Pseudotriangulations

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Day 5

Literature:

Pseudo-triangulations — a survey.

G.Rote, F.Santos, I.Streinu, 2008
Outline

1. Motivation: ray shooting
2. Pseudotriangulations: definitions and properties
3. Rigidity, Laman graphs
4. Rigidity: kinematics of linkages
5. Liftings of pseudotriangulations to 3 dimensions
1. Motivation: Ray Shooting in a Simple Polygon
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Walking in a triangulation:
Walk to starting point. Then walk along the ray.
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Walking in a triangulation:
Walk to starting point. Then walk along the ray.
$O(n)$ steps in the worst case.
Triangulations of a *convex* polygon
Triangulations of a **convex** polygon

A path crosses $O(\log n)$ triangles.

balanced triangulation
Triangulations of a *simple* polygon

balanced triangulation:
An edge crosses $O(\log n)$ triangles.

balanced *geodesic* triangulation:
An edge crosses $O(\log n)$ pseudotriangles.

[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]
Triangulations of a *simple* polygon

balanced triangulation: An edge crosses $O(\log n)$ triangles.

balanced *geodesic* triangulation: An edge crosses $O(\log n)$ pseudotriangles.

[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]
Going through a single pseudotriangle

balanced binary tree for each pseudo-edge:
→ $O(\log n)$ time per pseudotriangle
→ $O(\log^2 n)$ time total
Going through a single pseudotriangle

balanced binary tree for each pseudo-edge:
$\rightarrow O(\log n)$ time per pseudotriangle
$\rightarrow O(\log^2 n)$ time total

weighted binary tree:
$\rightarrow O(\log n)$ time total
2. Pseudotriangulations: Basic definitions and properties
A *pointed* vertex is incident to an angle $> 180^\circ$ (a reflex angle or big angle).

A straight-line graph is pointed if all vertices are pointed.
Pointed Vertices

A pointed vertex is incident to an angle $> 180^\circ$ (a reflex angle or big angle).

A straight-line graph is pointed if all vertices are pointed.

Where do pointed vertices arise?
Geodesic shortest paths

Shortest path (with given homotopy) turns only at pointed vertices. Addition of shortest path edges leaves intermediate vertices pointed.

→ geodesic triangulations of a simple polygon
[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink ’94]
Given: A set $V$ of vertices, a subset $V_p \subseteq V$ of \textit{pointed} vertices.

A \textit{pseudotriangulation} is a maximal (with respect to $\subseteq$) set of non-crossing edges with all vertices in $V_p$ pointed.
Pseudotriangulations

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A pseudotriangulation is a maximal (with respect to $\subseteq$) set of non-crossing edges with all vertices in $V_p$ pointed.
A pseudotriangle has three convex corners and an arbitrary number of reflex vertices (> 180°).
Pseudotriangulations

Given: A set $V$ of vertices, a subset $V_p \subseteq V$ of pointed vertices.

(1) A pseudotriangulation is a maximal (w.r.t. $\subseteq$) set $E$ of non-crossing edges with all vertices in $V_p$ pointed.
Pseudotriangulations

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(1) A pseudotriangulation is a maximal (w.r.t. $\subseteq$) set $E$ of non-crossing edges with all vertices in $V_p$ pointed.

(2) A pseudotriangulation is a partition of a convex polygon into pseudotriangles.
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Proof. $(2) \implies (1)$ No edge can be added inside a pseudotriangle without creating a nonpointed vertex.
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Proof. (2) $\implies$ (1) No edge can be added inside a pseudotriangle without creating a nonpointed vertex.

Proof. (1) $\implies$ (2) All convex hull edges are in $E$. $\rightarrow$ decomposition of the polygon into faces.

Need to show: If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.
Characterization of pseudotriangulations

Lemma. If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.
Lemma. *If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.*

Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.
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Lemma. If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.

Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.
A new edge is always added, unless the face is already a pseudotriangle (without inner obstacles).

[Roete, C. A. Wang, L. Wang, Xu 2003]
Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.

before after
Flipping of Edges

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The flip graph is connected. Its diameter is $O(n \log n)$.  

[Bespamyatnikh 2003]
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**BETTER THAN TRIANGULATIONS!**
Flipping

Every pseudoquadrangle has precisely two diagonals, which cut it into two pseudotriangles.

[Proof. Every tangent ray can be continued to a geodesic path running along the boundary to a corner, in a unique way.]
Vertex and face counts

Lemma. A pseudotriangulation with $x$ nonpointed and $y$ pointed vertices has $e = 3x + 2y - 3$ edges and $2x + y - 2$ pseudotriangles.

Corollary. A pointed pseudotriangulation with $n$ vertices has $e = 2n - 3$ edges and $n - 2$ pseudotriangles.
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Proof. A \( k \)-gon pseudotriangle has \( k - 3 \) large angles.

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\sum_{t \in T} (k_t - 3) + k_{outer} = y
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\[
\sum_{t} k_t + k_{\text{outer}} - 3|T| = y
\]

\[
e + 2 = (|T| + 1) + (x + y) \quad \text{(Euler)}
\]
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BETTER THAN TRIANGULATIONS!
**Vertex and face counts**

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**Corollary.** A pointed pseudotriangulation with \( n \) vertices has \( e = 2n - 3 \) edges and \( n - 2 \) pseudotriangles.

*BETTER THAN TRIANGULATIONS!*

**Corollary.** A non-crossing pointed graph with \( n \geq 2 \) vertices has at most \( 2n - 3 \) edges.
Pseudotriangulations/Geodesic Triangulations

Applications:

- kinetics of bar frameworks, robot motion planning, the “Carpenter’s Rule Problem” [Streinu 2000]
- ...
Applications (continued):

• art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]

• locally convex surfaces, reflex-free hull
  [Aichholzer, Aurenhammer, Krasser, Braß 2003]

• pseudotriangulations on the sphere, smooth counterexample surface to a conjecture of A. D. Alexandrov [G. Panina 2005]
What are the *graphs* of pseudotriangulations?

- planar
- $2n - 3$ edges
- ...?
Infinitesimal motions — rigid frameworks

A framework is a set of movable joints (vertices) connected by rigid bars (edges) of fixed length.

$n$ points $p_1, \ldots, p_n$.

1. (global) motion $p_i = p_i(t)$, $t \geq 0$
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2. infinitesimal motion (local motion)

$$v_i = \frac{d}{dt} p_i(t) = \dot{p}_i(0)$$

velocity vectors $v_1, \ldots, v_n$. 
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\[ v_i = \frac{d}{dt} p_i(t) = \dot{p}_i(0) \]

velocity vectors $v_1, \ldots, v_n$.

3. constraints:

$|p_i(t) - p_j(t)|$ is constant for every edge (bar) $ij$. 
Expansion

\[
\frac{1}{2} \cdot \frac{d}{dt} |p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle
\]

\[
\frac{v_i \cdot (p_j - p_i)}{|p_j - p_i|} \quad \text{and} \quad \frac{v_j \cdot (p_j - p_i)}{|p_j - p_i|}
\]

*expansion* (or *strain*) of the segment \(i,j\)
A framework is *infinitesimally rigid* if the system of equations

\[ \langle v_i - v_j, p_i - p_j \rangle = 0, \text{ for all edges } i,j \]

in the vector variables \( v_1, \ldots, v_n \) has only the trivial solutions: translations and rotations of the framework as a whole.
A framework is *infinitesimally rigid* if the system of equations

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in the vector variables \( v_1, \ldots, v_n \) has only the trivial solutions: translations and rotations of the framework as a whole.

[ Alternative: pin an edge \( ij \) by setting \( v_i = v_j = 0 \).

\[ \implies \text{only } (0, 0, \ldots, 0) \text{ is a trivial solution.} \]
Rigid frameworks

An infinitesimally rigid framework is rigid.

This framework is rigid, but not infinitesimally rigid:
Generically rigid frameworks

A given graph can be rigid in most embeddings, but it may have special non-rigid embeddings:

A graph is *generically rigid* if it is infinitesimally rigid in almost all embeddings.

This is a *combinatorial property* of the graph.
Minimally rigid frameworks

**Theorem.** A graph with \( n \) vertices is *minimally rigid* in the plane (with respect to \( \subseteq \)) iff it has the *Laman property*:

- It has \( 2n - 3 \) edges.
- Every subset of \( k \geq 2 \) vertices spans at most \( 2k - 3 \) edges.

\[ n = 10, \ e = 17 \]

\[ n = 6, \ e = 9 \]

[1961 Laman]
A pointed pseudotriangulation is a Laman graph

Proof: Every subset of $k \geq 2$ vertices is pointed and has therefore at most $2k - 3$ edges.

[Streinu 2001]
Every planar Laman graph is a pointed pseudotriangulation

**Theorem.** Every planar Laman graph has a realization as a pointed pseudotriangulation. The outer face can be chosen arbitrarily.

[Haas, Rote, Santos, B. Servatius, H. Servatius, Streinu, Whiteley 2003]
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Proof I: Induction, using *Henneberg constructions*

Proof II: via Tutte embeddings for directed graphs
Every planar Laman graph is a pointed pseudotriangulation

**Theorem.** *Every planar Laman graph has a realization as a pointed pseudotriangulation. The outer face can be chosen arbitrarily.*

[Haas, Rote, Santos, B. Servatius, H. Servatius, Streinu, Whiteley 2003]

**Proof I:** Induction, using *Henneberg constructions*

**Proof II:** via Tutte embeddings for directed graphs

**Theorem.** *Every rigid planar graph has a realization as a pseudotriangulation (not necessarily pointed).*

[Orden, Santos, B. Servatius, H. Servatius 2003]
Every Laman graph can be built up by a sequence of Henneberg construction steps, starting from a single edge.
Proof I: Henneberg constructions
4. RIGIDITY AND KINEMATICS

Unfolding of polygons — expansive motions

The Carpenter’s Rule Problem:

**Theorem.** Every polygonal arc in the plane can be brought into straight position, without self-overlap.

Every polygon in the plane can be unfolded into convex position. [Connelly, Demaine, Rote 2000], [Streinu 2000]
The Carpenter’s Rule Problem:

**Theorem.** Every polygonal arc in the plane can be brought into straight position, without self-overlap.

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[Connelly, Demaine, Rote 2000], [Streinu 2000]

**Proof outline:**

1. Find an *expansive* infinitesimal motion.
2. Find a global motion.
Expansive Motions

No distance between any pair of vertices decreases.

Expansive motions cannot lead to self-crossings.
Expansive Motions

No distance between any pair of vertices decreases.

Expansive motions cannot lead to self-crossings.

... need to show that an expansive motion exists ...
Every Polygon has an Expansive Motion

**Proof I:** (Outline)

Existence of an expansive motion

\[\updownarrow\] (duality)

Self-stresses (rigidity)

Self-stresses on planar frameworks

\[\updownarrow\] (Maxwell-Cremona correspondence)

polyhedral terrains

[ Connelly, Demaine, Rote 2000 ]

**Proof II:** via pseudotriangulations and the Pseudotriangulation Polytope

[ Streinu 2000 ] [ Rote, Santos, Streinu 2003 ]
Expansive motions exist

Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]

(There are in general rigid substructures.)
Expansive motions for a chain (or a polygon)

- Add edges to form a pseudotriangulation
- Remove a convex hull edge
- $\rightarrow$ expansive mechanism

Theorem. Every polygonal arc in the plane can be brought into straight position, without self-overlap.

Every polygon in the plane can be unfolded into convex position.

[Connelly, Demaine, Rote 2000], [Streinu 2000]
5. LIFTINGS OF PSEUDOTRIANGULATIONS
Locally convex liftings — the reflex-free hull

nearly convex
saddle
nearly reflex
reflex
flat
convex

an approach for recognizing pockets in biomolecules

[Ahn, Cheng, Cheong, Snoeyink 2002]
Locally convex surfaces

A function over a polygonal domain $P$ is *locally convex* if it is convex on every segment in $P$. 
Locally convex surfaces

A function over a polygonal domain $P$ is *locally convex* if it is convex on every segment in $P$. 
Locally convex functions on a polygon

Given a polygon $P$ and a height value $h_i$ for all vertices plus some additional points $p_i$ in the polygon, find the highest locally convex function $f : P \to \mathbb{R}$ with $f(p_i) \leq h_i$.

If $P$ is convex, this is the lower convex hull of the three-dimensional point set $(p_i, h_i)$.

In general, the result is a piecewise linear function defined on a pseudotriangulation of $(P, S)$. (Interior vertices may be missing.)

→ **regular pseudotriangulations**

[Aichholzer, Aurenhammer, Braß, Krasser 2003]

This can be extended to 3-polytopes.

[Aurenhammer, Krasser 2005]
OPEN QUESTIONS

• Pseudotriangulations in 3-space?
  (Rigid graphs are not well-understood in 3-space.)

• How many pseudotriangulations does a point set have?

• Can every pseudotriangulation be (re)drawn on a polynomial-size grid?