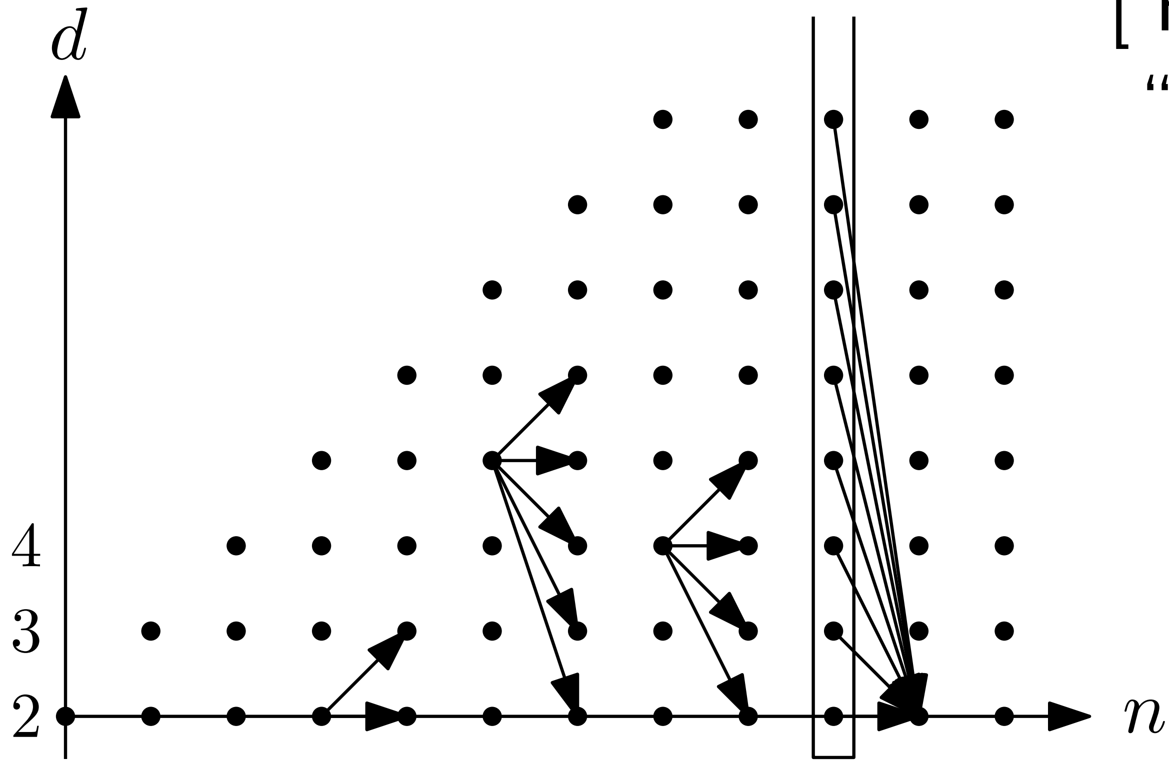
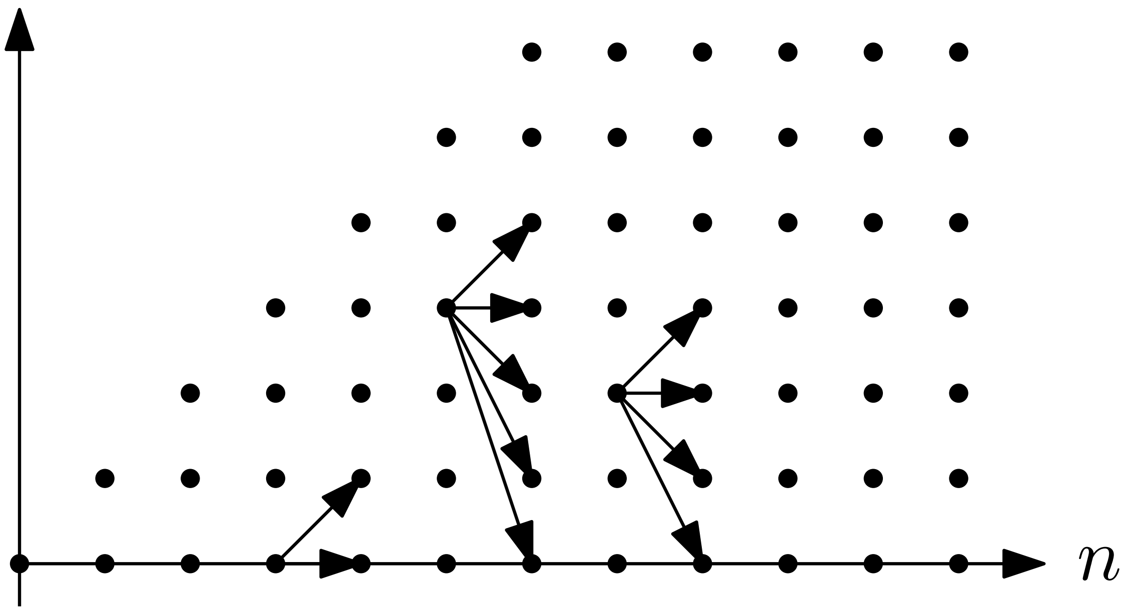


triangulation
$\updownarrow$
lattice path

Production matrices



count paths in a layered graph

The answer is

$$\begin{pmatrix}1 & 0 & 0 & \dots\end{pmatrix} \underbrace{\begin{pmatrix}1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}^n}_{\text{the ``production matrix'' } P} \begin{pmatrix}1 \\ 0 \\ 0 \\ \vdots\end{pmatrix}$$

Production matrices for enumeration

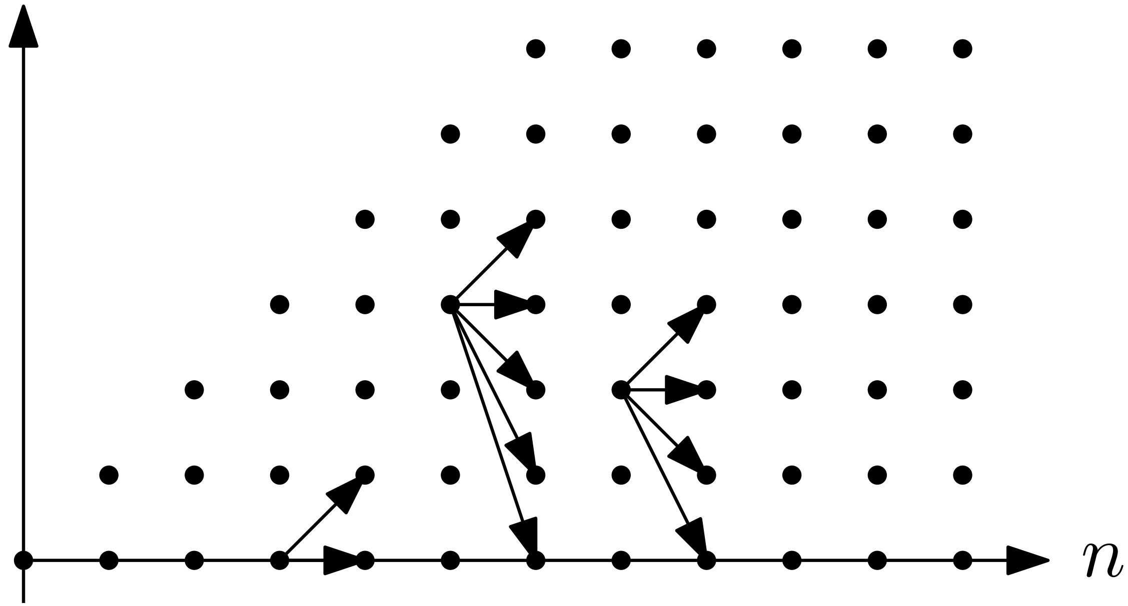
were introduced by Emeric Deutsch, Luca Ferrari, and Simone Rinaldi (2005).

were used for counting noncrossing graphs for points in convex position: Huemer, Seara, Silveira, and Pilz (2016)
Huemer, Pilz, Seara, and Silveira (2017)

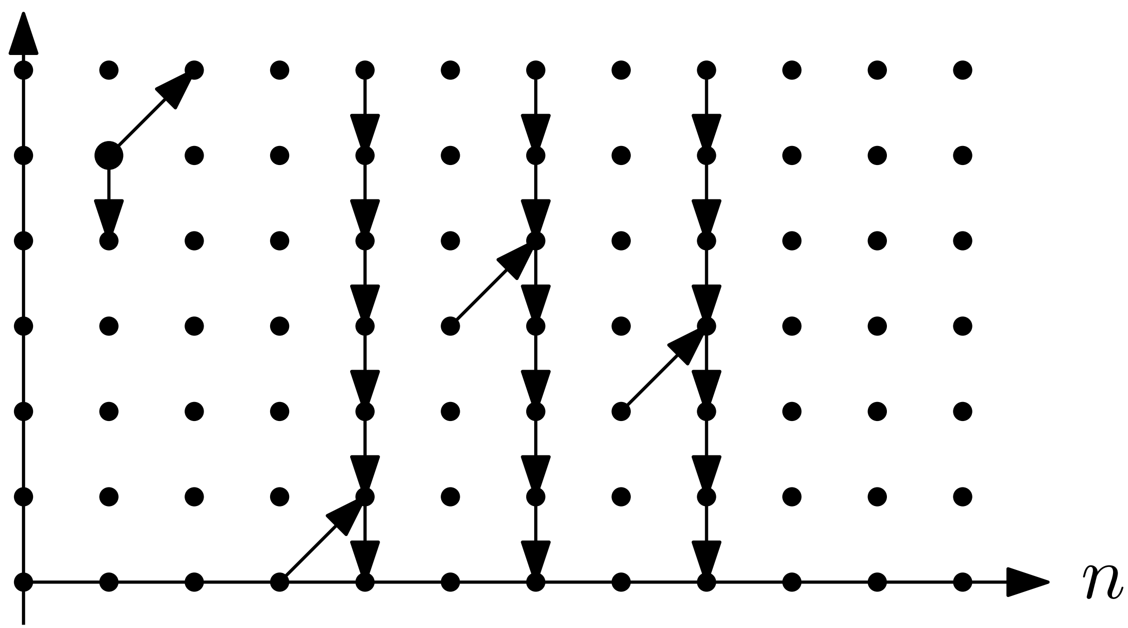
$$\begin{pmatrix} 0 & 1 & 1 & 1 & \dots \\ 1 & 0 & 1 & 1 & \dots \\ 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 5 & \dots \\ 1 & 2 & 3 & 4 & \dots \\ 0 & 1 & 2 & 3 & \dots \\ 0 & 0 & 1 & 2 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 4 & 5 & \dots \\ 0 & 1 & 3 & 4 & \dots \\ 0 & 0 & 1 & 3 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

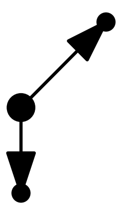
matchings — spanning trees — forests

Making the degree finite



use
vertical edges for
partial summation

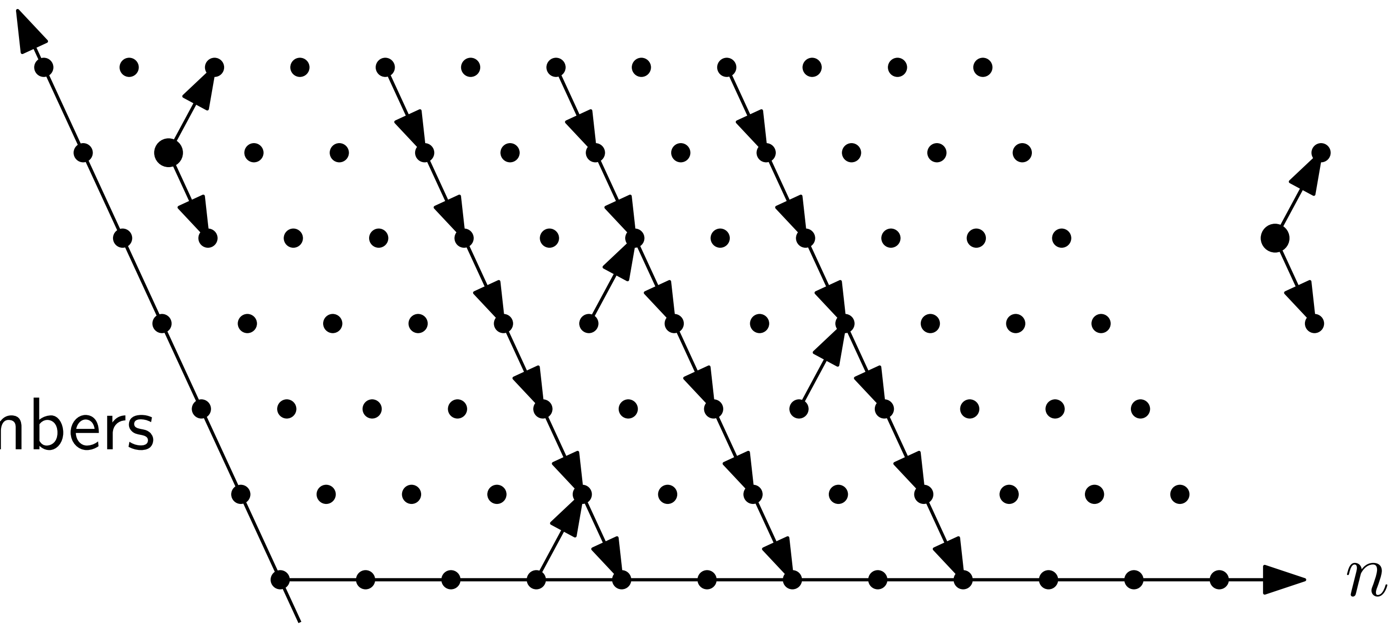


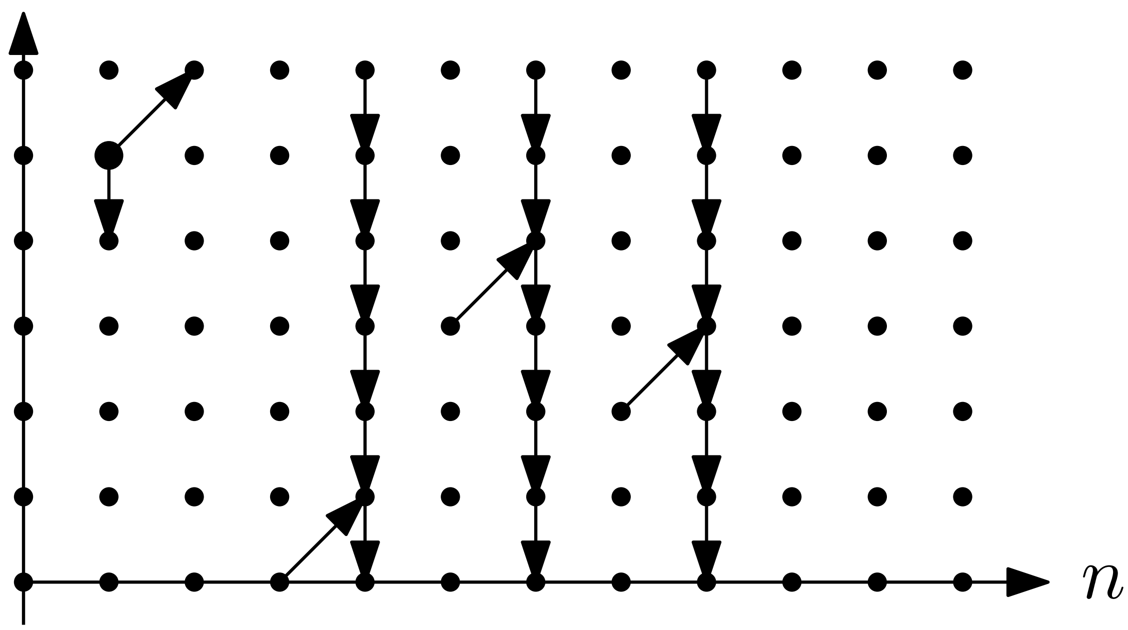


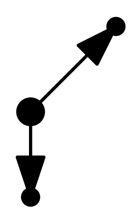
Number of paths
is preserved.

Making the degree finite

Shearing
→ Dyck paths
→ Catalan numbers







Number of paths
is preserved.

Other examples: graphs, paths

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 3 & 3 & 3 & 3 & \dots \\ 0 & 2 & 4 & 4 & 4 & 4 & \dots \\ 0 & 0 & 2 & 4 & 4 & 4 & \dots \\ 0 & 0 & 0 & 2 & 4 & 4 & \dots \\ 0 & 0 & 0 & 0 & 2 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

geometric graphs

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

paths

Huemer, Seara, Silveira, and Pilz (2016)
Huemer, Pilz, Seara, and Silveira (2017)

Other examples: graphs, paths

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 3 & 3 & 3 & 3 & 3 & \dots \\ 0 & 2 & 4 & 4 & 4 & 4 & \dots \\ 0 & 0 & 2 & 4 & 4 & 4 & \dots \\ 0 & 0 & 0 & 2 & 4 & 4 & \dots \\ 0 & 0 & 0 & 0 & 2 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}$$

geometric graphs

$$\left(\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 0 & 1 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}\right)$$

paths

two "states"

Huemer, Seara, Silveira, and Pilz (2016)
Huemer, Pilz, Seara, and Silveira (2017)