Pseudotriangulations, Polytopes, and How to Expand Linkages

Günter Rote
Universitat Lliure de Berlin

[joint] work of/with Bob Connelly, Erik Demaine, Paco Santos, Ileana Streinu.
Unfolding of polygons

**Theorem.** Every polygonal arc in the plane can be brought into straight position, without self-overlap.

Every polygon in the plane can be unfolded into convex position.
Infinitesimal Motion

$n$ vertices $p_1, \ldots, p_n$.

1. (global) motion $p_i = p_i(t), \ t \geq 0$
Infinitesimal Motion

$n$ vertices $p_1, \ldots, p_n$.

1. (global) motion $p_i = p_i(t)$, $t \geq 0$

2. infinitesimal motion (local motion)

$$v_i = \frac{d}{dt} p_i(t) = \dot{p}_i(0)$$

Velocity vectors $v_1, \ldots, v_n$. 
Expansion

\[ \frac{1}{2} \cdot \frac{d}{dt} |p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle =: \exp_{ij} \]

expansion (or strain) \( \exp_{ij} \) of the segment \( ij \)
The Rigidity Map

\[ M : (\nu_1, \ldots, \nu_n) \mapsto (\exp_{ij})_{ij \in E} \]
The Rigidity Map

\[ M : (v_1, \ldots, v_n) \mapsto (\exp_{ij})_{ij \in E} \]

The rigidity matrix:

\[ M = \begin{pmatrix} \text{the} \\ \text{rigidity} \\ \text{matrix} \end{pmatrix} \bigg\} \ E \\
\bigg\} \ \ 2|V| \]
Expansive Motions

\[ \exp_{ij} = 0 \text{ for all } bars \ ij \]

(preservation of length)

\[ \exp_{ij} \geq 0 \text{ for all other pairs } (struts) \ ij \]

(expansiveness)

\[ \left[ \exp_{ij} > 0 \right] \]

(strict expansiveness)
Expansive motions cannot overlap
Proof Outline

1. Prove that expansive motions exist.

2. Select an expansive motion and provide a global motion.
1. Prove that expansive motions exist. [ 2 PROOFS ]

2. Select an expansive motion and provide a global motion.
Proof Outline

Existence of an expansive motion
  ✤ (duality)
Self-stresses (rigidity)
Self-stresses on planar frameworks
    ✤ (Maxwell-Cremona correspondence)
polyhedral terrains

[ Connelly, Demaine, Rote 2000 ]
The Expansion Cone

The set of expansive motions forms a convex polyhedral cone $\bar{X}_0$ in $\mathbb{R}^{2n}$, defined by homogeneous linear equations and inequalities of the form

$$\langle v_i - v_j, p_i - p_j \rangle \begin{cases} = \\ \geq \\ [>] \end{cases} 0$$
Bars, Struts, Frameworks, Stresses

Assign a stress $\omega_{ij} = \omega_{ji} \in \mathbb{R}$ to each edge.

Equilibrium of forces in vertex $i$:

$$\sum_j \omega_{ij} (p_j - p_i) = 0$$

$\omega_{ij} \leq 0$ for struts: Struts can only push.

$\omega_{ij} \in \mathbb{R}$ for bars: Bars can push or pull.
Motions and Stresses

Linear Programming duality:

There is a strictly expansive motion if and only if there is no non-zero stress.

\[ \langle v_i - v_j, p_i - p_j \rangle \begin{cases} = 0 & \text{for a bar } ij \\ > 0 & \text{for a strut } ij \end{cases} \]

\[ \sum_j \omega_{ij} (p_j - p_i) = 0, \text{ for all } i \]

\[ \omega_{ij} \in \mathbb{R}, \text{ for a bar } ij \]

\[ \omega_{ij} \leq 0, \text{ for a strut } ij \]
Motions and Stresses

Linear Programming duality:

There is a strictly expansive motion if and only if there is no non-zero stress.

\[
\langle v_i - v_j, p_i - p_j \rangle \begin{cases} = 0 & \text{for a bar } ij \\ > 0 & \text{for a strut } ij \end{cases}
\]

\[
\sum_j \omega_{ij} (p_j - p_i) = 0, \text{ for all } i
\]

\[
M^T \omega = 0
\]

\[
\omega_{ij} \in \mathbb{R}, \text{ for a bar } ij
\]

\[
\omega_{ij} \leq 0, \text{ for a strut } ij
\]
Making the Framework Planar

- subdivide edges at intersection points
- collapse multiple edges
The Maxwell-Cremona Correspondence [1850]

3-d lifting (polyhedral terrain)

↕️

self-stresses on a planar framework
The Maxwell-Cremona Correspondence
[1850]

3-d lifting (polyhedral terrain)

\[ \leftrightarrow \]

self-stresses on a planar framework

\[ \leftrightarrow \]

orthogonal dual
Valley and Mountain Folds

\( \omega_{ij} > 0 \)
valley
bar or strut

\( \omega_{ij} < 0 \)
mountain
bar
Look a the highest peak!

Every polygon has $> 3$ convex vertices

$\rightarrow$ 3 valleys $\rightarrow$ 3 bars.
The general case

There is at least one vertex with angle $> \pi$. 
The only remaining possibility

a convex polygon
Constructing a Global Motion

• Define a point \( v := v(p) \) in the *interior* of the expansion cone, by a suitable non-linear convex objective function.

• \( v(p) \) depends smoothly on \( p \).

• Solve the differential equation \( \dot{p} = v(p) \)
Constructing a Global Motion

Alternative approach: Select an *extreme ray* of the expansion cone.

Streinu [2000]:
Extreme rays correspond to pseudotriangulations.

[show animation]
Part II: Pseudotriangulations
Part II: Pseudotriangulations

Pseudotriangulations!

Assumption: Points in general position.
Pseudotriangles

A pseudotriangulation has three convex corners and an arbitrary number of reflex vertices.
Pseudotriangulations/Geodesic Triangulations

Other applications:

• data structures for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, and Snoeyink 1994] and visibility [Pocchiola and Vegter 1996]

• kinetic collision detection [Agarwal, Basch, Erickson, Guibas, Hershberger, Zhang 1999–2001] [Kirkpatrick, Snoeyink, and Speckmann 2000] [Kirkpatrick & Speckmann 2002 this afternoon]

• art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]
Minimum (or Pointed) Pseudotriangulations (PPT)

A *pointed* vertex is incident to an angle $> 180^\circ$.
A *maximal* non-crossing and pointed set of edges decomposes the convex hull into $n - 2$ pseudotriangles using $2n - 3$ edges.
Characterization of Pointed Pseudotriangulations

An edge set with any two of the following properties:

- $2n - 3$ edges (or $n - 2$ faces)
- decomposition into pseudotriangles
- non-crossing, and every vertex is pointed.

[Streinu 2002]
Characterization of Trees

An edge set with any two of the following properties:

- $n - 1$ edges
- connected
- acyclic
Characterization of Pointed Pseudotriangulations

An edge set with any two of the following properties:

- \(2n - 3\) edges (or \(n - 2\) faces)
- decomposition into pseudotriangles
- non-crossing, and every vertex is pointed.
Characterization of Pointed Pseudotriangulations

An edge set with any two of the following properties:

- \(2n - 3\) edges (or \(n - 2\) faces)
- decomposition into pseudotriangles
- non-crossing, and every vertex is pointed.

Caveat: Removing edges from a triangulation does not necessarily lead to a pointed pseudotriangulation.
Rigidity Properties of Pseudotriangulations

- Pseudotriangulations are minimally rigid.
- A Henneberg-type construction
- Removing a hull edge gives an expansive mechanism with 1 degree of freedom.

[Streinu 2002]
Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.
Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.

![Before and After Diagram](image)

The flip graph is connected. Its diameter is $O(n^2)$.

[Brönnimann, Kettner, Pocchiola, Snoeyink 2001]
Part III: Cones and Polytopes

[Rote, Santos, Streinu 2002]

- The expansion cone
  \[ \bar{X}_0 = \{ \exp_{ij} \geq 0 \} \]

- The perturbed expansion cone
  = the PPT polyhedron
  \[ \bar{X}_f = \{ \exp_{ij} \geq f_{ij} \} \]

- The PPT polytope
  \[ X_f = \{ \exp_{ij} \geq f_{ij}, \quad \exp_{ij} = f_{ij} \text{ for } ij \text{ on boundary} \} \]
Pinning of Vertices

Trivial Motions: Motions of the point set as a whole (translations, rotations).

Pin a vertex and a direction. ("tie-down")

\[ v_1 = 0 \]
\[ v_2 \parallel p_2 - p_1 \]

This eliminates 3 degrees of freedom.
Extreme Rays of the Expansion Cone

Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000] Rigid substructures can be identified.
A Polyhedron for Pseudotriangulations

Wanted:
A perturbation of the constraints “\( \exp_{ij} \geq 0 \)” such that the vertices are in 1-1 correspondence with pseudotriangulations.
**Heating up the Bars**

\[ \Delta T = |x|^2 \]

Length increase \[ \geq \int_{x \in p_i p_j} |x|^2 \, ds \]
Heating up the Bars

\[ \Delta T = |x|^2 \]

Length increase \[ \geq \int_{x \in p_i p_j} |x|^2 \, ds \]
Heating up the Bars

\[ \Delta T = |x|^2 \]

Length increase \[ \geq \int_{x \in p_i p_j} |x|^2 \, ds \]

\[ \exp_{ij} \geq |p_i - p_j| \cdot \int_{x \in p_i p_j} |x|^2 \, ds \]
Heating up the Bars

\[ \Delta T = |x|^2 \]

Length increase \( \geq \int_{x \in p_ip_j} |x|^2 \, ds \)

\[ \exp_{ij} \geq |p_i - p_j| \cdot \int_{x \in p_ip_j} |x|^2 \, ds \]

\[ \exp_{ij} \geq |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2) \cdot \frac{1}{3} \]
Heating up the Bars — Points in Convex Position
The Perturbed Expansion Cone  

$\bar{X}_f = \{ (v_1, \ldots, v_n) \mid \exp_{ij} \geq f_{ij} \}$

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

$[x, y, z] = $ signed area of the triangle $xyz$

$a, b$: two arbitrary points.
Tight Edges

For $v = (v_1, \ldots, v_n) \in \bar{X}_f$, 

$$E(v) := \{ij \mid \exp_{ij} = f_{ij}\}$$

is the set of tight edges at $v$.

Maximal sets of tight edges $\equiv$ vertices of $\bar{X}_f$. 
What are good values of $f_{ij}$?

Which configurations of edges can occur in a set of tight edges?

We want:

- no crossing edges

- no 3-star with all angles $\leq 180^\circ$

It is sufficient to look at 4-point subsets.
Good Values $f_{ij}$ for 4 points

$f_{ij}$ is given on six edges. Any five values $\exp_{ij}$ determine the last one. Check if the resulting value $\exp_{ij}$ of the last edge is feasible ($\exp_{ij} \geq f_{ij}$) → checking the sign of an expression.
Good Values $f_{ij}$ for 4 points

A 4-tuple $p_1, p_2, p_3, p_4$ has a unique self-stress (up to a scalar factor).

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}, \text{ for all } 1 \leq i < j \leq 4$$

$\omega_{ij} > 0$ for boundary edges.

$\omega_{ij} < 0$ for interior edges.
Why the stress?

If the equation

$$\sum_{1 \leq i < j \leq 4} \omega_{ij} f_{ij} = 0$$

holds, then $f_{ij}$ are the expansion values $\exp_{ij}$ of a motion $(v_1, v_2, v_3, v_4)$.

Actually, “if and only if”.
Why the stress?

If the equation

\[ \sum_{1 \leq i < j \leq 4} \omega_{ij} f_{ij} = 0 \]

holds, then \( f_{ij} \) are the expansion values \( \exp_{ij} \) of a motion \((v_1, v_2, v_3, v_4)\).

Actually, “if and only if”.

\[
[ M^T \omega = 0, \ f = \exp = Mv ]
\]
Good Perturbations

We need

\[ \sum_{1 \leq i < j \leq 4} \omega_{ij} f_{ij} > 0 \]

for all 4-tuples of points.

→ For every vertex \( v \), \( E(v) \) is non-crossing and pointed.

→ \( \bar{X}_f \) is a simple polyhedron.
The PPT-polyhedron

Every vertex is incident to $2n - 3$ edges.

Edge $\equiv$ removing a segment from $E(v)$.

Removing an interior segment leads to an adjacent pseudotriangulation (flip).

Removing a hull segment is an extreme ray. \qed
Proof of
\[
\omega_{12}f_{12} + \omega_{13}f_{13} + \omega_{14}f_{14} + \omega_{23}f_{23} + \omega_{24}f_{24} + \omega_{34}f_{34} > 0
\]

\[
R(a, b) := \sum_{1 \leq i < j \leq 4} \omega_{ij} \cdot [a, p_i, p_j][b, p_i, p_j]
\]

\[R \equiv 1!\]

\[R \text{ is linear in } a \text{ and linear in } b. \ R(p_i, p_j) = 1 \text{ is sufficient.} \]

\[R(p_1, p_2): \text{ all } f_{ij} = 0 \text{ except } f_{34}\]

\[
R(p_1, p_2) = \omega_{34}f_{34} = \frac{\det(p_1, p_3, p_4) \det(p_2, p_3, p_4)}{\det(p_3, p_4, p_1) \det(p_3, p_4, p_2)} = 1.
\]

\[\square\]
The PPT polytope

Cut out all rays:
Change $\exp_{ij} > f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.
The PPT polytope

Cut out all rays:
Change $\exp_{ij} > f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

The Expansion Cone $\bar{X}_0$:
collapse parallel rays into one ray. $\rightarrow$ pseudotriangulations minus one hull edge. Rigid subcomponents are identified.
Expansive motions for a chain (or a polygon)

- Add edges to form a pseudotriangulation
- Remove a convex hull edge
- $\rightarrow$ expansive mechanism
Which $f_{ij}$ to choose?

- \( f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2) \)

- \( f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j] \)

Go to the space of the \((\exp_{ij})\) variables instead of the \((v_i)\) variables.

\[
\exp = M v
\]
Characterization of the space \((\exp_{ij})_{i,j}\)

A set of values \((\exp_{ij})_{1\leq i<j\leq n}\) forms the expansion values of a motion \((v_1, \ldots, v_n)\) if and only if the equation

\[
\sum_{1\leq i<j\leq 4} \omega_{ij} \exp_{ij} = 0
\]

holds for all 4-tuples.
A canonical representation

\[ \sum_{1 \leq i < j \leq 4} \omega_{ij} \exp_{ij} = 0, \text{ for all } 4\text{-tuples} \]

\[ \exp_{ij} \geq f_{ij}, \text{ for all pairs } i, j \]
A canonical representation

$$\sum_{1 \leq i < j \leq 4} \omega_{ij} \exp_{ij} = 0, \text{ for all 4-tuples}$$

$$\exp_{ij} \geq f_{ij}, \text{ for all pairs } i, j$$

$$\sum_{1 \leq i < j \leq 4} \omega_{ij} f_{ij} = 1, \text{ for all 4-tuples}$$

Substitute $d_{ij} := \exp_{ij} - f_{ij}$:

$$\sum_{1 \leq i < j \leq 4} d_{ij} \exp_{ij} = -1, \text{ for all 4-tuples} \quad (1)$$

$$d_{ij} \geq 0, \text{ for all } i, j \quad (2)$$
The Associahedron
Catalan Structures

- Triangulations of a convex polygon / edge flip
- Binary trees / rotation

\[
(a \ast (b \ast (c \ast d))) \ast e \lor ((a \ast b) \ast (c \ast d)) \ast e
\]
Catalan Structures

- Triangulations of a convex polygon / edge flip
- Binary trees / rotation
- $$(a * (b * (c * d))) * e / ((a * b) * (c * d)) * e$$
- non-crossing alternating trees
- . . . . . . . . . . . . . . . . . . . . . . . .
The Secondary Polytope

Triangulation $T \mapsto (x_1, \ldots, x_n)$.

$x_i := \text{total area of all triangles incident to } p_i$

vertices $\equiv$ regular triangulations of $(p_1, \ldots, p_n)$

$(p_1, \ldots, p_n)$ in convex position:
pseudotriangulations $\equiv$ triangulations $\equiv$ regular triangulations.

$\rightarrow$ two realizations of the associahedron.

These two associahedra are affinely equivalent.
Expansive Motions in One Dimension

\[ \{(v_i) \in \mathbb{R}^n \mid v_j - v_i \geq f_{ij} \text{ for } 1 \leq i < j \leq n\} \]

\[ f_{il} + f_{jk} > f_{ik} + f_{jl}, \text{ for all } i < j < k < l. \]

\[ f_{il} > f_{ik} + f_{kl}, \text{ for all } i < k < l. \]

For example, \( f_{ij} := (i - j)^2 \)

related to the Monge Property.
Non-crossing alternating trees

non-crossing: no two edges $ik, jl$ with $i < j < k < l$.

alternating: no two edges $ij, jk$ with $i < j < k$.

[Gelfand, Graev, and Postnikov 1997], in a dual setting.
[Postnikov 1997], [Zelevinsky ?]
The Associahedron
Open Questions

1. the meaning of $\sum \omega_{ij} f_{ij} = 1$

2. Is there essentially only one solution of $\sum \omega_{ij} f_{ij} > 0$?

3. canonical pseudotriangulations

4. pseudotriangulations in 3-space
The meaning of

\[ \sum_{1 \leq i < j \leq 4} \omega_{ij} f_{ij} = 1 \]

“I believe there is some underlying homology in this situation. Given the fact that motions and stresses also fit into a setting of cohomology and homology as well, the authors might, at least, mention possible homology descriptions.”

[a referee, about the definition of \( \omega_{ij} \) ]
The meaning of

$$\sum_{1 \leq i < j \leq 4} \omega_{ij} f_{ij} = 1$$

$$\omega_{ij} = \frac{1}{[p_i, p_j, p_k] \cdot [p_i, p_j, p_l]}$$

One can define a similar formula for $\omega$ for the $k$-wheel.
\[ \sum_{ij \in E} \omega_{ij} f_{ij} = 1 \] for the $k$-wheel

\[
\begin{align*}
\omega_{i,i+1} &= \frac{1}{[p_i, p_{i+1}, p_0] \cdot [p_1, p_2, \ldots, p_k]} \\
\omega_{0i} &= \frac{1}{[p_{i-1}, p_i, p_0] \cdot [p_i, p_{i+1}, p_0]} \cdot \frac{[p_{i-1}, p_i, p_{i+1}]}{[p_1, p_2, \ldots, p_k]} 
\end{align*}
\]
Open Questions

1. the meaning of $\sum \omega_{ij} f_{ij} = 1$

2. Is there essentially only one solution of $\sum \omega_{ij} f_{ij} > 0$?

3. canonical pseudotriangulations

4. pseudotriangulations in 3-space
Canonical pseudotriangulations

Maximize/minimize $\sum_{i=1}^{n} c_i \cdot v_i$ over the PPT-polytope.

$c_i := p_i$:

Delaunay triangulation  Max/Min $\sum p_i \cdot v_i$
(affine invariant)
Edge flipping criterion for canonical pseudotriangulations
Pseudotriangulations in 3-space?

Rigid graphs are not well-understood in 3-space.