Pseudotriangulations: A Survey and Recent Results

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Journées de Géometrie Algorithmique, September 2003, Giens

Part I:

0. Introduction, definitions, basic properties
1. Planar Laman graphs
2. The PPT-polytope

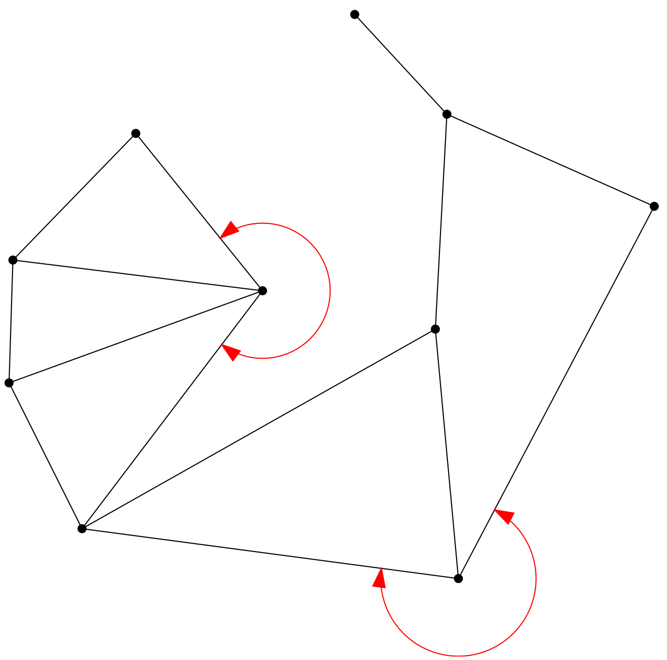
Part II:

3. Stresses and reciprocals
4. Liftings and surfaces

Part III: 5. kinetic data structures, counting and enumeration problems, visibility graphs, flips, combinatorial questions

0. BASIC PROPERTIES. Pointed Vertices

A *pointed* vertex is incident to an angle $> 180°$.



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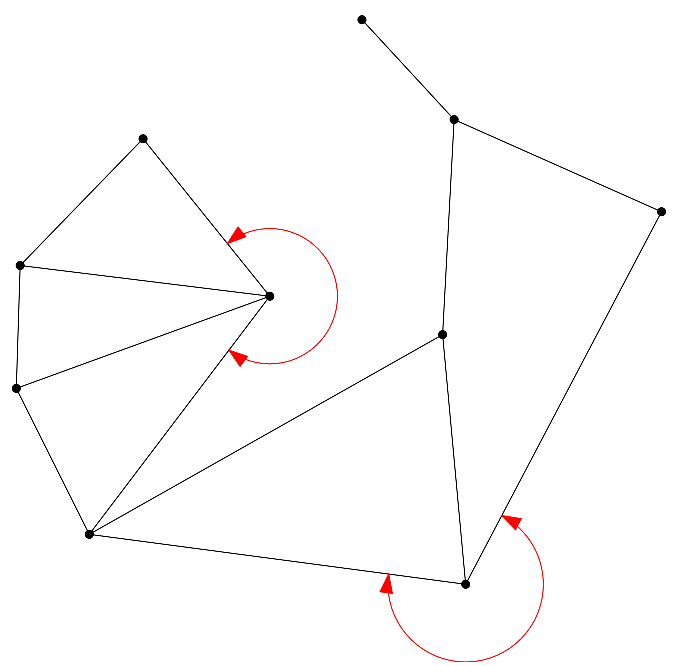

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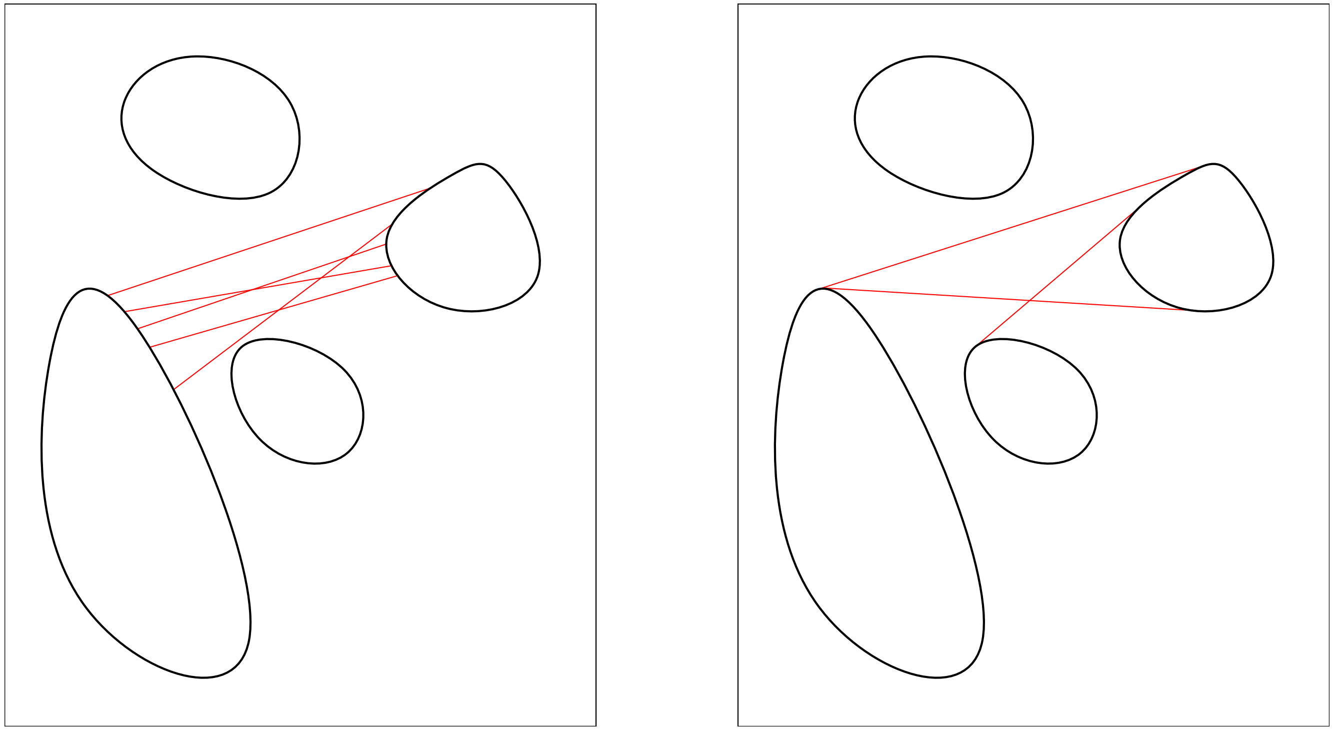
Where do pointed vertices arise?

Visibility among convex obstacles

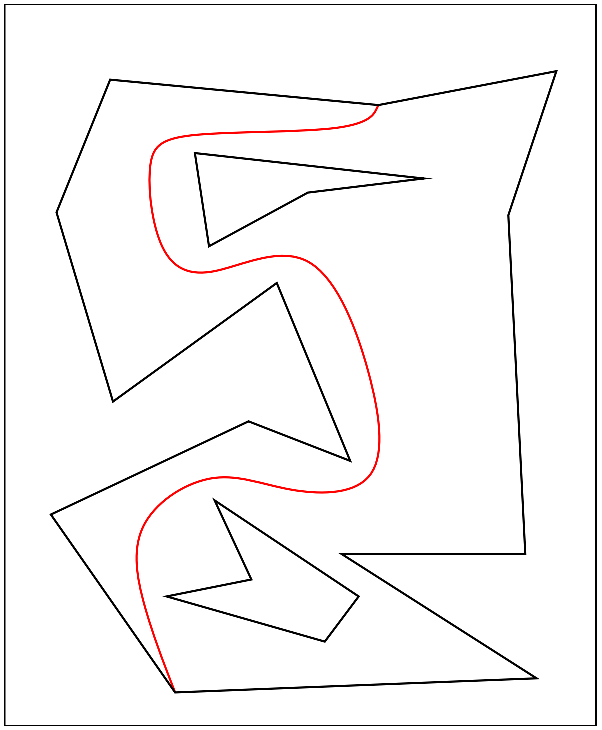
Equivalence classes of *visibility segments*. Extreme segments are *bitangents* of convex obstacles.

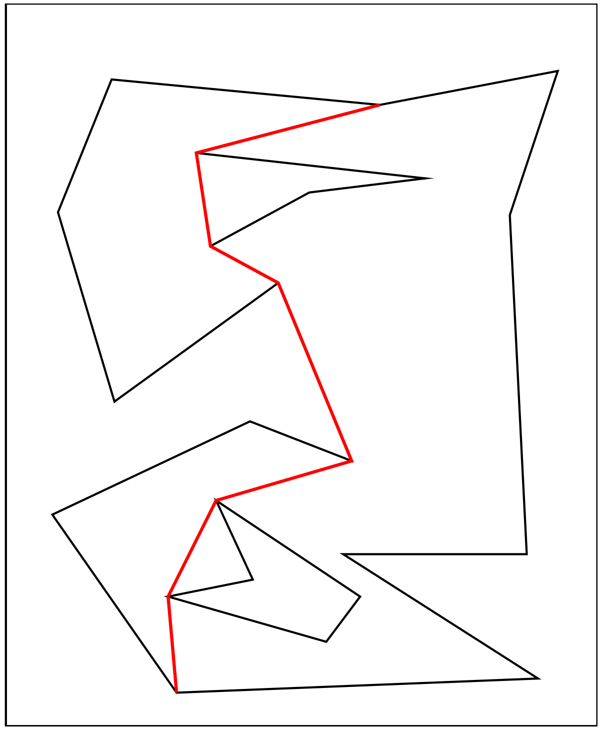


[Pocchiola and Vegter 1996]

Geodesic shortest paths

Shortest path (with given homotopy) turns only at pointed vertices. Addition of shortest path edges leaves intermediate vertices pointed.





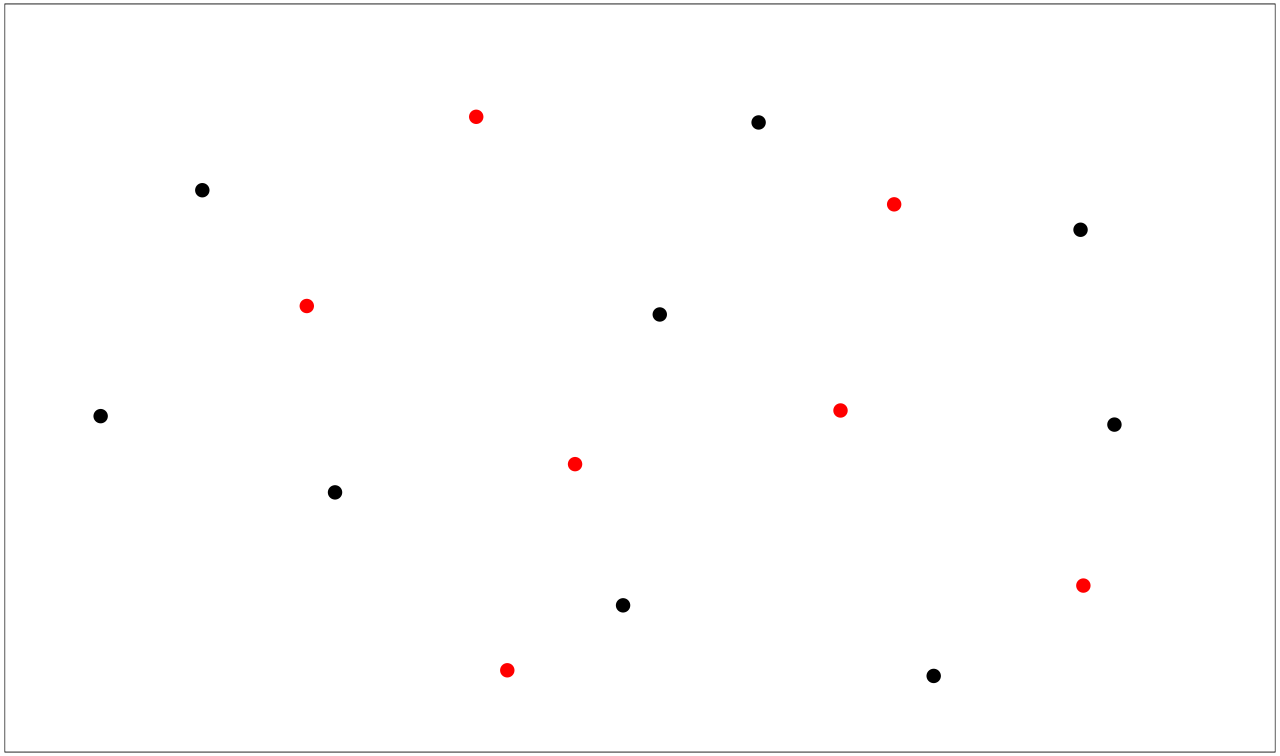
$\longrightarrow$ *geodesic* triangulations of a simple polygon

[Chazelle,Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]

Pseudotriangulations

Given: A set V of vertices, a subset $V_p \subseteq V$ of *pointed vertices*.

A *pseudotriangulation* is a maximal (with respect to $\subseteq$) set of non-crossing edges with all vertices in V_p pointed.



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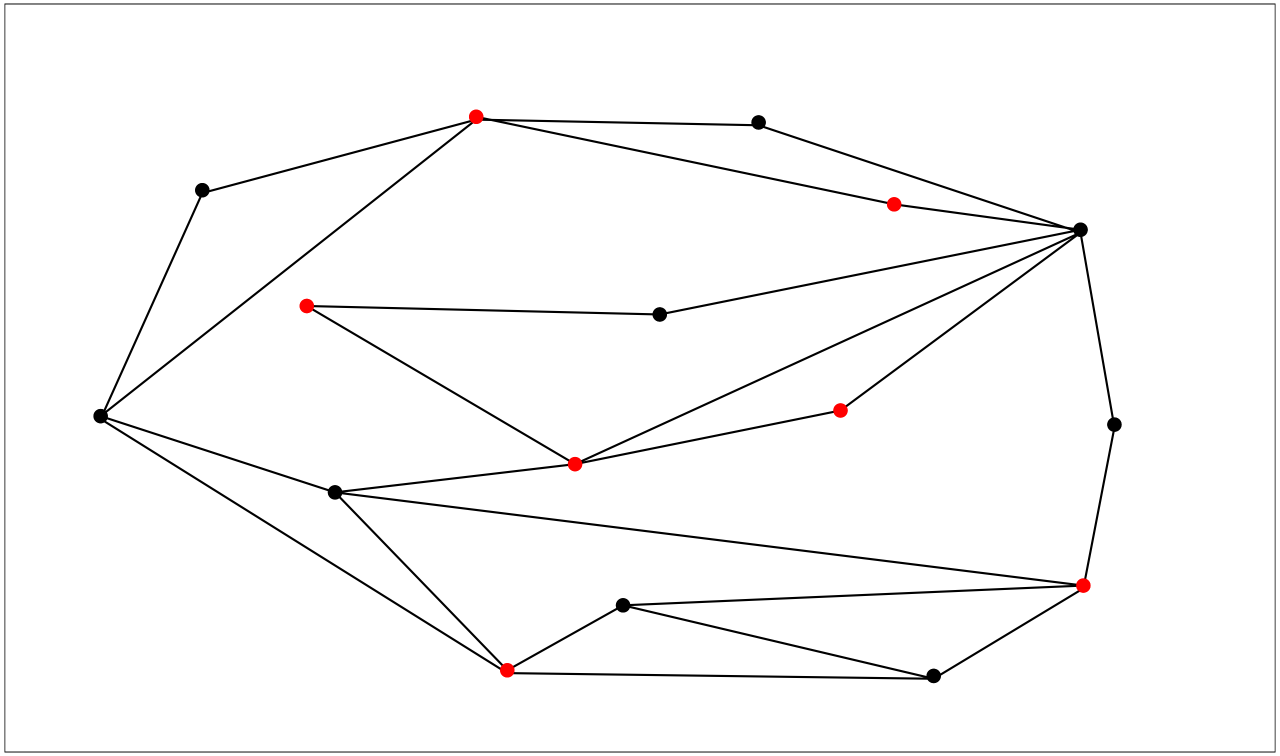

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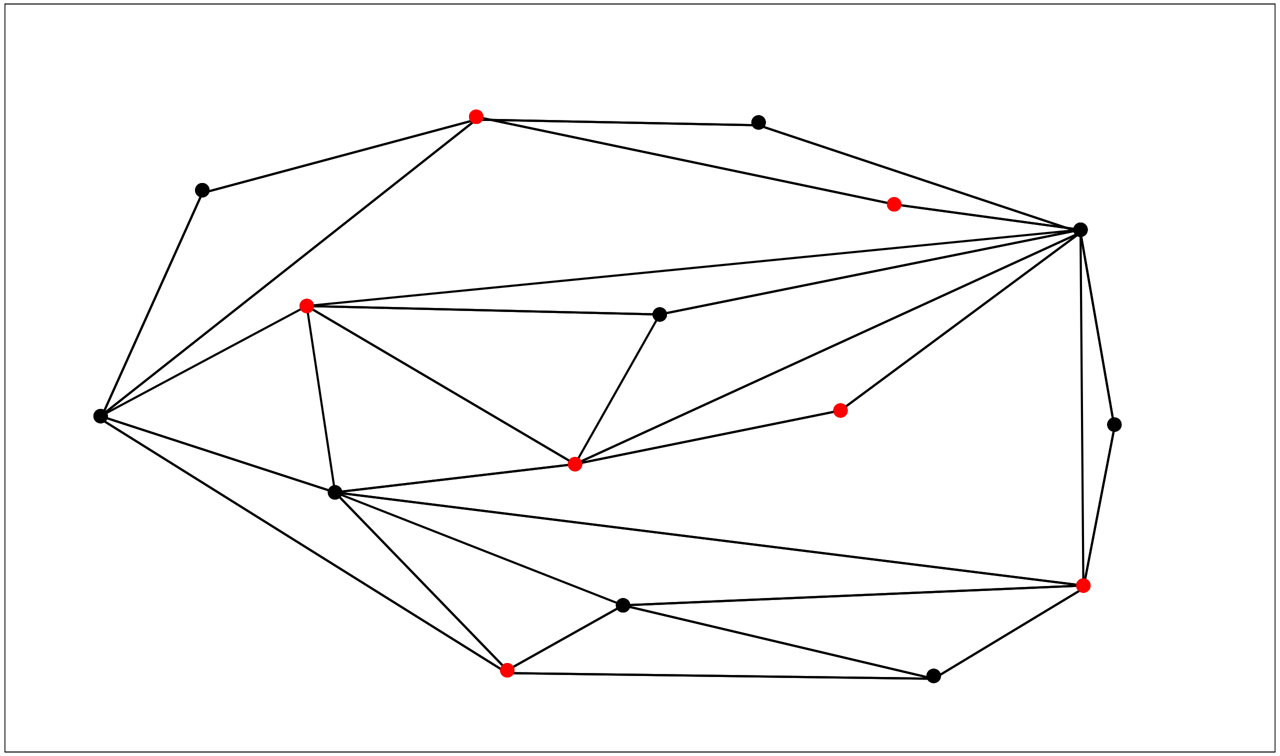

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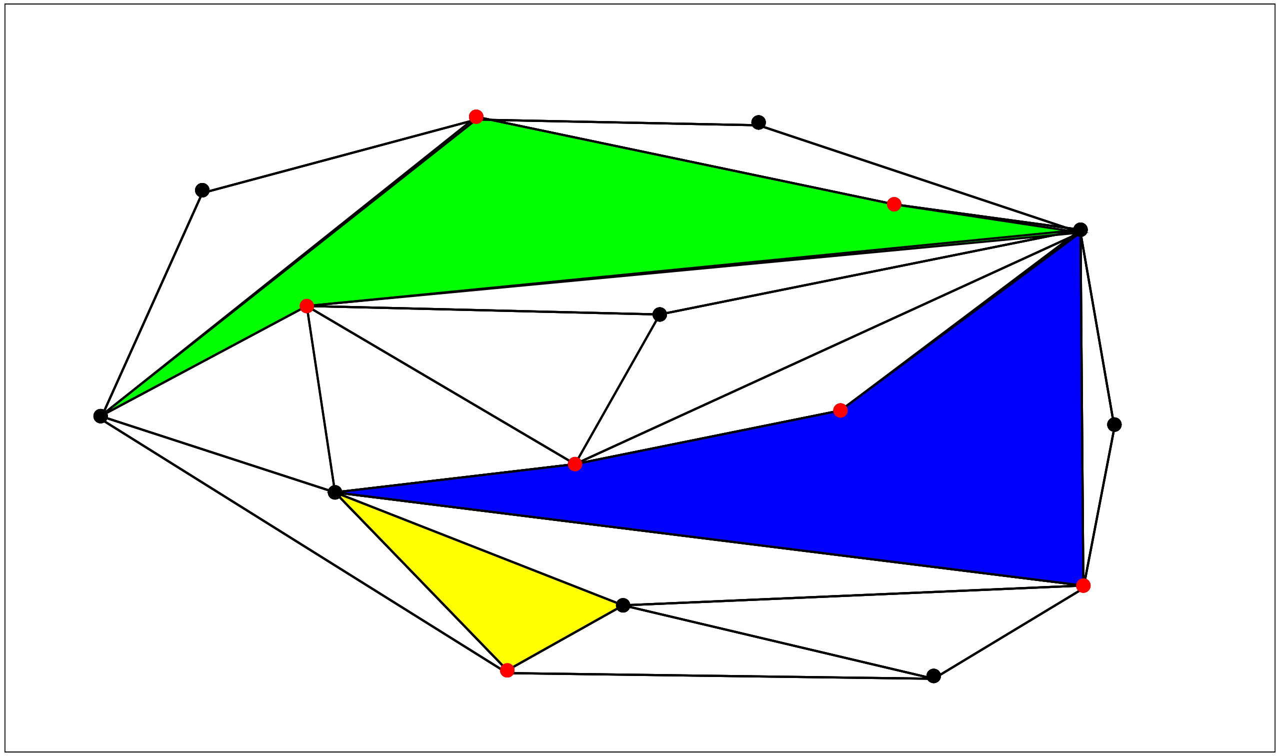
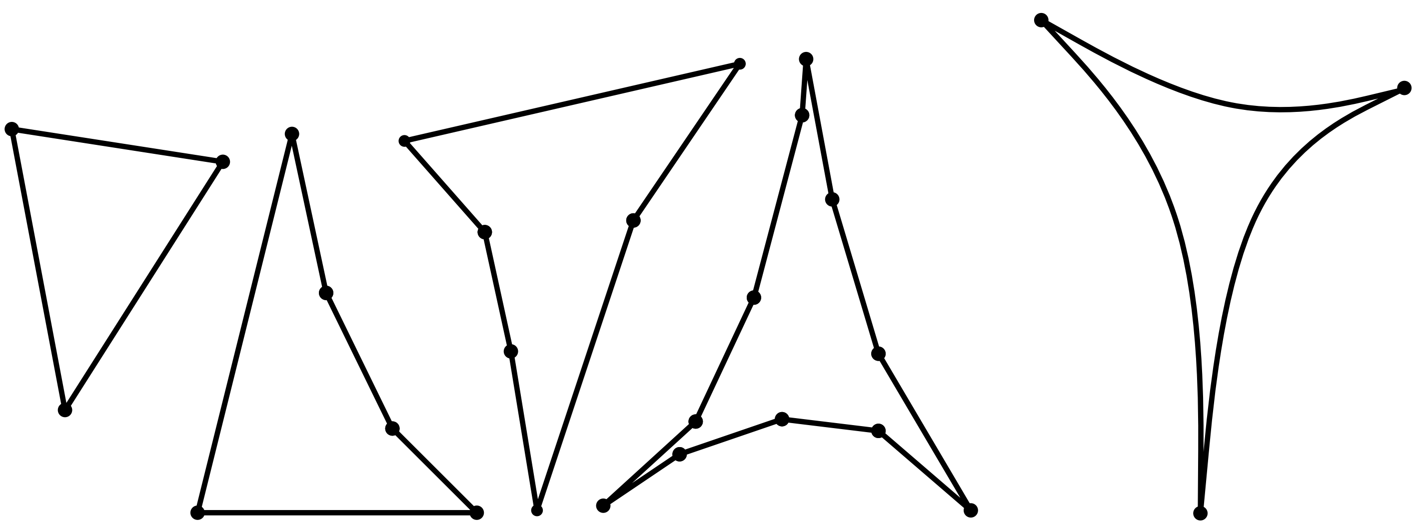


Pseudotriangles

A pseudotriangle has three convex *corners* and an arbitrary number of reflex vertices ($> 180^\circ$).



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Proof. $(1) \implies (2)$ All convex hull edges are in E.
$\rightarrow$ decomposition of the polygon into faces.
Need to show: If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.

Characterization of pseudotriangulations

Lemma. *If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.*

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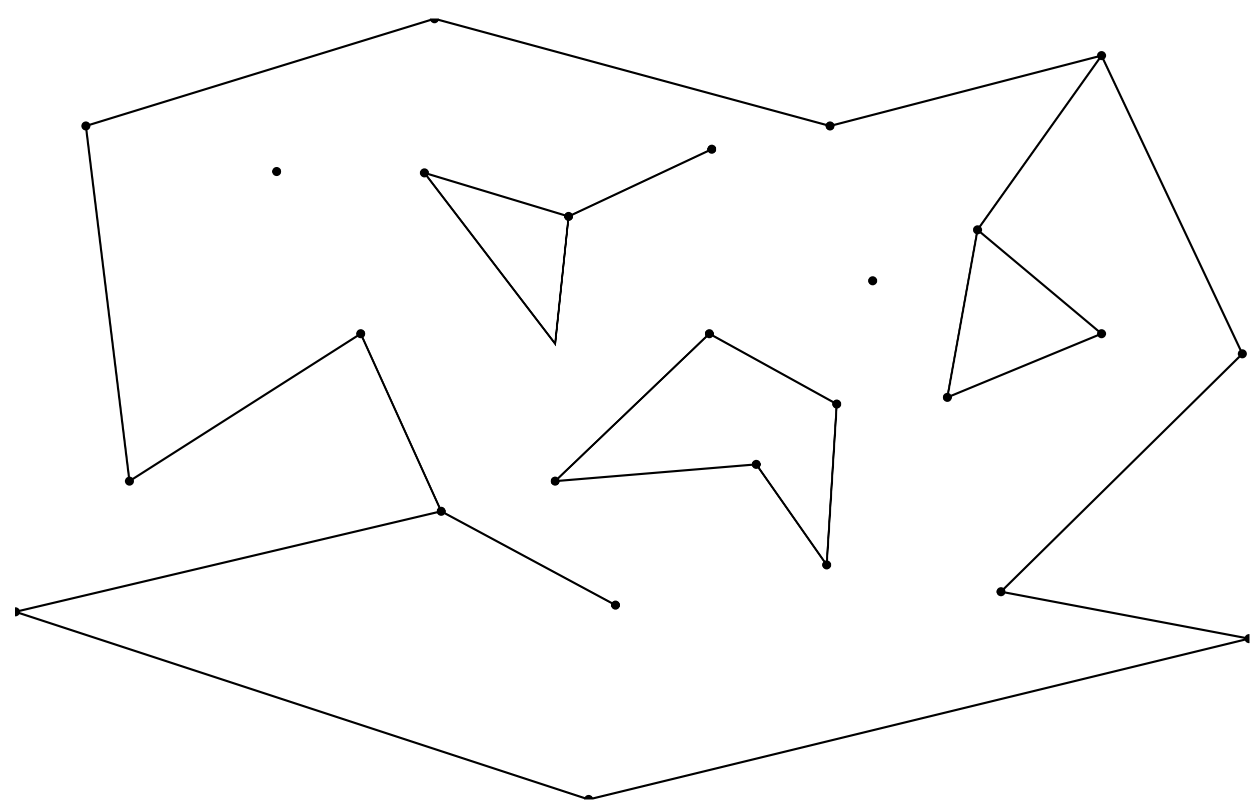
Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.



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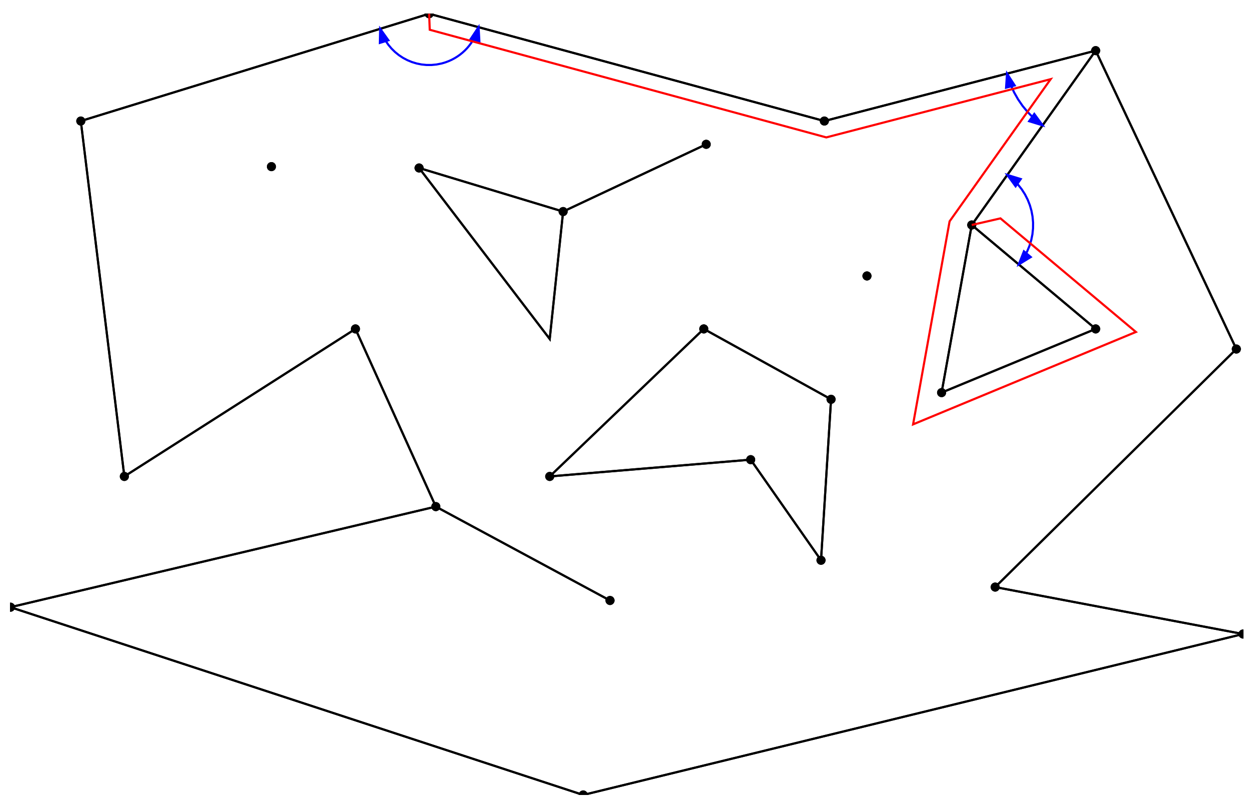

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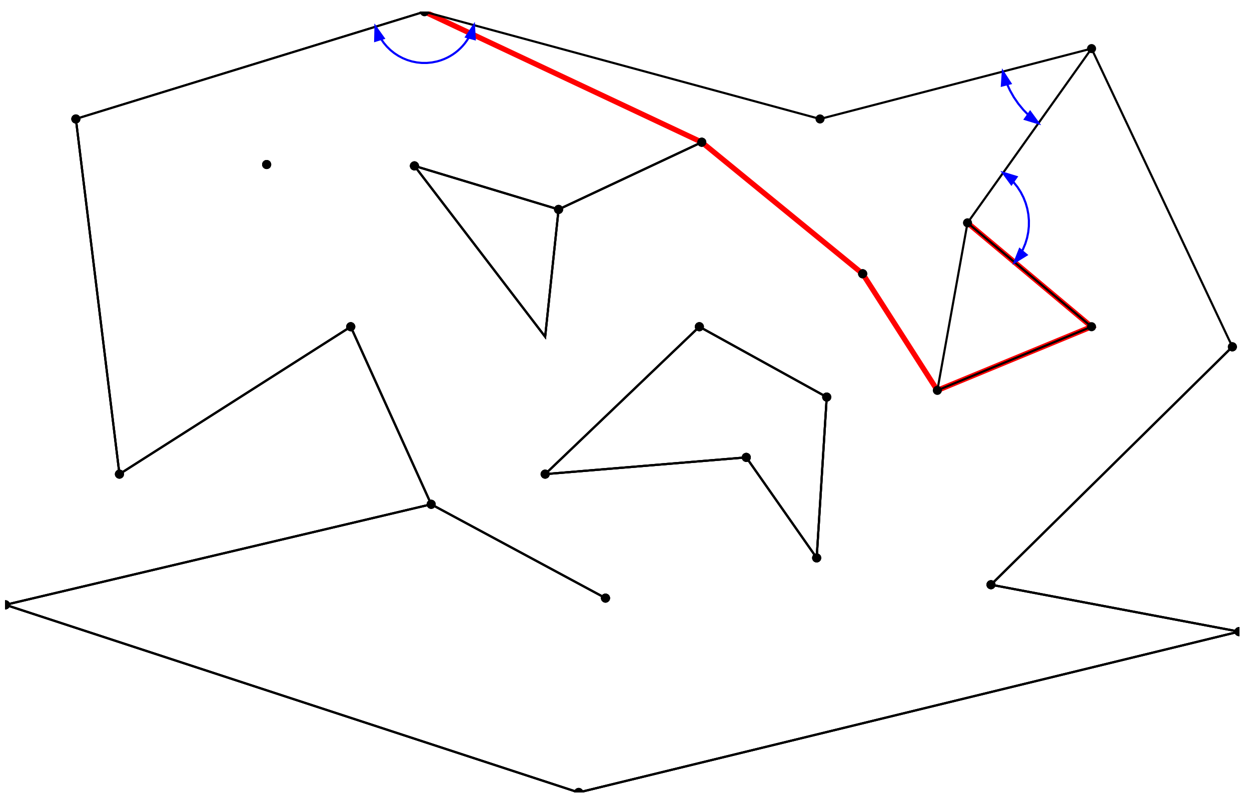
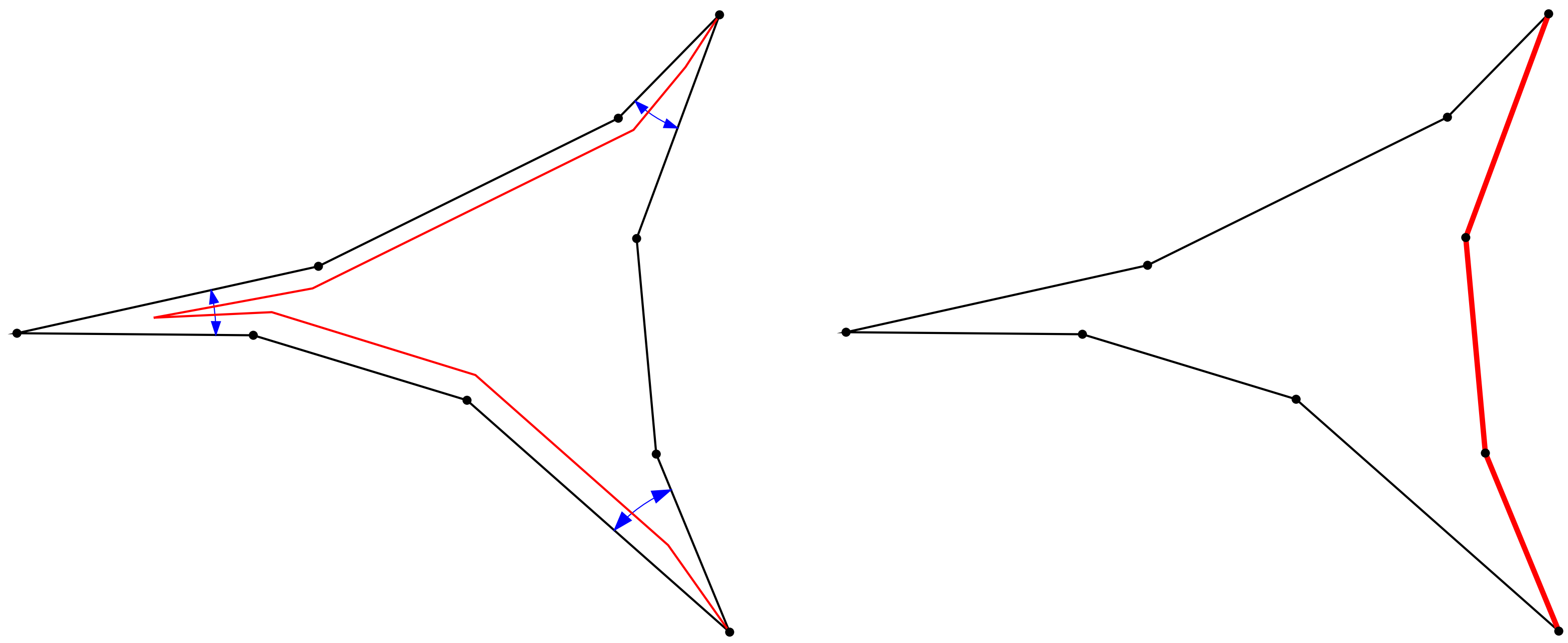


Characterization of pseudotriangulations, continued

A new edge is always added, unless the face is already a pseudotriangle (without inner obstacles).



[Rote, C. A. Wang, L. Wang, Xu 2003]

Vertex and face counts

A pseudotriangulation with x nonpointed and y pointed vertices has $e = 3x + 2y - 3$ edges and $2x + y - 2$ pseudotriangles.

A pointed pseudotriangulation with n vertices has $e = 2n - 3$ edges and $n - 2$ pseudotriangles.

Proof. A k-gon pseudotriangle has $k - 3$ large angles.

$$\sum_{t \in T} (k_t - 3) + k_{\text{outer}} = y$$

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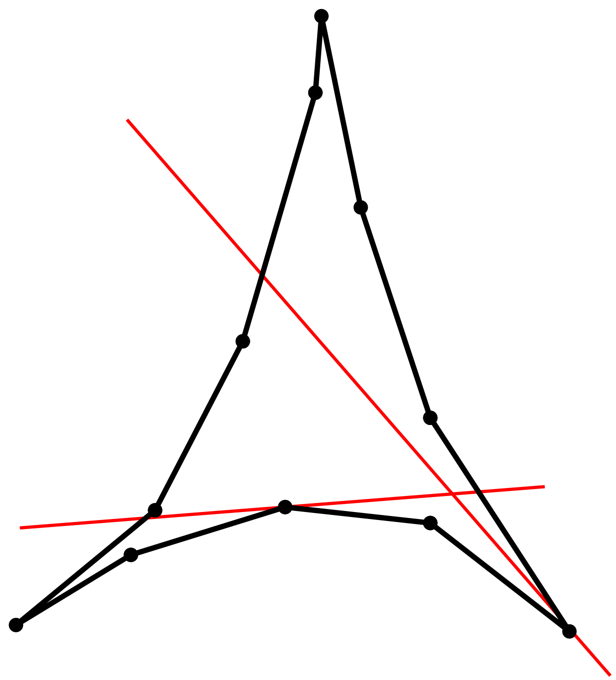
$$\underbrace{\sum_t k_t + k_{\mathrm{outer}}}_{2e} - 3|T| = y$$

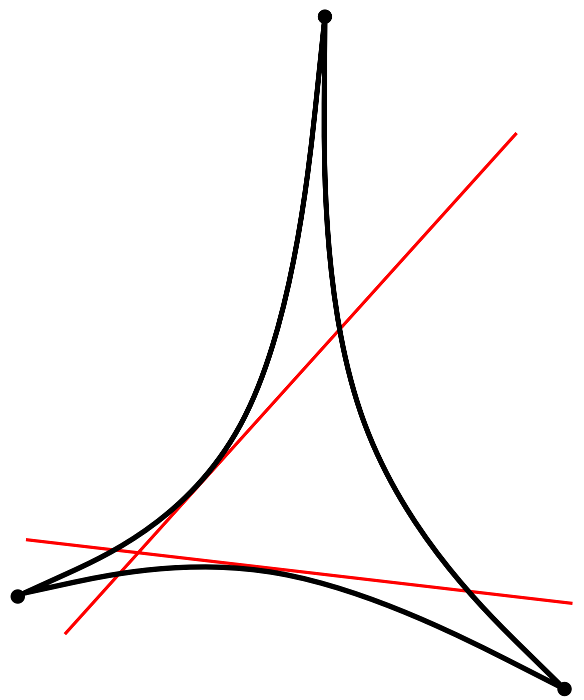
$$e + 2 = (|T| + 1) + (x + y) \qquad \text{(Euler)}$$

Tangents of pseudotriangles

"Proof. $(2) \implies (1)$ No edge can be added inside a pseudo-triangle without creating a nonpointed vertex."

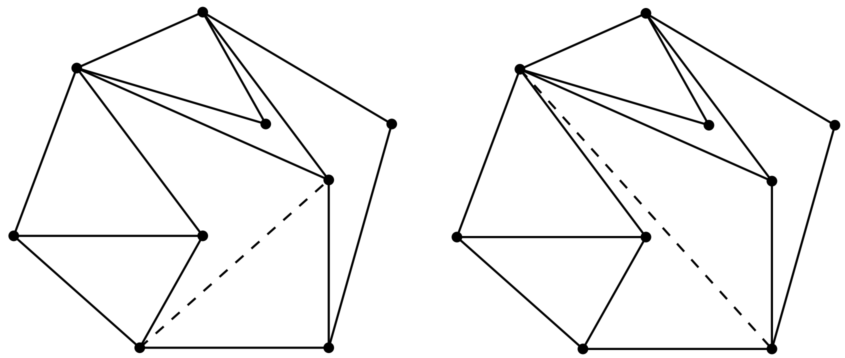
For every direction, there is a unique line which is "tangent" at a reflex vertex or "cuts through" a corner.





Flipping of Edges

Any interior edge can be flipped against another edge. That edge is unique.



before after

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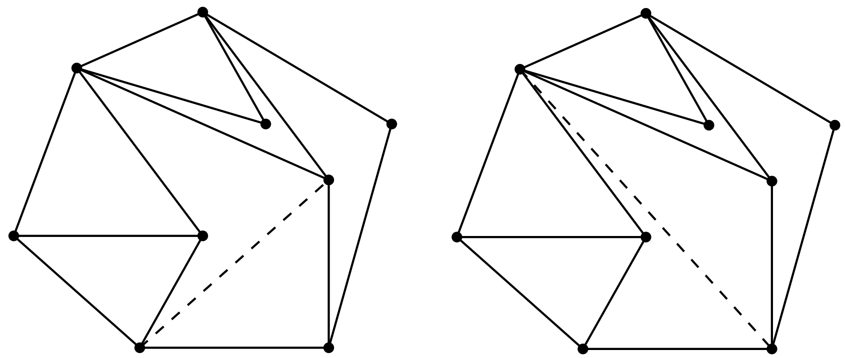

before after

The flip graph is connected.
Its diameter is $O(n \log n)$. [Bespamyatnikh 2003]

1. RIGIDITY, PLANAR LAMAN GRAPHS
Infinitesimal motions — rigid frameworks

n vertices $p_1, \ldots, p_n$.

1. (global) *motion* $p_i = p_i(t)$, $t \geq 0$

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Infinitesimal motions — rigid frameworks

n vertices $p_1, \ldots, p_n$.

1. (global) *motion* $p_i = p_i(t)$, $t \geq 0$

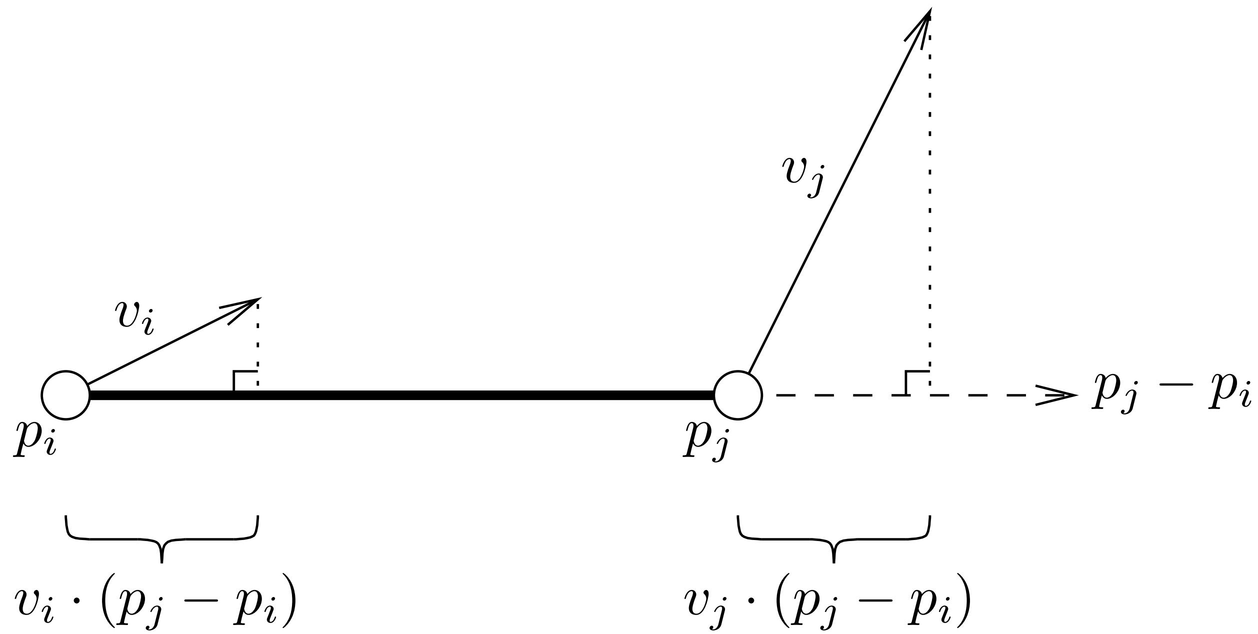
2. *infinitesimal motion* (local motion)

$$v_i = \frac{d}{dt} p_i(t) = \dot{p}_i(0)$$

Velocity vectors $v_1, \ldots, v_n$.

Expansion

$$\frac{1}{2} \cdot \frac{d}{dt}|p_i(t) - p_j(t)|^2 = \langle v_i - v_j, p_i - p_j \rangle =: \exp_{ij}$$



expansion (or *strain*) $\exp_{ij}$ of the segment ij

The rigidity map

of a framework $((V, E), (p_1, \ldots, p_n))$:

$$M\colon (v_1, \ldots, v_n) \mapsto (\exp_{ij})_{ij \in E}$$

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The rigidity matrix:

$$M = \underbrace{\begin{pmatrix} \text{the} \\ \text{rigidity} \\ \text{matrix} \end{pmatrix}}_{2|V|} \left.\vphantom{\begin{pmatrix} a \\ a \\ a \end{pmatrix}}\right\} E$$

Infinitesimally rigid frameworks

A framework is *infinitesimally rigid* if

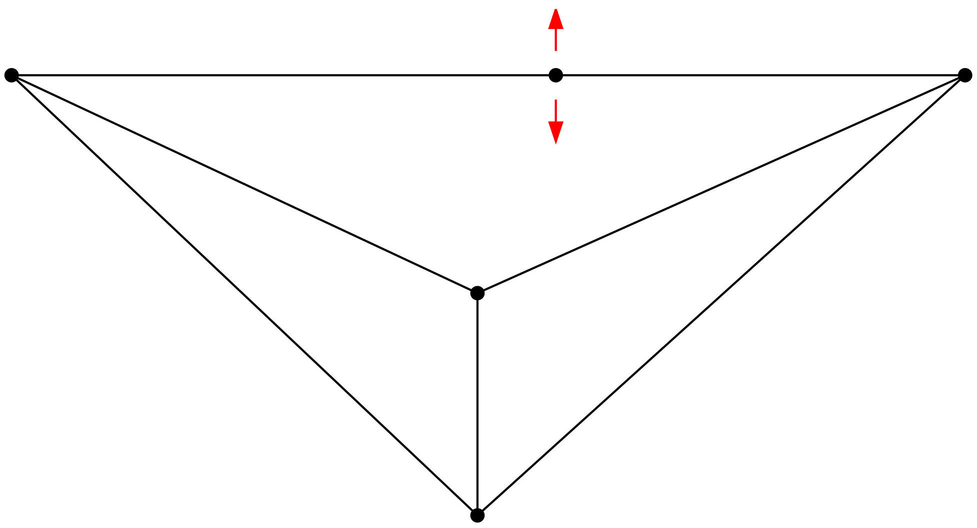
$$M(v) = 0$$

has only the trivial solutions: translations and rotations of the framework as a whole.

Rigid frameworks

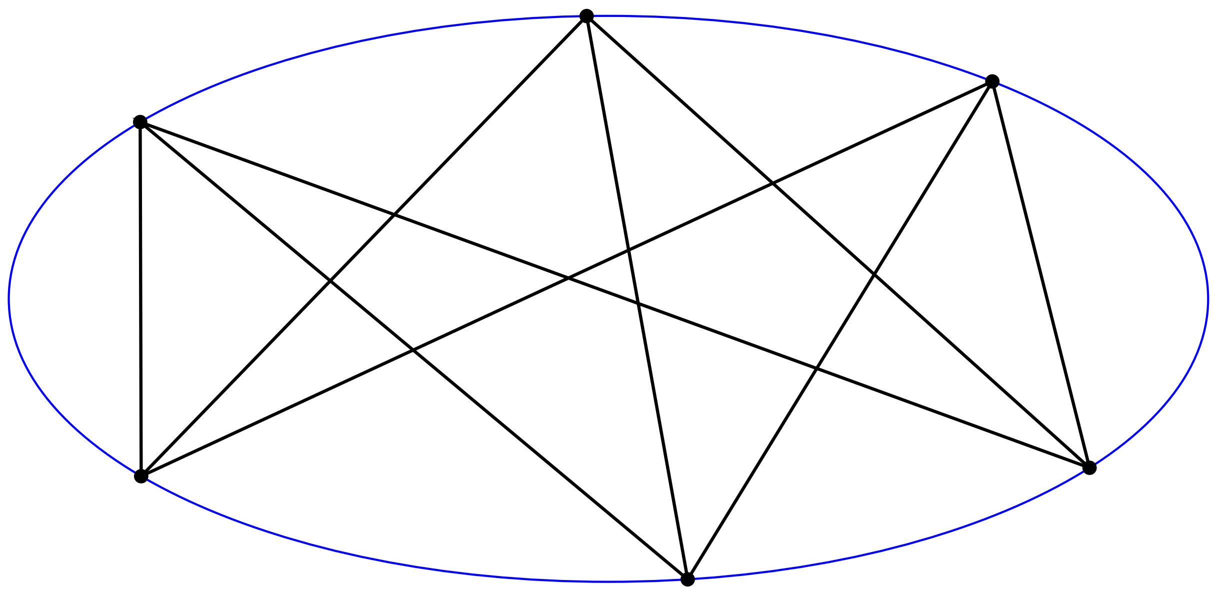
An infinitesimally rigid framework is rigid.

This framework is rigid, but not infinitesimally rigid:



Generically rigid frameworks

A given graph can be rigid in most embeddings, but it may have special non-rigid embeddings:



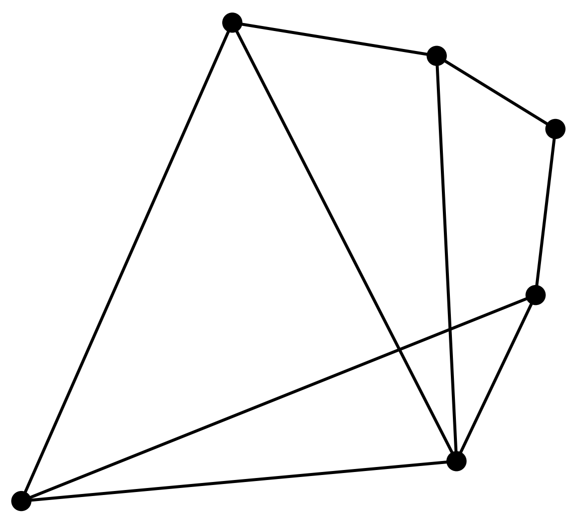
A graph is *generically rigid* if it is infinitesimally rigid in almost all embeddings.

This is a *combinatorial property* of the graph.

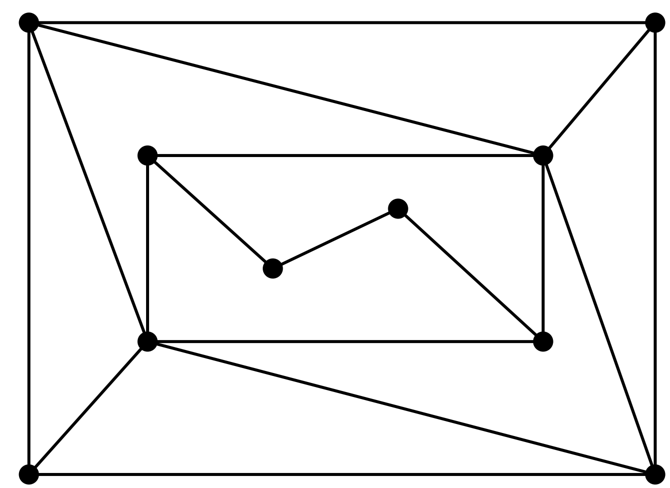
Minimally rigid frameworks

A graph with n vertices is *minimally rigid* in the plane (with respect to $\subseteq$) iff it has the *Laman property*:

- It has $2n-3$ edges.
- Every subset of $k \geq 2$ vertices spans at most $2k-3$ edges.



$n=6,\ e=9$



$n=10,\ e=17$

[Laman 1961]

A pointed pseudotriangulation is a Laman graph

Proof: Every subset of $k \geq 2$ vertices is pointed and has therefore at most $2k - 3$ edges.

[Streinu 2001]

Every planar Laman graph is a pointed pseudotriangulation

Theorem. *Every planar Laman graph has a realization as a pointed pseudotriangulation. The outer face can be chosen arbitrarily.*

Proof I: Induction, using *Henneberg constructions*

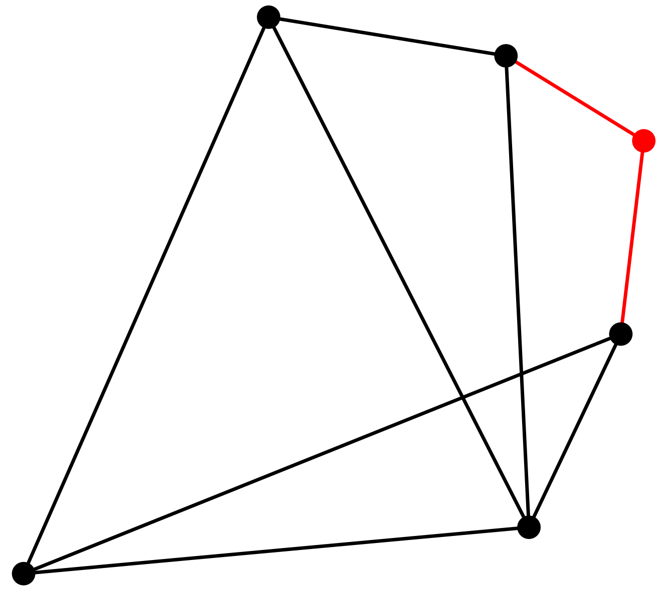
Proof II: via Tutte embeddings for directed graphs

[Haas, Rote, Santos, B. Servatius, H. Servatius, Streinu, Whiteley 2003]

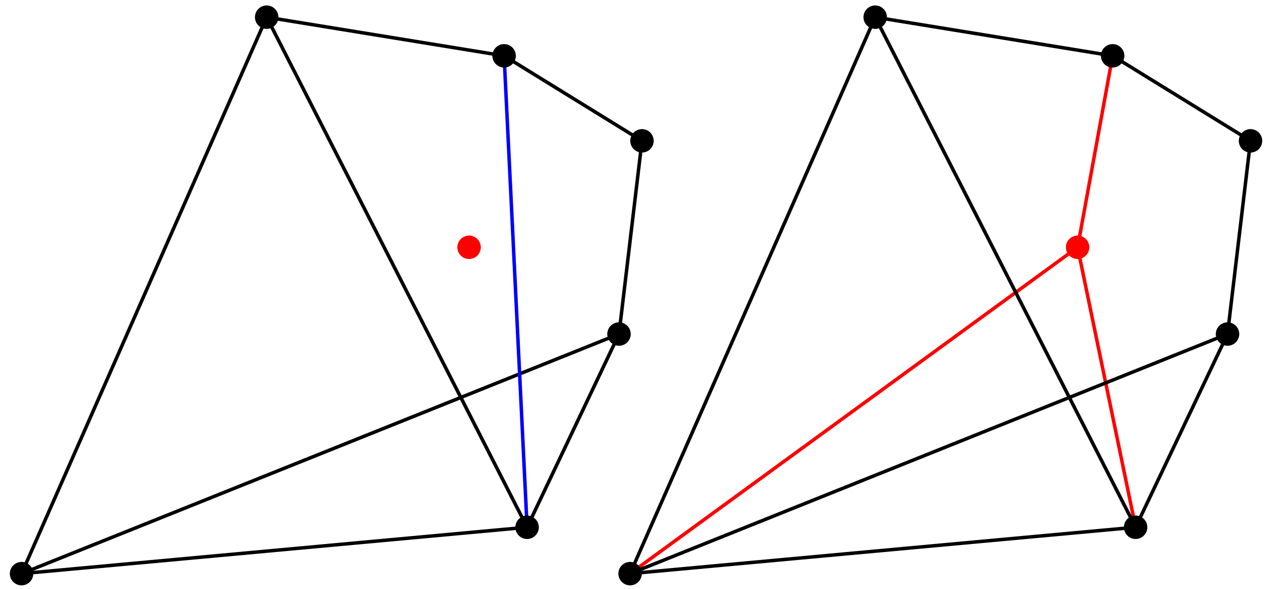
Theorem. *Every rigid planar graph has a realization as a pseudotriangulation.*

[Orden, Santos, B. Servatius, H. Servatius 2003]

Henneberg constructions

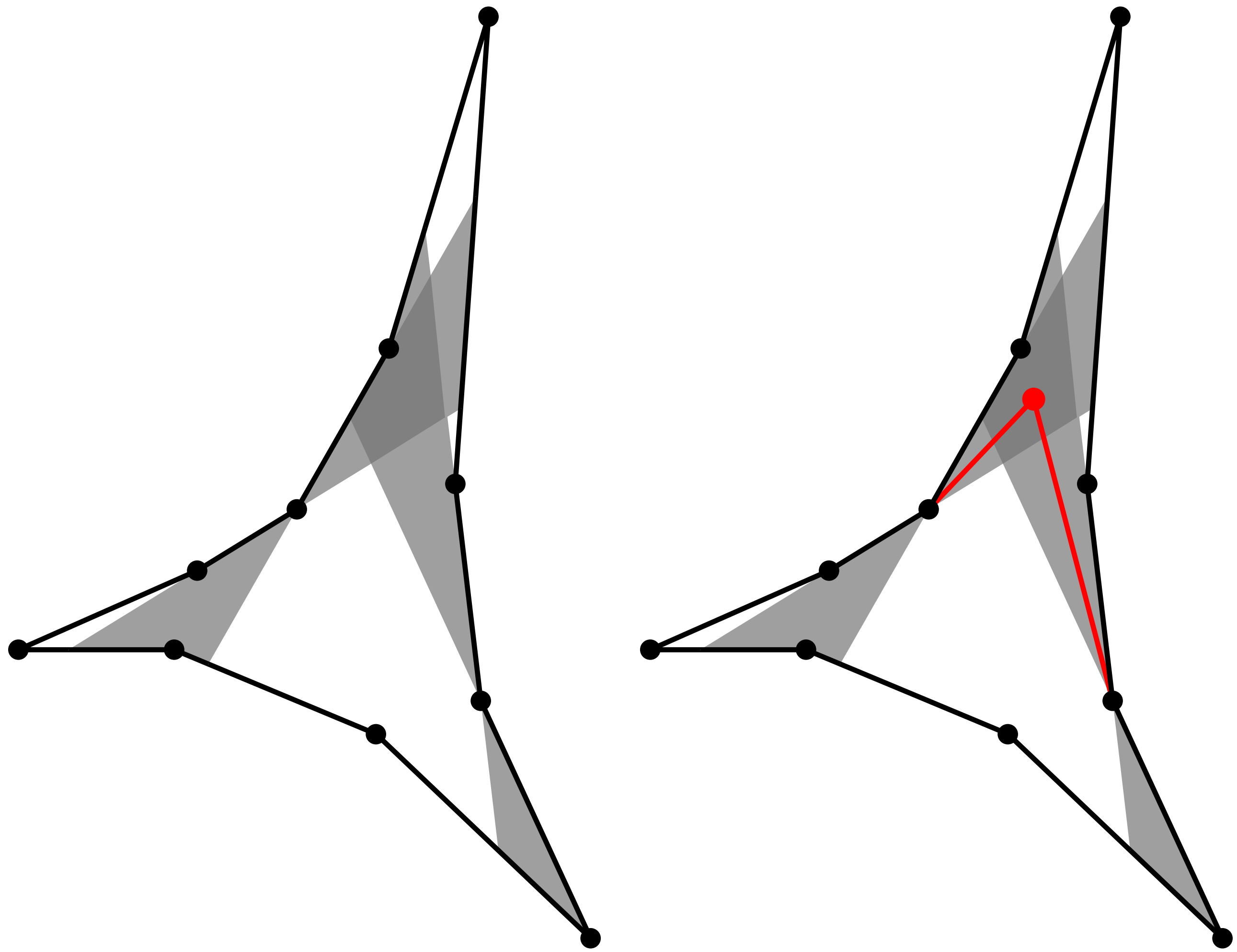


Type I



Type II

Proof I: Henneberg constructions



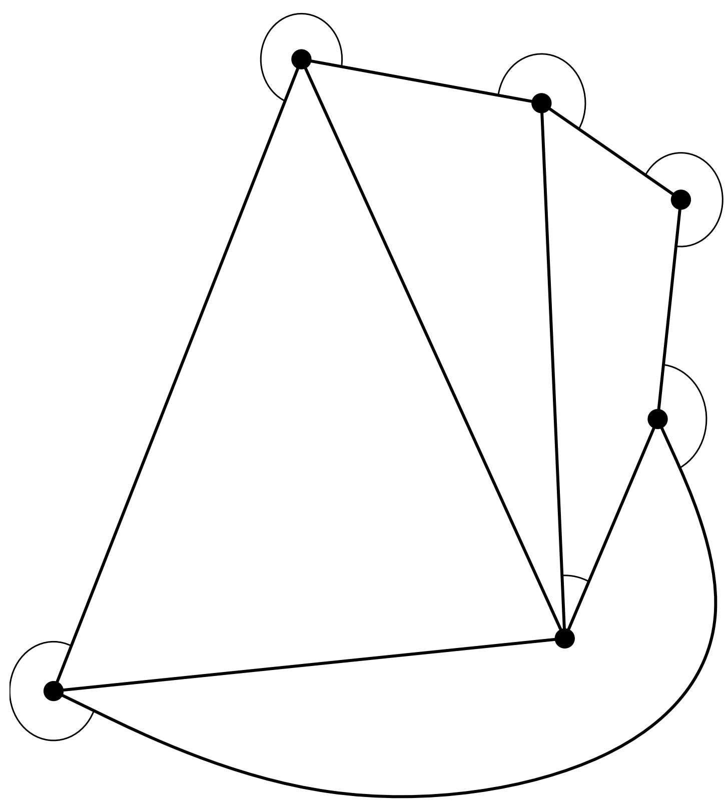
Proof II: embedding Laman graphs via directed Tutte embeddings

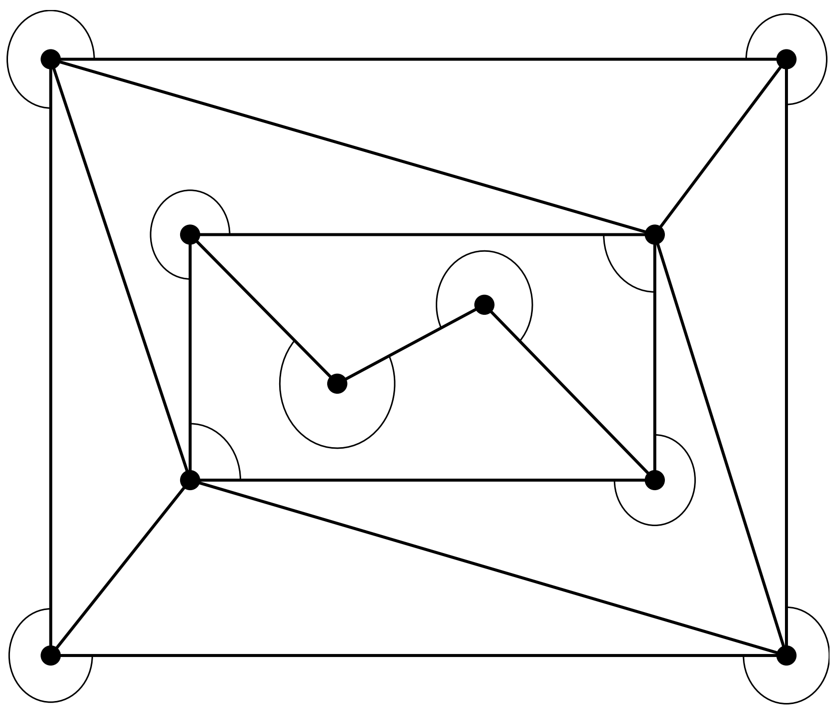
Step 1: Find a *combinatorial pseudotriangulation* (CPT): Mark every angle of the embedding either as *small* or *big*.

- Every interior face has 3 small angles.
- The outer face has no small angles.
- Every vertex is incident to one big angle.

Step 2: Find a geometric realization of the CPT.

Combinatorial pseudotriangulations





Step 2—Tutte's barycenter method

Fix the vertices of the outer face in convex position. Every interior vertex p_i should lie at the barycenter of its neighbors.

$$\sum_{(i,j)\in E} \omega_{ij}(p_j - p_i) = 0, \quad \text{for every vertex } i$$

$\omega_{ij} \geq 0$, but ω need not be symmetric.

Theorem. *If every interior vertex has three vertex disjoint paths to the outer boundary, using arcs with $\omega_{ij} > 0$, the solution is a planar embedding.*

[Tutte 1961], [Floater and Gotsman 1999],

[Colin de Verdière, Pocchiola, Vegter 2003]

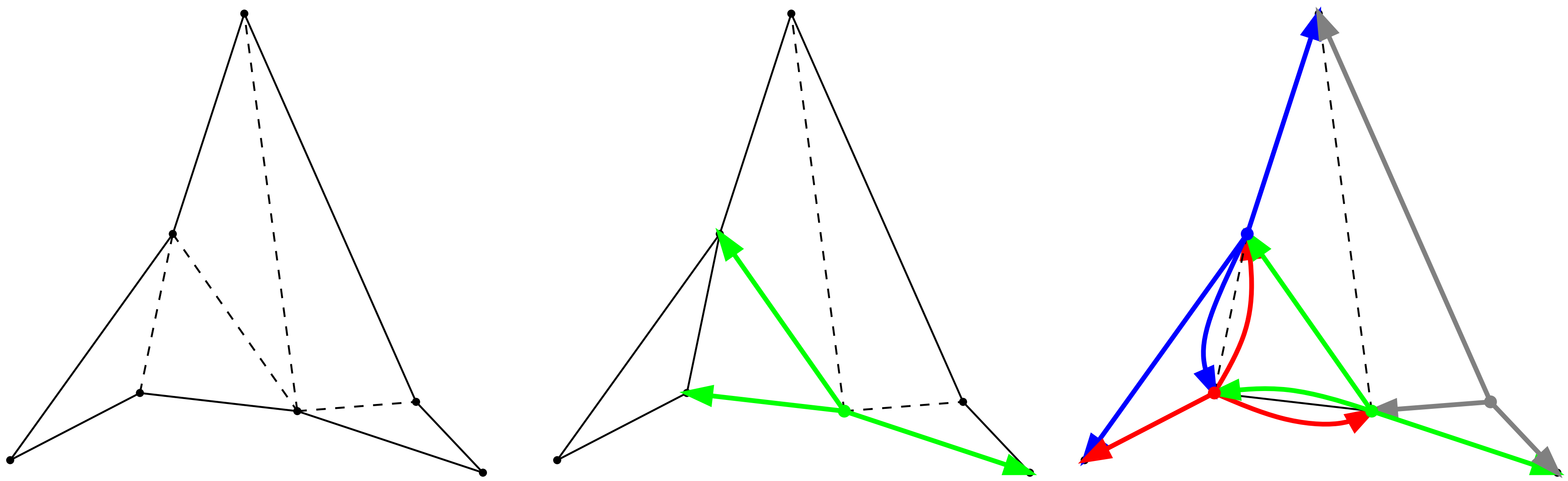
→ animation of spider-web embedding (requires Cinderella 2.0 software)

Selection of outgoing arcs

3 outgoing arcs for every interior vertex:

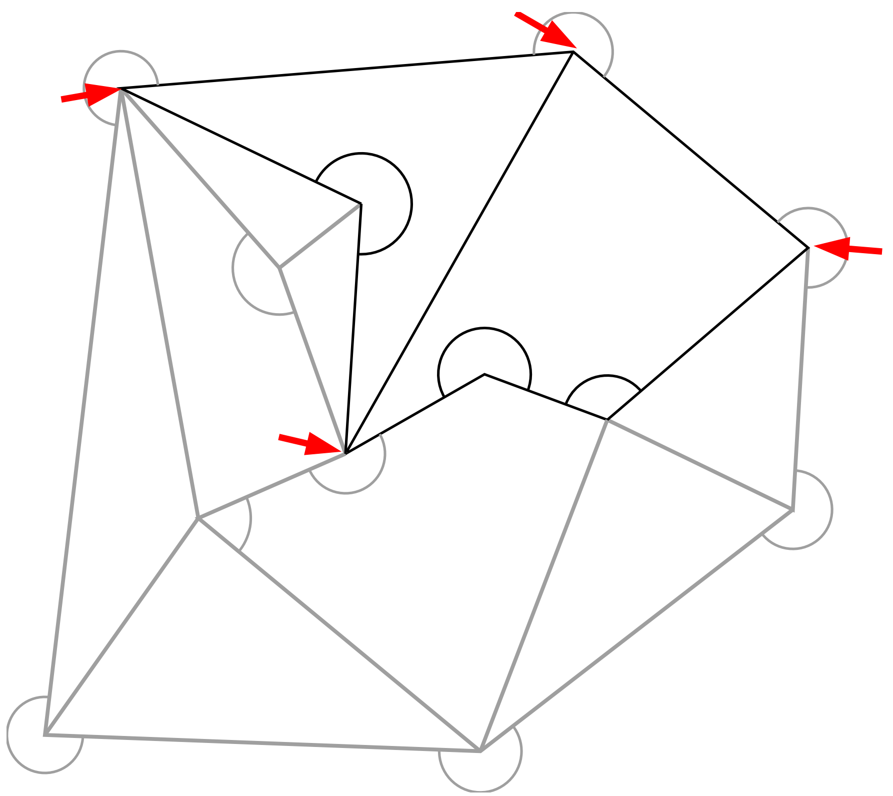
Triangulate each pseudotriangle arbitrarily.
For each reflex vertex, select

- the two incident boundary edges
- an interior edge of the pseudotriangulation



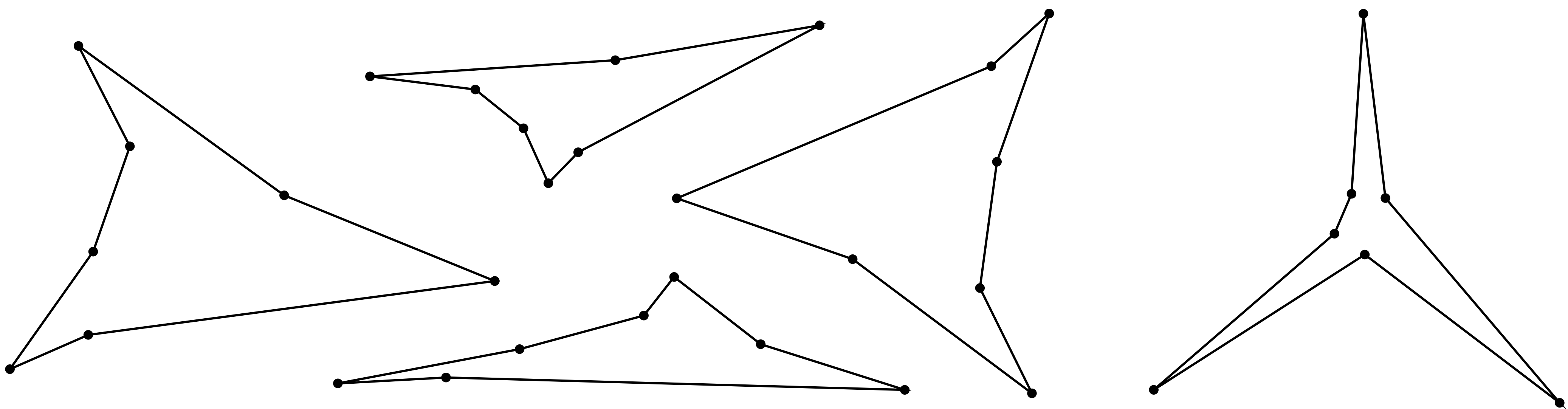
3-connectedness

Lemma. *Every induced subgraph of a planar Laman graph with a CPT has at least 3 outside "corners".*



Specifying the shape of pseudotriangles

The shape of every pseudotriangle (and the outer face) can be arbitrarily specified up to affine transformations.



2. THE PPT-POLYTOPE
Unfolding of polygons — expansive motions

Theorem. *Every polygonal arc in the plane can be brought into straight position, without self-overlap.*

Every polygon in the plane can be unfolded into convex position.

[Connelly, Demaine, Rote 2001], [Streinu 2001]

Unfolding polygons—proof outline

Existence of an expansive motion

$\Updownarrow$ (duality)

Self-stresses (rigidity)
Self-stresses on planar frameworks

$\Updownarrow$ (Maxwell-Cremona correspondence)

polyhedral terrains

[Connelly, Demaine, Rote 2001]

Expansive motions

$\exp_{ij} = 0$ for all *bars* ij

(preservation of length)

$\exp_{ij} \geq 0$ for all other pairs (*struts*) ij

(expansiveness)

The expansion cone

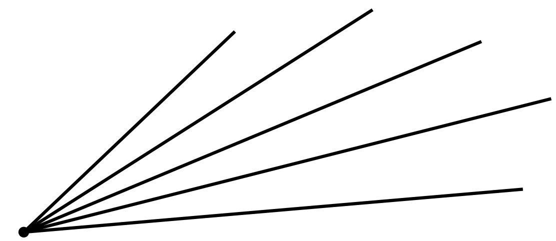
The set of expansive motions forms a convex polyhedral cone $\bar{X}_0$ in $\mathbb{R}^{2n}$, defined by homogeneous linear equations and inequalities of the form

$$\langle v_i - v_j, p_i - p_j \rangle \left\{ \begin{array}{c} = \\ \geq \end{array} \right\} 0$$

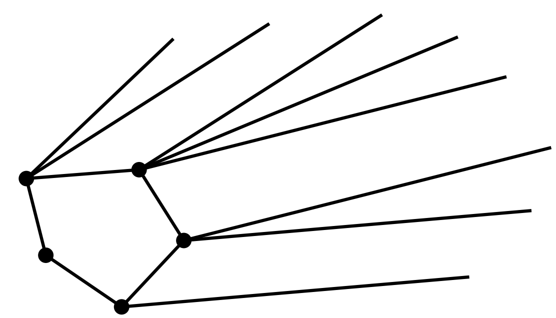
Cones and polytopes

[Rote, Santos, Streinu 2002]

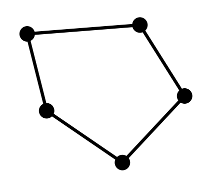
- The *expansion cone*
  $\bar{X}_0 = \{ \exp_{ij} \geq 0 \}$



- The *perturbed expansion cone*
  $=$ the *PPT polyhedron*
  $\bar{X}_f = \{ \exp_{ij} \geq f_{ij} \}$



- The *PPT polytope*
  $X_f = \{ \exp_{ij} \geq f_{ij},$
  $\exp_{ij} = f_{ij}$ for ij on boundary $\}$



Pinning of Vertices

Trivial Motions: Motions of the point set as a whole (translations, rotations).

Pin a vertex and a direction. ("tie-down")

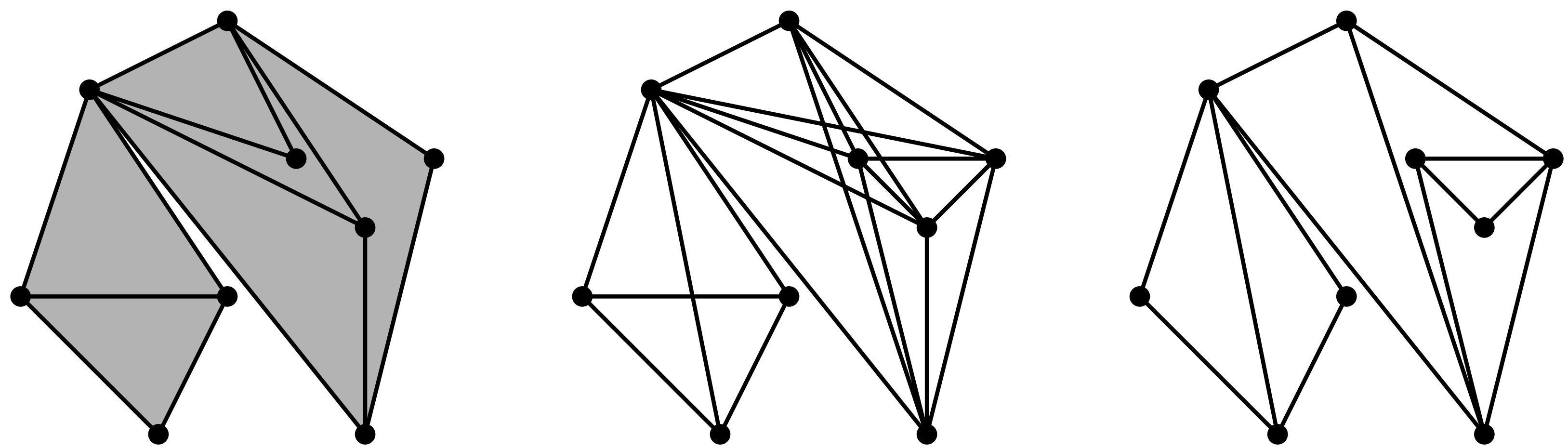
$$v_1 = 0$$

$$v_2 \parallel p_2 - p_1$$

This eliminates 3 degrees of freedom.
$\longrightarrow$ a $2n-3$-dimensional polyhedron.

Extreme rays of the expansion cone

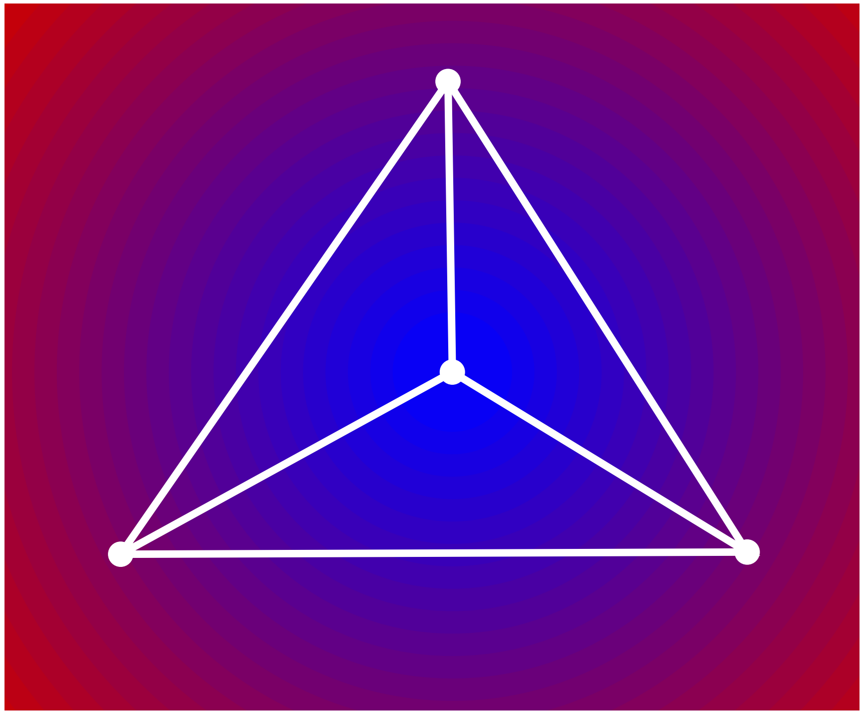
Pseudotriangulations with one convex hull edge removed yield expansive mechanisms. [Streinu 2000]
Rigid substructures can be identified.



A Polyhedron for Pseudotriangulations

Wanted:
A perturbation of the constraints "$\exp_{ij} \geq 0$" such that the vertices are in 1-1 correspondence with pseudotriangulations.

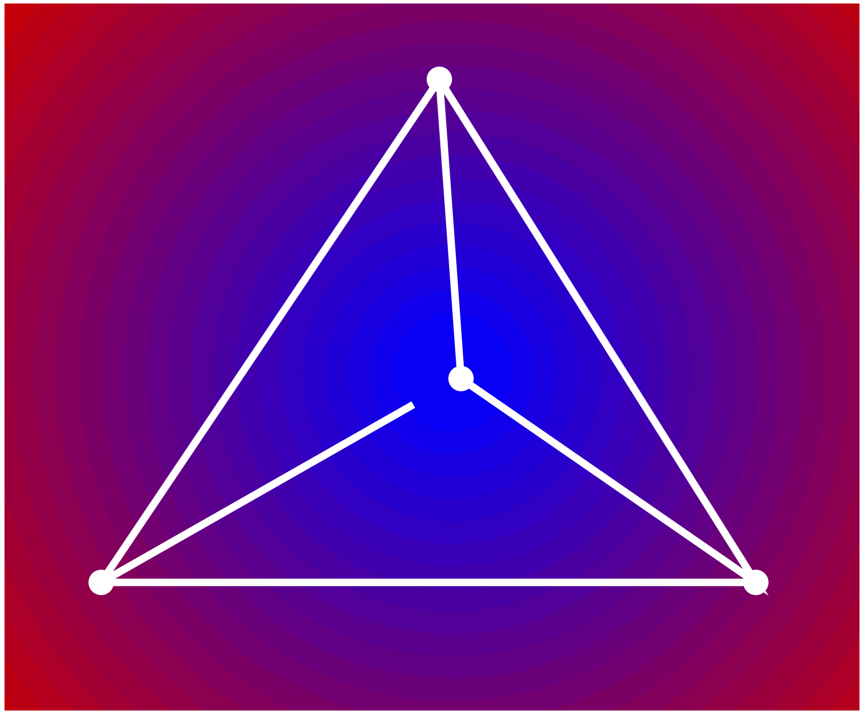
Heating up the bars



$$\Delta T = |x|^2$$

Length increase $\geq \int_{x \in p_i p_j} |x|^2 \, ds$

Heating up the Bars



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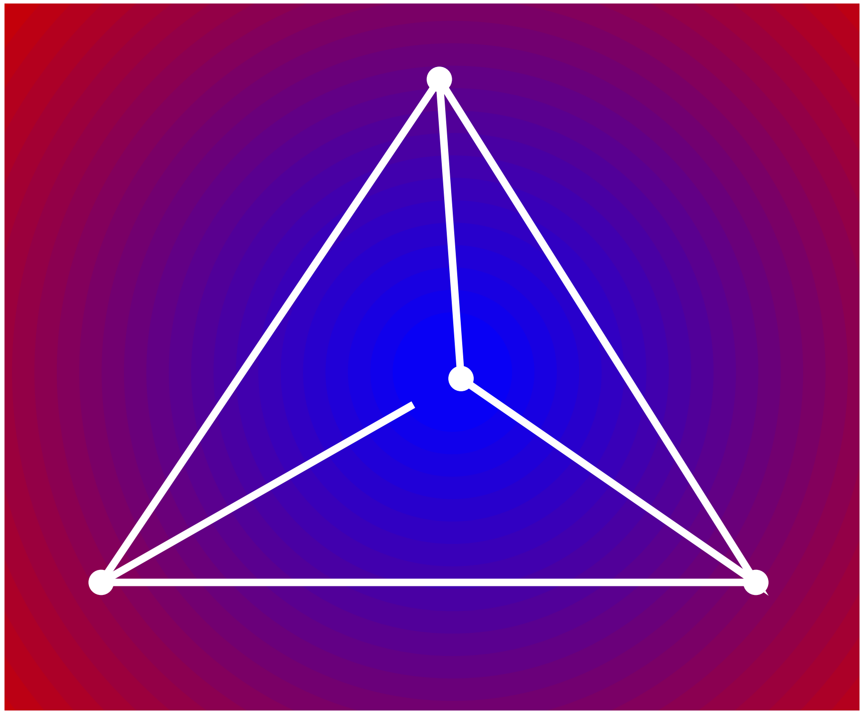
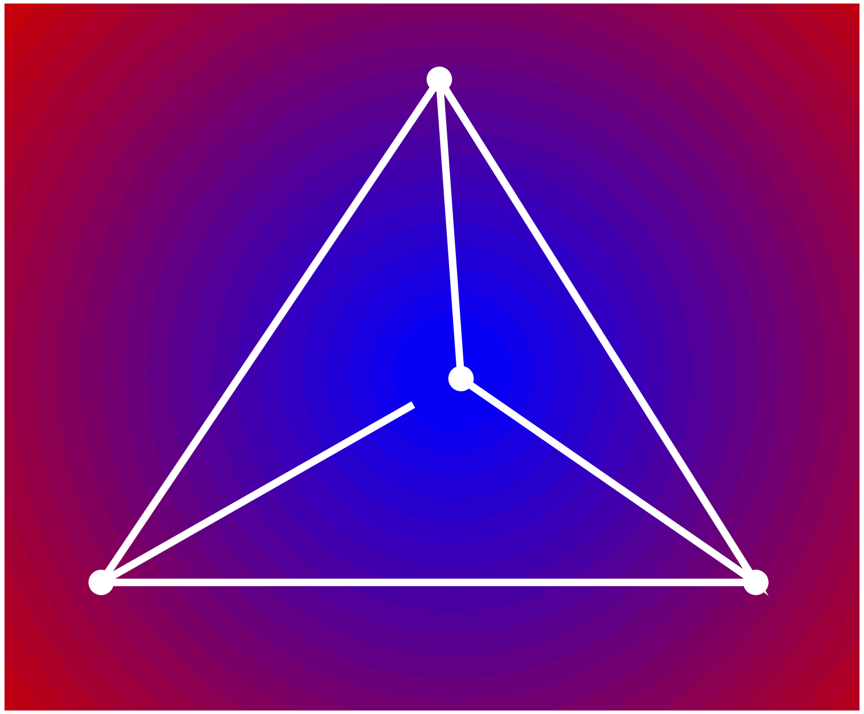


$$\Delta T = |x|^2$$

$$\text{Length increase} \geq \int_{x \in p_i p_j} |x|^2 \, ds$$

$$\exp_{ij} \geq |p_i - p_j| \cdot \int_{x \in p_i p_j} |x|^2 \, ds$$

Heating up the Bars



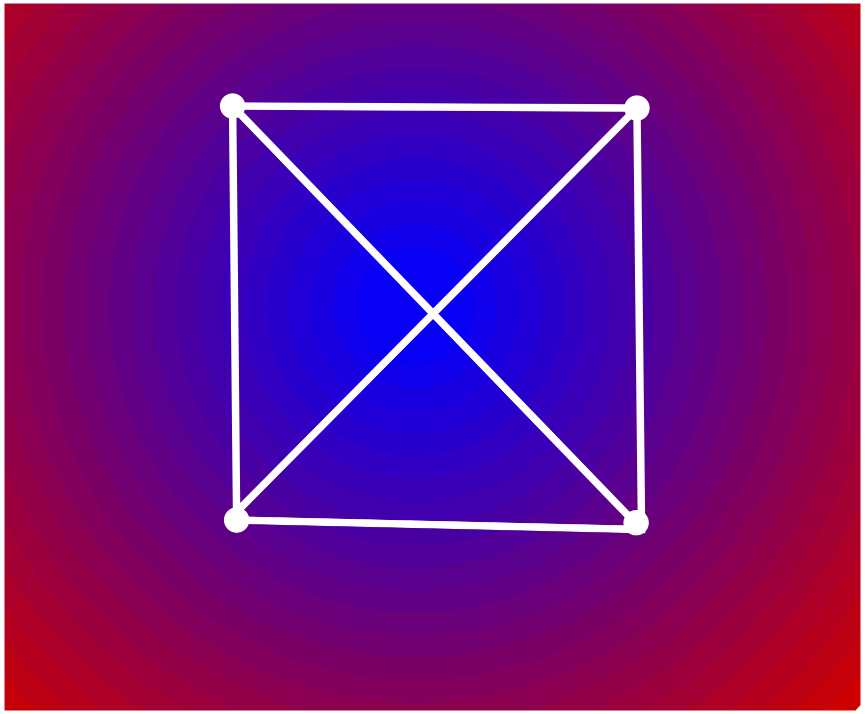
$$\Delta T = |x|^2$$

Length increase $\geq \int_{x \in p_i p_j} |x|^2 \, ds$

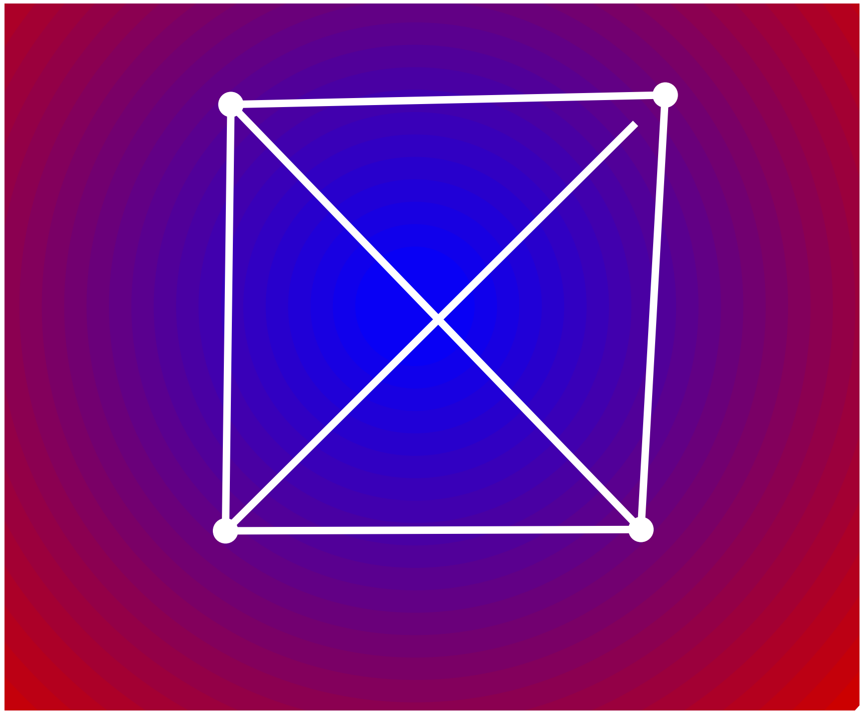
$$\exp_{ij} \geq |p_i - p_j| \cdot \int_{x \in p_i p_j} |x|^2 \, ds$$

$$\exp_{ij} \geq |p_i - p_j|^2 \cdot \left(|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2\right) \cdot \frac{1}{3}$$

Heating up the Bars — Points in Convex Position



$\Rightarrow$



The Perturbed Expansion Cone = PPT Polyhedron

$$\bar{X}_f = \{\, (v_1, \ldots, v_n) \mid \exp_{ij} \geq f_{ij} \,\}$$

- $f_{ij} := |p_i - p_j|^2 \cdot (|p_i|^2 + \langle p_i, p_j \rangle + |p_j|^2)$
- $f'_{ij} := [a, p_i, p_j] \cdot [b, p_i, p_j]$

  $[x, y, z] =$ signed area of the triangle xyz

  a, b: two arbitrary points.

Tight Edges

For $v = (v_1, \ldots, v_n) \in \bar{X}_f$,

$$E(v) := \{\, ij \mid \exp_{ij} = f_{ij} \,\}$$

is the *set of tight edges* at v.

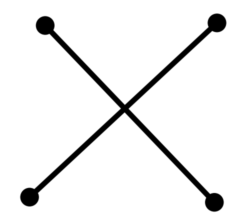
Maximal sets of tight edges $\equiv$ vertices of $\bar{X}_f$.

What are good values of f_{ij}?

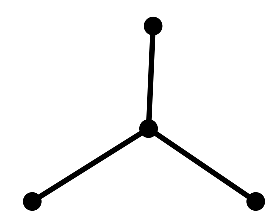
Which configurations of edges can occur in a set of tight edges?

We want:

• no crossing edges

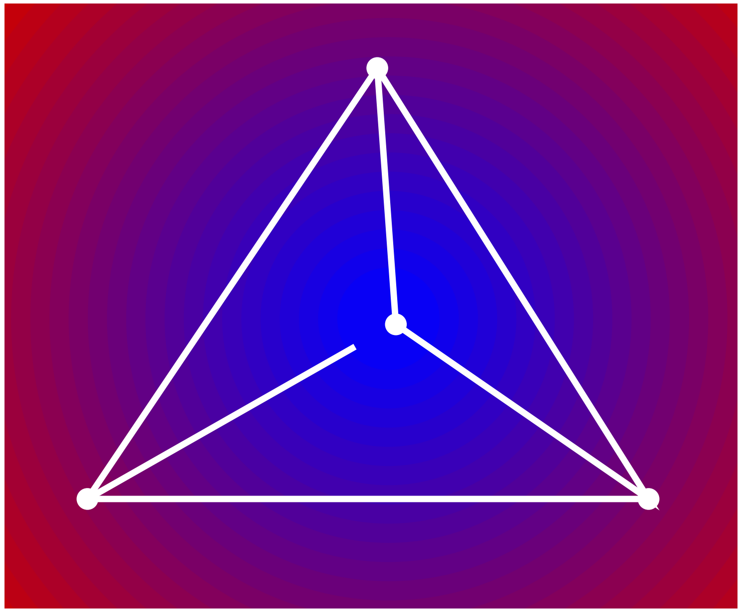


• no 3-star with all angles $\leq 180^\circ$



It is sufficient to look at 4-point subsets.

Good Values f_{ij} for 4 points



f_{ij} is given on six edges.
Any five values $\exp_{ij}$ determine the last one.
Check if the resulting value $\exp_{ij}$ of the last edge is feasible ($\exp_{ij} \geq f_{ij}$) $\rightarrow$ checking the sign of an expression.

$\square$

The PPT-polyhedron

Every vertex is incident to $2n-3$ edges.

Edge $\equiv$ removing a segment from $E(v)$.

Removing an interior segment leads to an adjacent pseudotri-angulation (flip).

Removing a hull segment is an extreme ray. $\square$

The PPT polytope

Cut out all rays:

Change $\exp_{ij} \geq f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

The PPT polytope

Cut out all rays:
Change $\exp_{ij} \geq f_{ij}$ to $\exp_{ij} = f_{ij}$ for hull edges.

The Expansion Cone $\bar{X}_0$:

collapse parallel rays into one ray. $\longrightarrow$ pseudotriangulations minus one hull edge. Rigid subcomponents are identified.

The PT polytope

Vertices correspond to *all* pseudotriangulations, pointed or not.

Change inequalities $\exp_{ij} \geq f_{ij}$ to

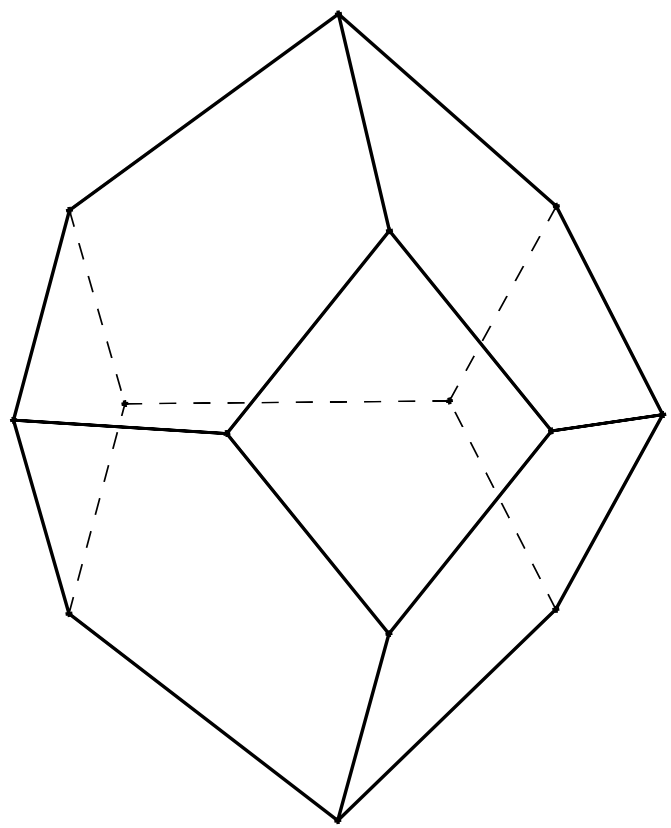
$$\exp_{ij} + (s_i + s_j)\|p_j - p_i\| \geq f_{ij}$$

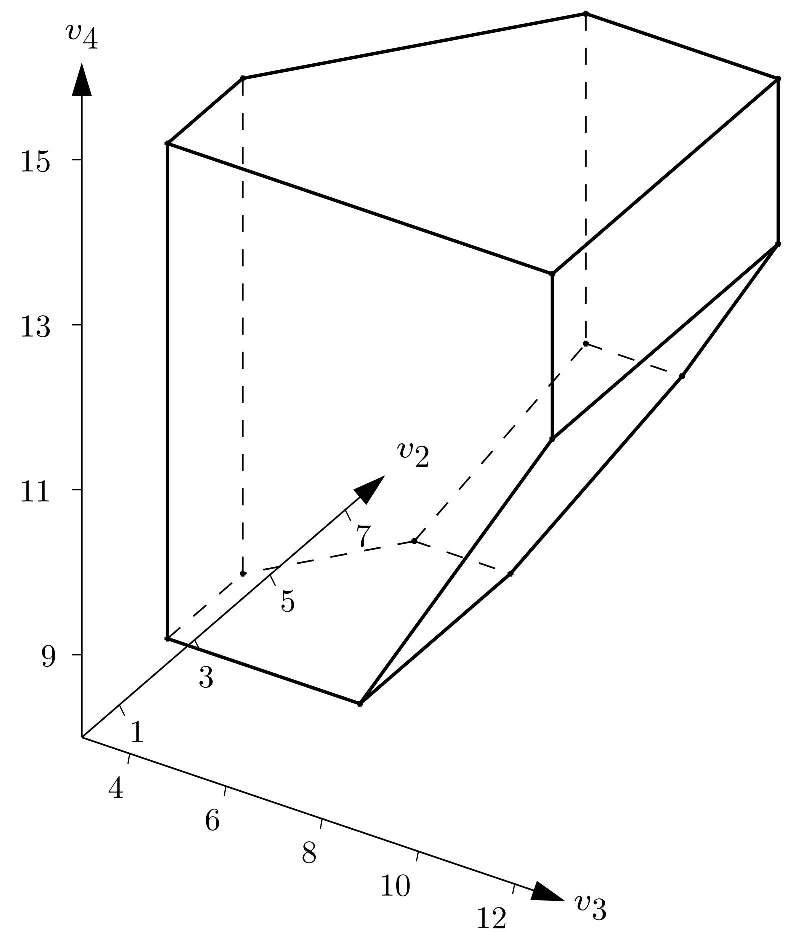
with a "slack variable" s_i for every vertex.
$s_i = 0$ indicates that vertex i is pointed.

Faces are in one-to-one correspondence with all non-crossing graphs.

[Orden, Santos 2002]

The associahedron





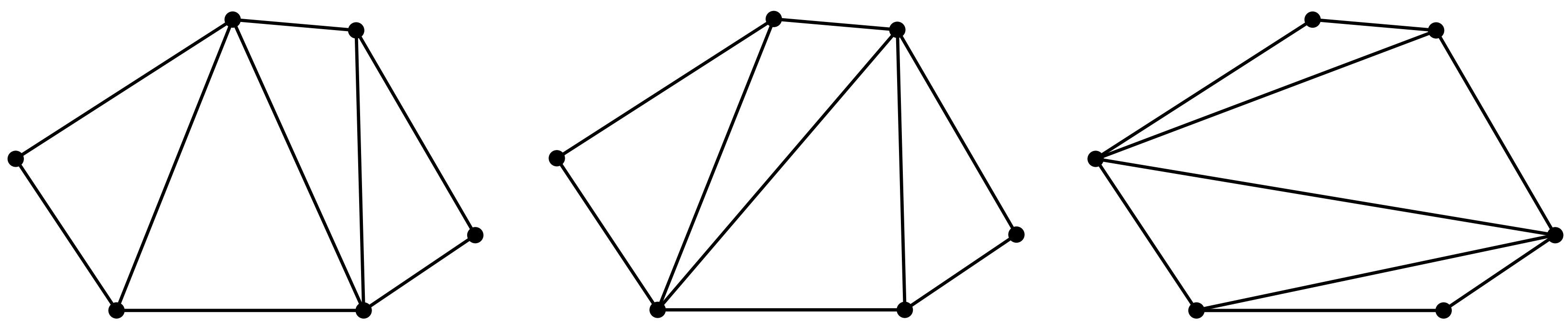
Catalan structures

- Triangulations of a convex polygon / edge flip
- Binary trees / rotation
- $(a * (b * (c * d))) * e$ / $((a * b) * (c * d)) * e$
- . . . .

Canonical pseudotriangulations

Maximize/minimize $\sum_{i=1}^{n} c_i \cdot v_i$ over the PPT-polytope.

$c_i := p_i$:



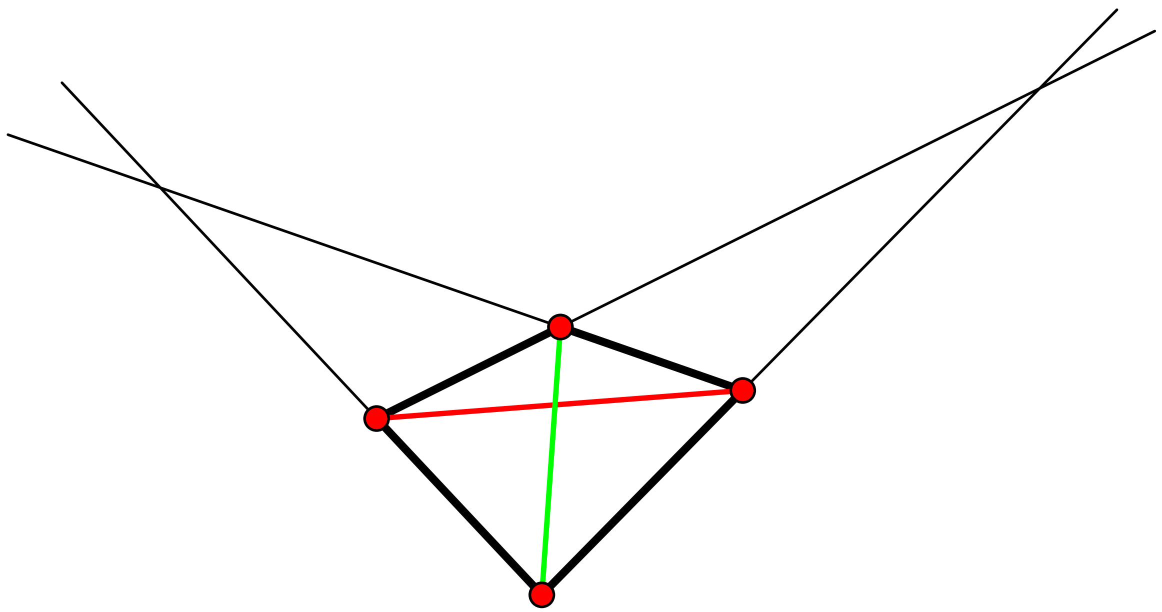
(a) (b) (c)

Delaunay triangulation

Max/Min $\sum p_i \cdot v_i$
(not affinely invariant)

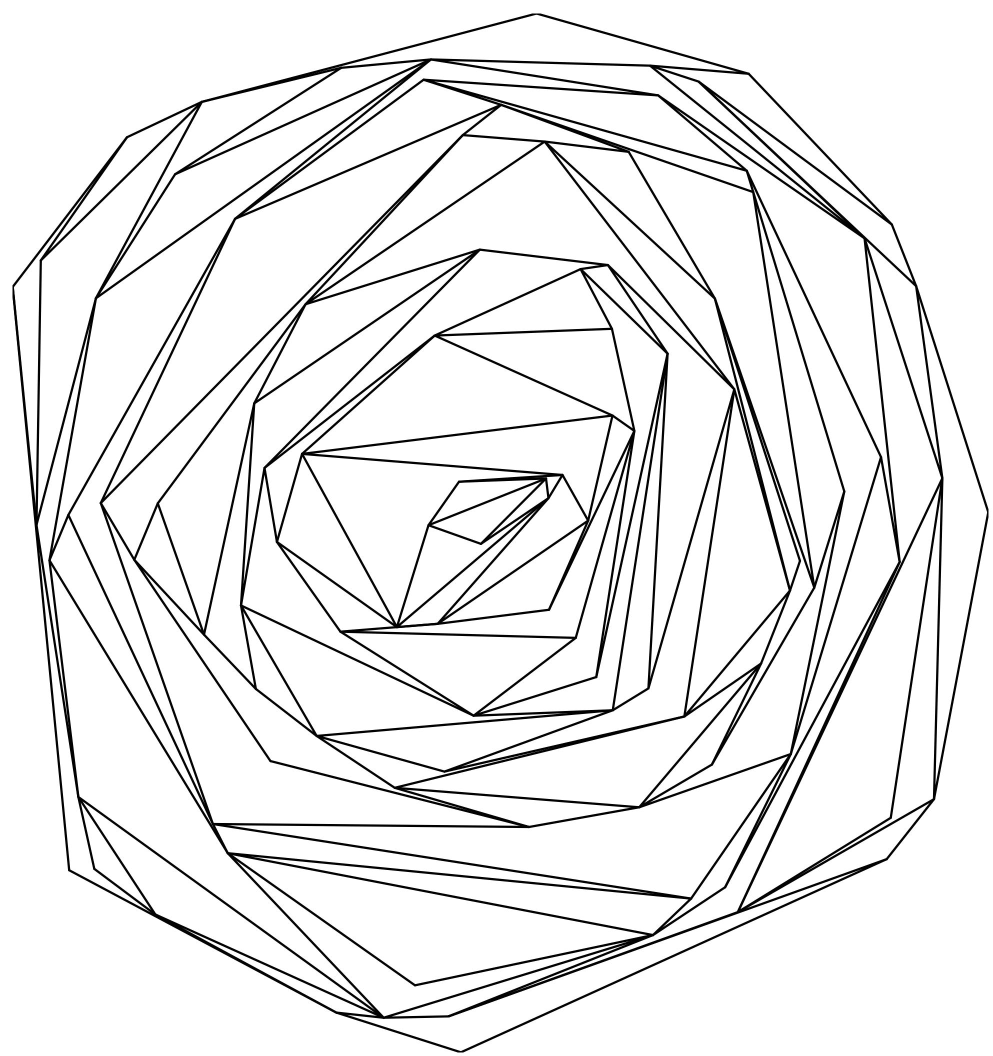
(Can be constructed as the lower/upper convex hull of lifted points.)

Edge flipping criterion for canonical pseudotriangulations of 4 points in convex position

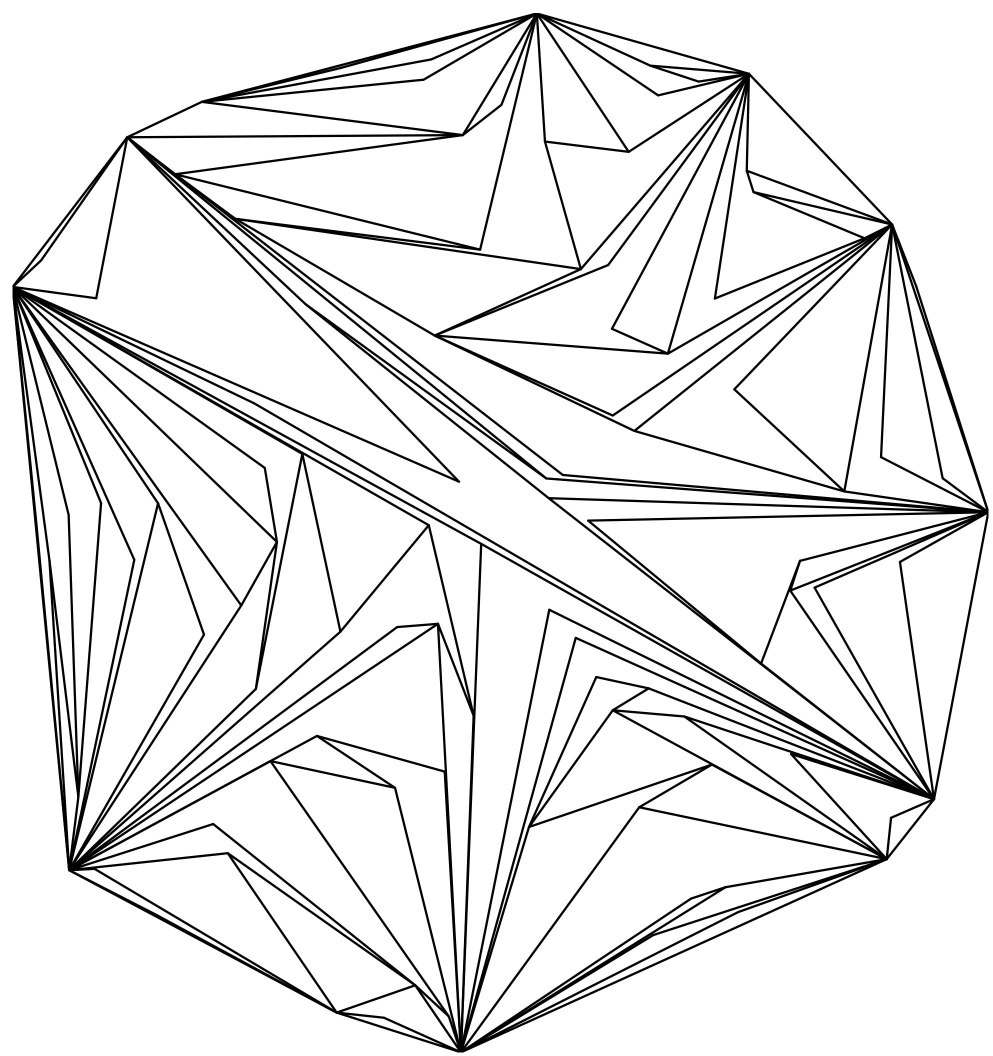


Maximize/minimize the product of the areas.
Invariant under affine transformations.

The “Delone pseudotriangulation” for 100 random points



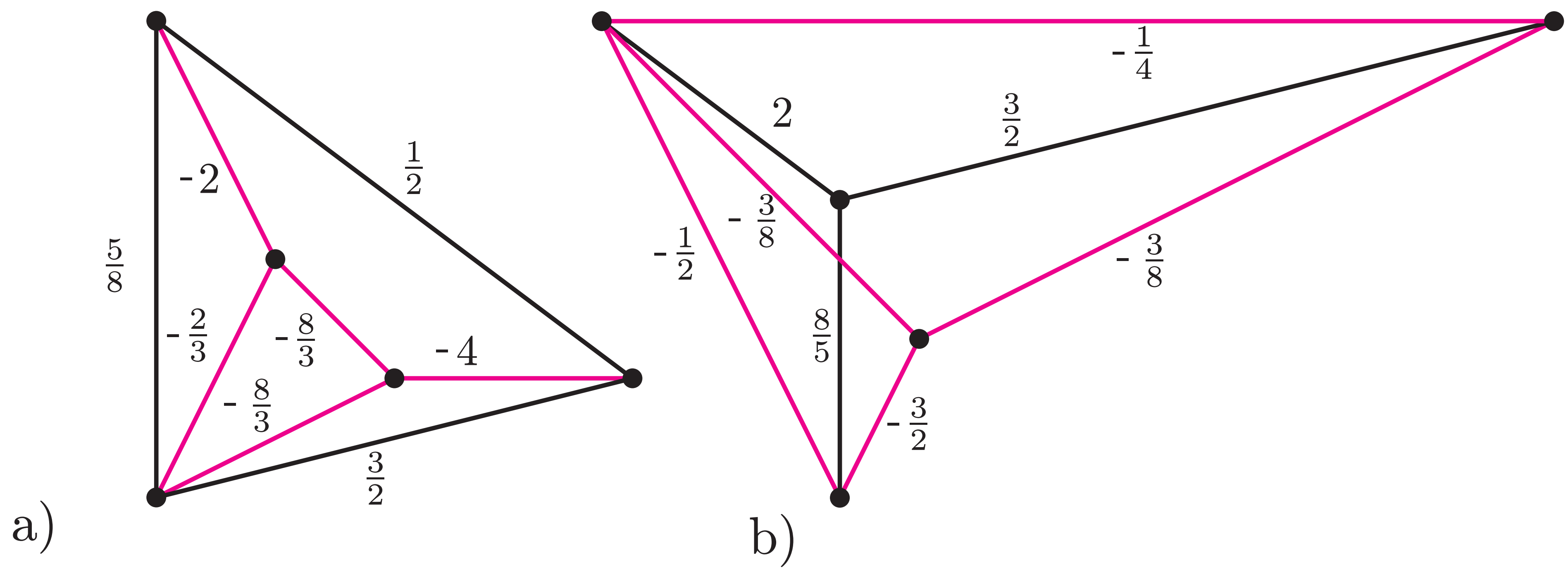
The “Anti-Delone pseudotriangulation” for 100 random points



3. STRESSES AND RECIPROCALS
Reciprocal frameworks

Given: A plane graph G and its planar dual G^*.

A framework (G, p) is *reciprocal* to (G^*, p^*) if corresponding edges are parallel.

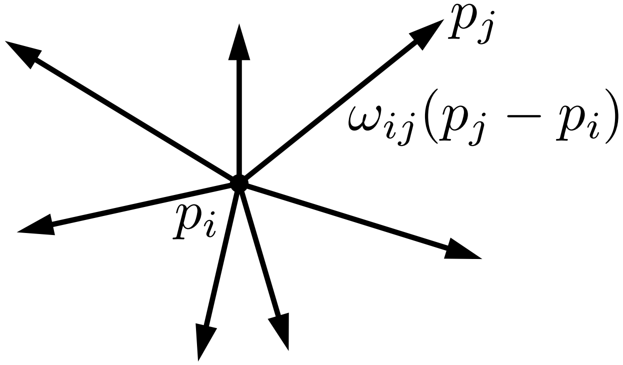


$\rightarrow$ dynamic animation of reciprocal diagrams with *Cinderella*

Self-stresses

A *self-stress* in a framework is given by a set of internal forces (compressions and tensions) on the edges in *equilibrium* at every vertex i:

$$\sum_{j:(i,j)\in E} \omega_{ij}(p_j - p_i) = 0$$



The force of edge (i,j) on vertex i is

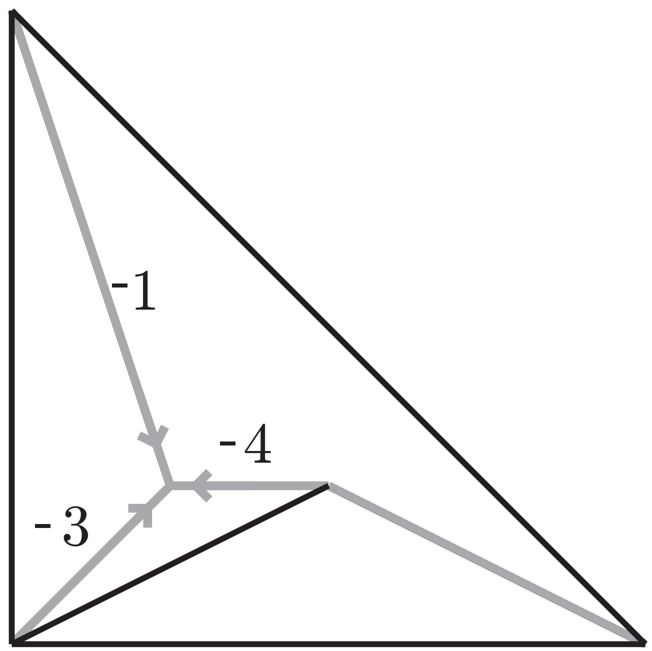
$$\omega_{ij}(p_j - p_i).$$

The force of edge (i,j) on vertex j is

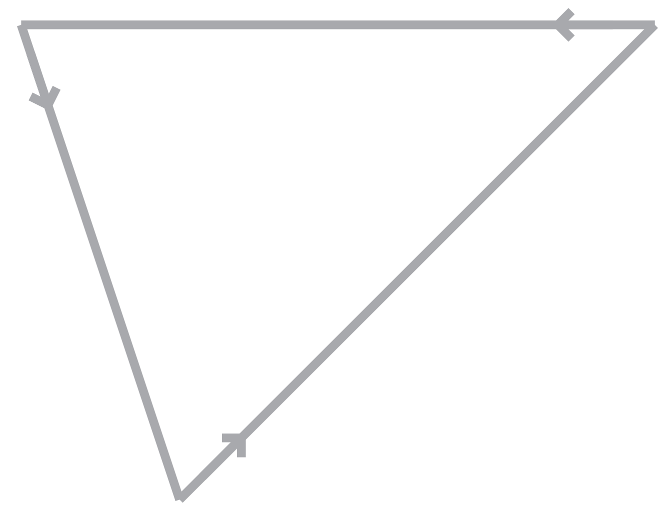
$$\omega_{ji}(p_i - p_j) = -\omega_{ij}(p_j - p_i). \qquad (\omega_{ij} = \omega_{ji})$$

Self-stresses and reciprocal frameworks

An equilibrium at a vertex gives rise to a polygon of forces:



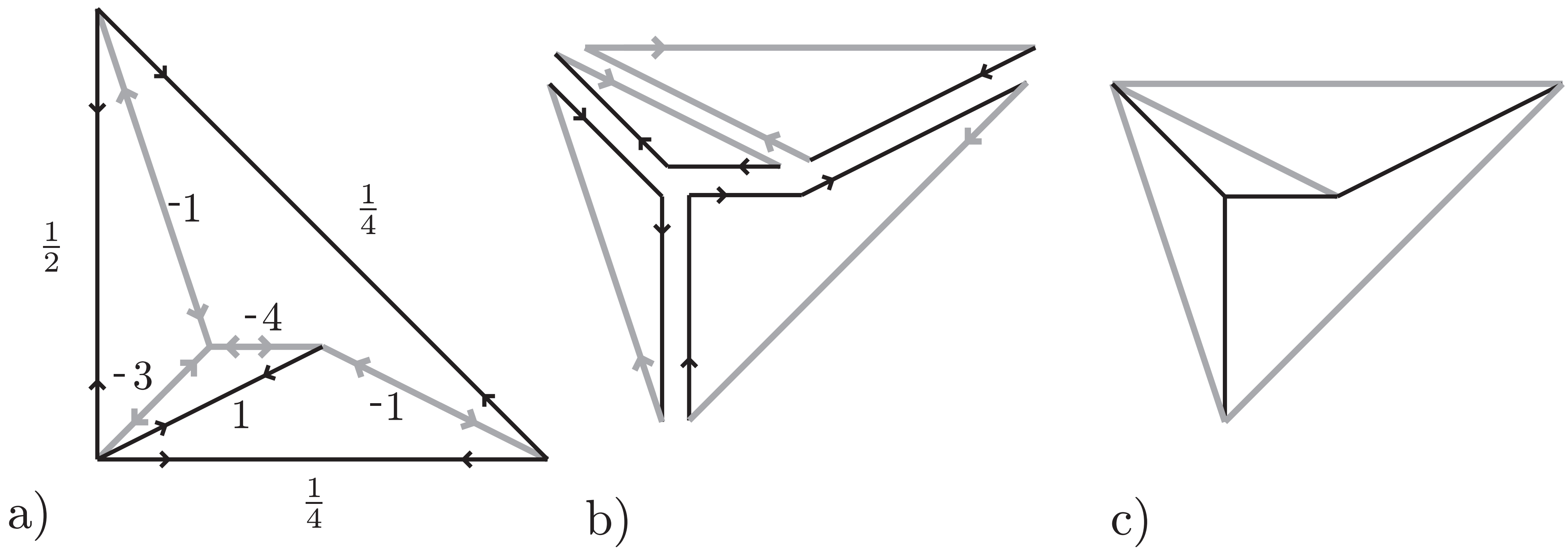
a)



b)

These polygons can be assembled to the reciprocal diagram.

Assembling the reciprocal framework



$\omega^*_{ij} := 1/\omega_{ij}$ defines a self-stress on the reciprocal.

The Maxwell-Cremona Correspondence [1864/1872]

self-stresses on a
planar framework

$\Updownarrow$ one-to-one correspondence

reciprocal diagram

The Maxwell-Cremona Correspondence [1864/1872]

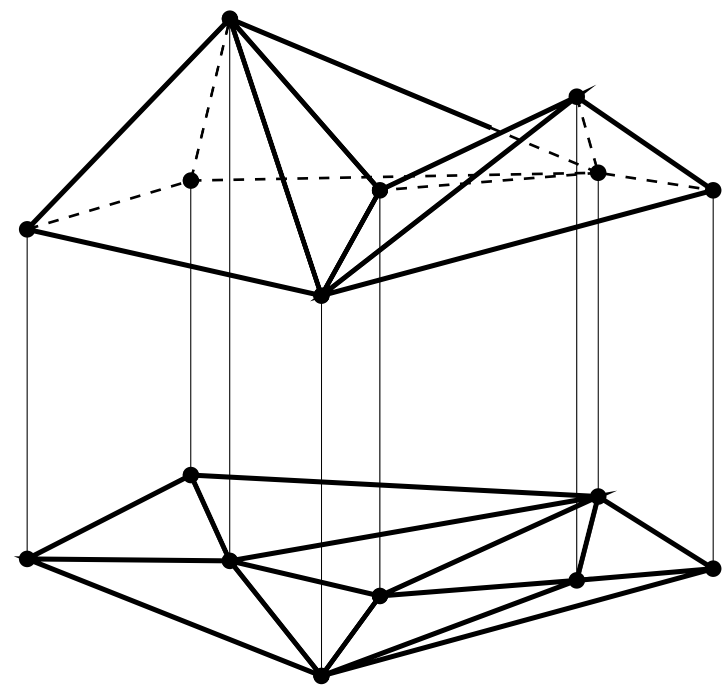
self-stresses on a planar framework

$\Updownarrow$ one-to-one correspondence

reciprocal diagram

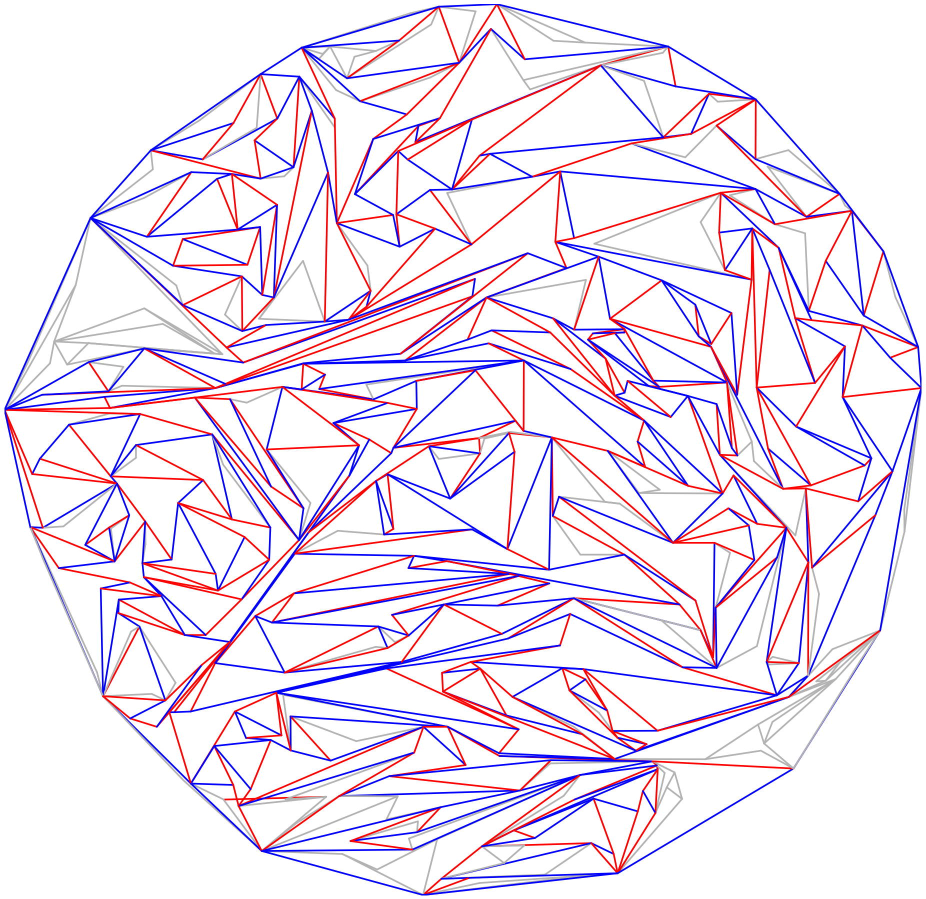
$\Updownarrow$ one-to-one correspondence

3-d lifting (polyhedral terrain)



Minimally dependent graphs (rigidity circuits)

A Laman graph plus one edge has a unique self-stress (up to scalar multiplication).



$\rightarrow$ It has a unique reciprocal (up to scaling).

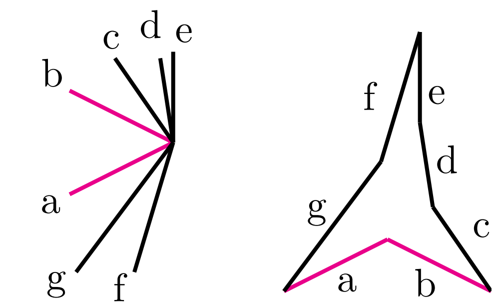
Planar frameworks with planar reciprocals

Theorem. *Let G be a pseudotriangulation with $2n-2$ edges (and hence with a single nonpointed vertex). Then G^* is non-crossing.*
Moreover, if the stress on G is nonzero on all edges, G^* is also a pseudotriangulation with $2n-2$ edges.

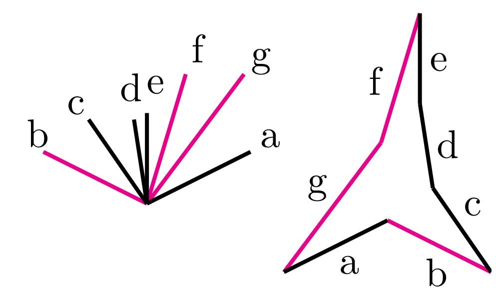
[Orden, Rote, Santos, B. Servatius, H. Servatius, Whiteley 2003]

Possible sign patterns around vertices

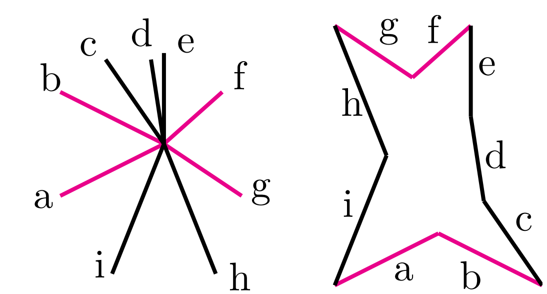
pointed, with two sign changes
(none at the big angle)



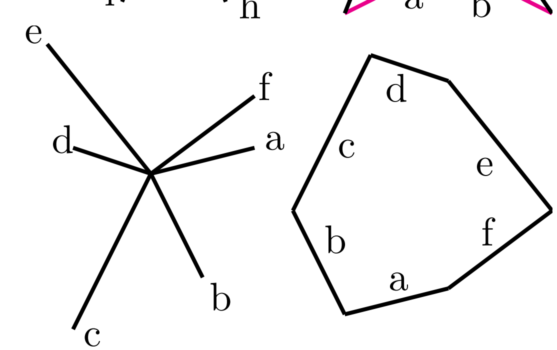
pointed, with four sign changes
(including one at the big angle)



nonpointed, with four sign changes



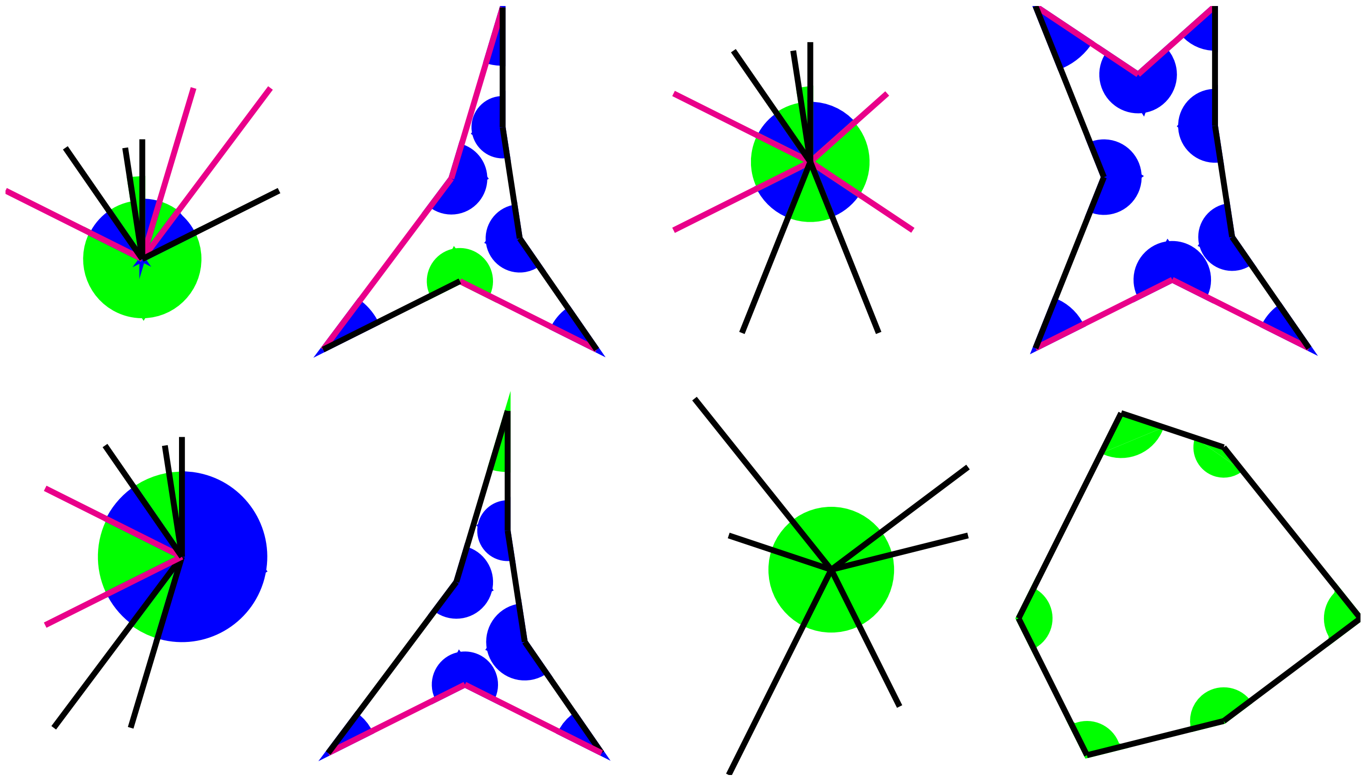
nonpointed, with no sign changes



Vertex-proper and Face-proper angles

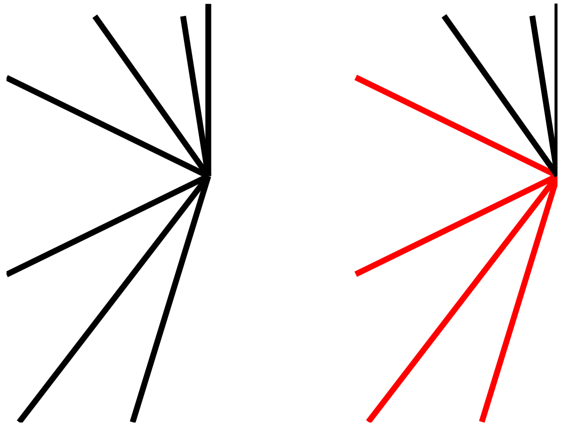
A face-proper angle is a big angle with equal signs or a small angle with a sign change.

A vertex-proper angle is a small angle with equal signs or a big angle with a sign change.



Counting angles

Lemma. *At every pointed vertex, there are at least* 3 *face-proper angles in a self-stress.*



Counting angles

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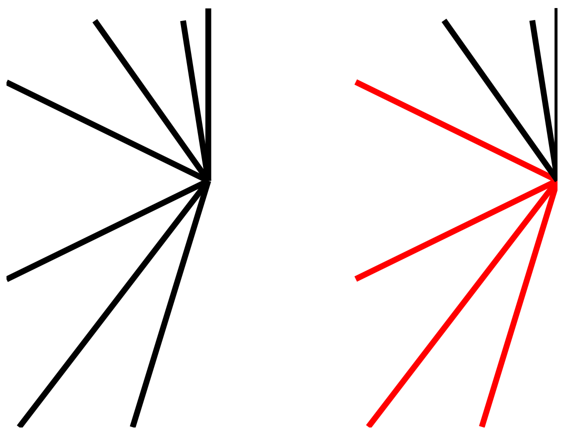
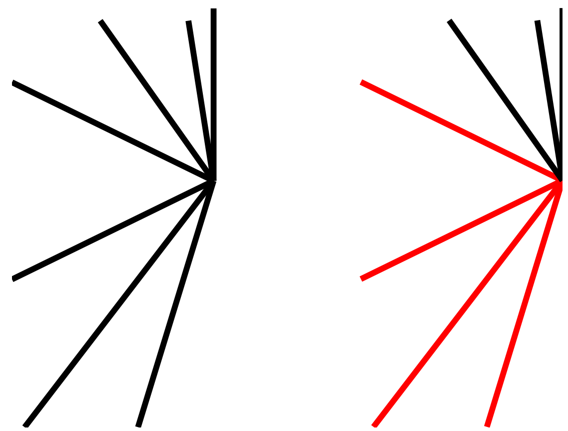


Lemma. *In every pseudotriangle, there is at least* 1 *vertex-proper angle.*

Counting angles

Lemma. *At every pointed vertex, there are at least* 3 *face-proper angles in a self-stress.*



Lemma. *In every pseudotriangle, there is at least* 1 *vertex-proper angle.*

$$2e = \#\text{angles} \geq 3(n-1) + (n-1) = 2(2n-2) = 2e$$

$\longrightarrow$ equality throughout!

Counting angles—conclusion

Every pointed vertex has exactly 3 face-proper angles.
$\longrightarrow$ reciprocal face is a pseudotriangle.

The non-pointed vertex has no face-proper angles.
$\longrightarrow$ reciprocal face is convex $=$ the outer face.

Every pseudotriangle has exactly 1 vertex-proper angle.
$\longrightarrow$ reciprocal vertex is pointed.

The outer face has no vertex-proper angles.
$\longrightarrow$ reciprocal vertex is nonpointed.

Counting angles—conclusion

Every pointed vertex has exactly 3 face-proper angles.
$\longrightarrow$ reciprocal face is a pseudotriangle.

The non-pointed vertex has no face-proper angles.
$\longrightarrow$ reciprocal face is convex $=$ the outer face.

Every pseudotriangle has exactly 1 vertex-proper angle.
$\longrightarrow$ reciprocal vertex is pointed.

The outer face has no vertex-proper angles.
$\longrightarrow$ reciprocal vertex is nonpointed.

If some edges have zero stress, the reciprocal can have more than one non-pointed vertex.

General pairs of non-crossing reciprocal frameworks

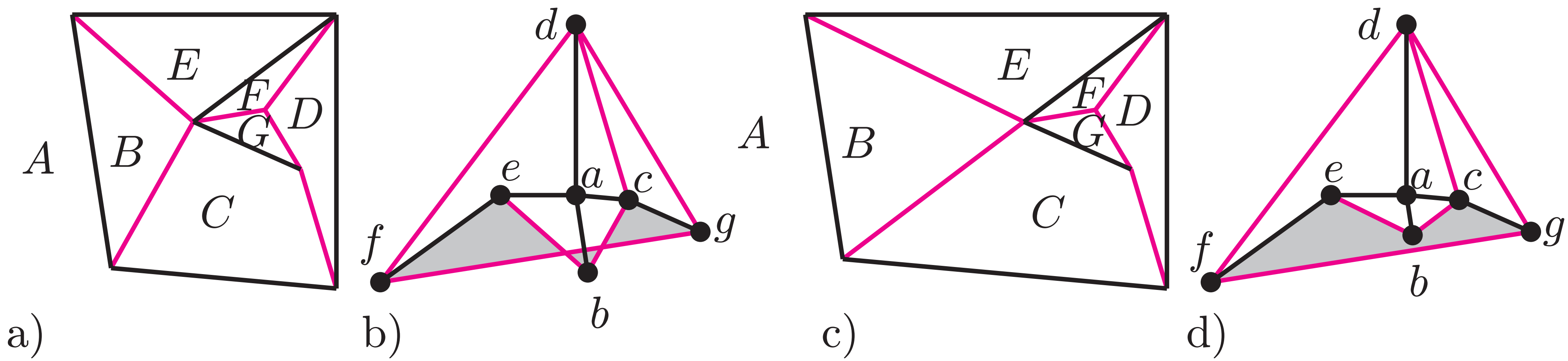
G and G^* can have more than one non-pointed vertex and can contain *pseudoquadrangles*.

Necessary conditions:

- Vertices must be as above, with a unique non-pointed vertex that has no sign changes.
- All other non-pointed vertices must have 4 sign changes.
- Analogous *face conditions*.

General pairs of non-crossing reciprocals

These combinatorial vertex conditions are also sufficient for a non-crossing reciprocal, except possibly for “self-crossing” pseudoquadrangles.



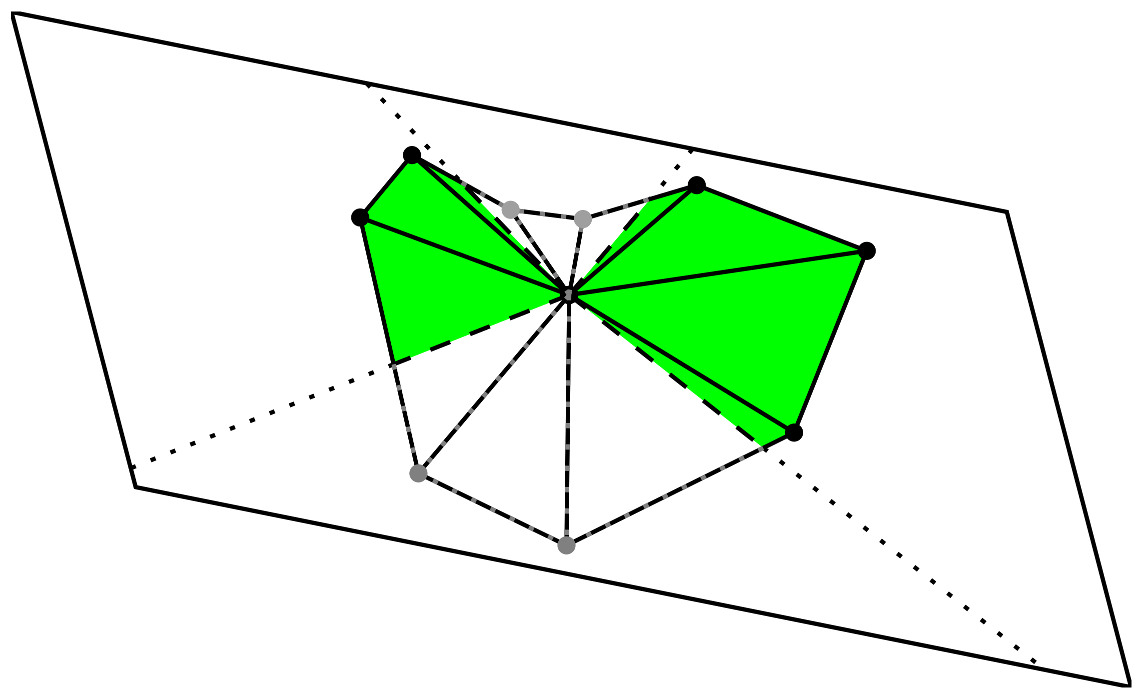
4. LIFTINGS AND SURFACES

4a. Liftings of non-crossing reciprocals

4b. Locally convex liftings

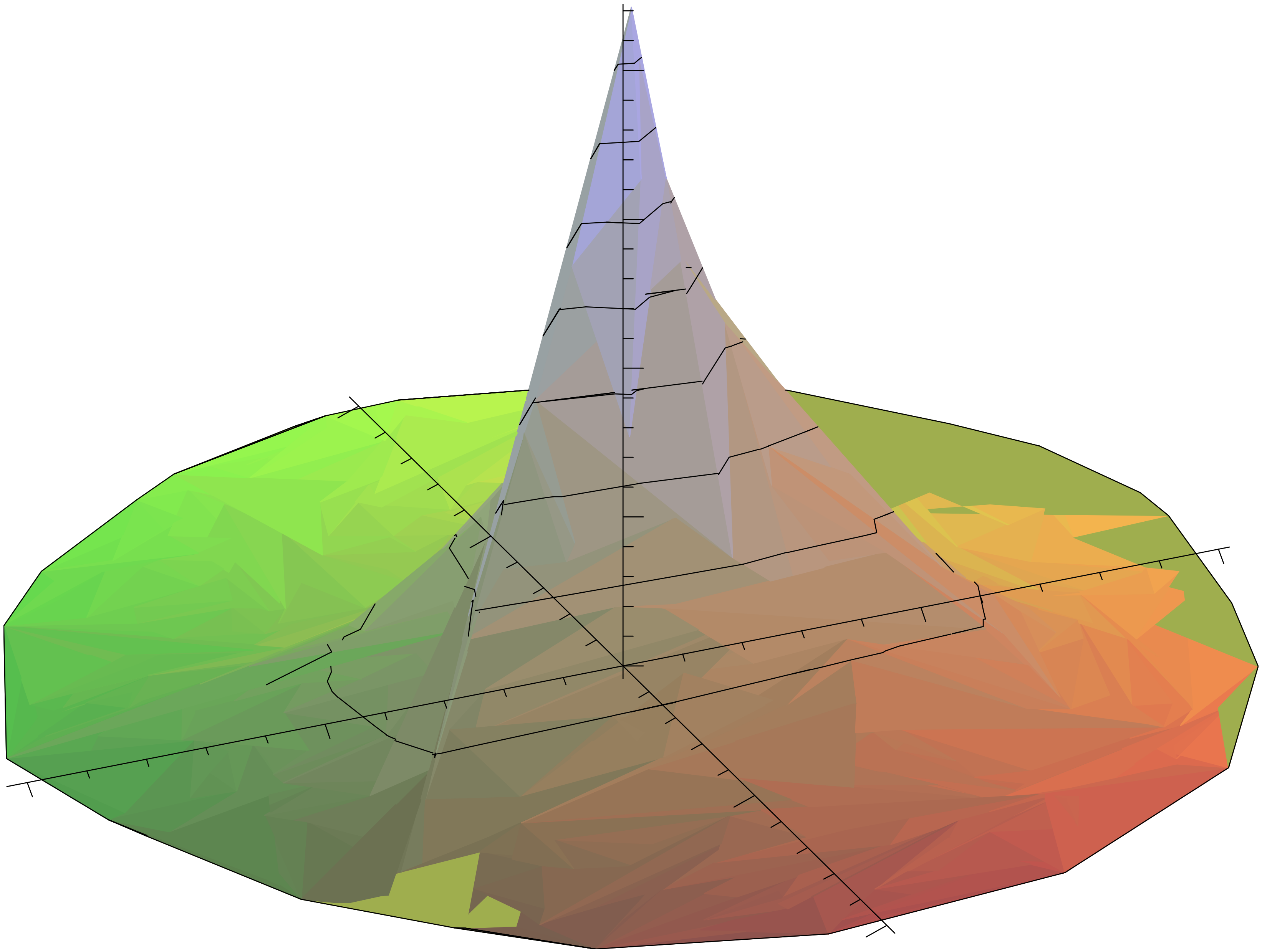
4a. Liftings of non-crossing reciprocals

Theorem. *If G and G^* are non-crossing reciprocals, the lifting has a unique maximum. There are no other critical points. Every other point p is a "twisted saddle": Its neighborhood is cut into four pieces by some plane through v (but not more).*

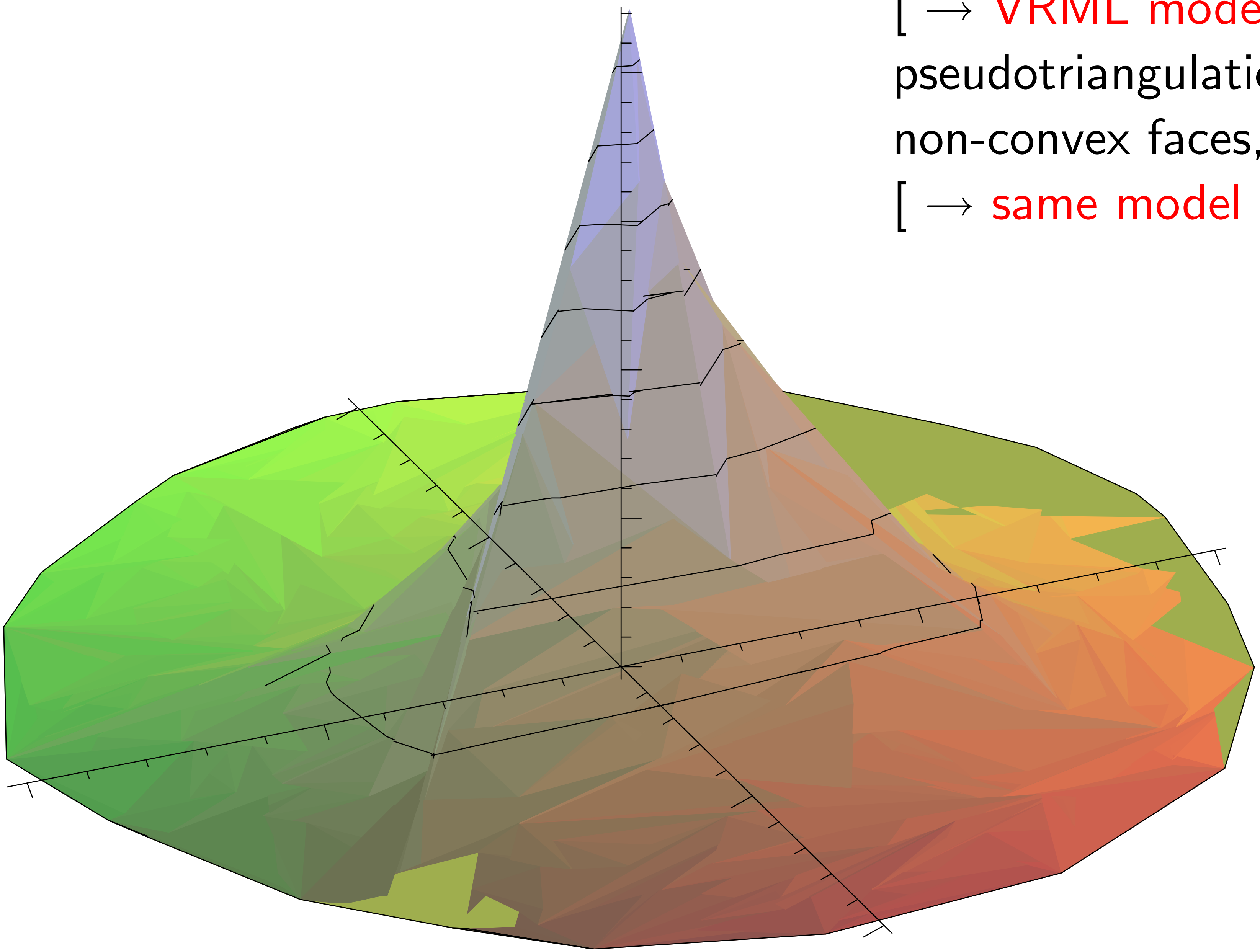


"Negative curvature" everywhere except at the peak!

Liftings of non-crossing reciprocals



Liftings of non-crossing reciprocals

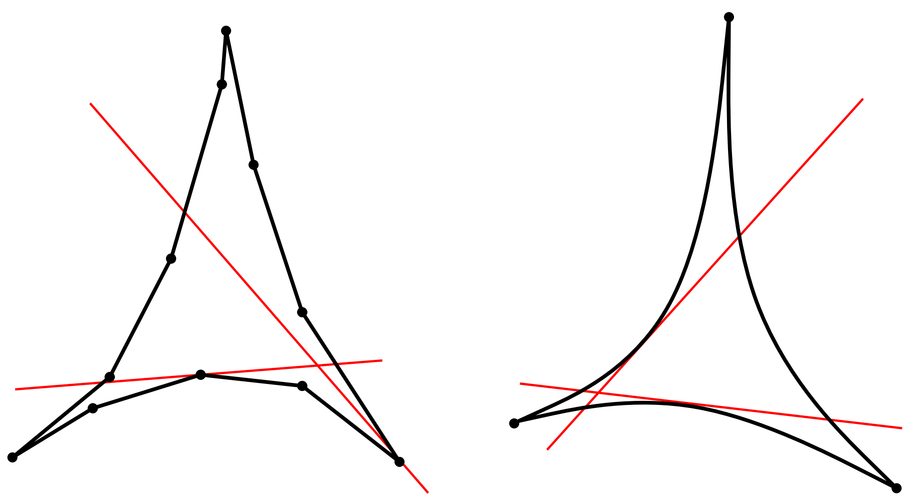


[→ VRML model of a different pseudotriangulation (with non-convex faces, too!)]
[→ same model without light]

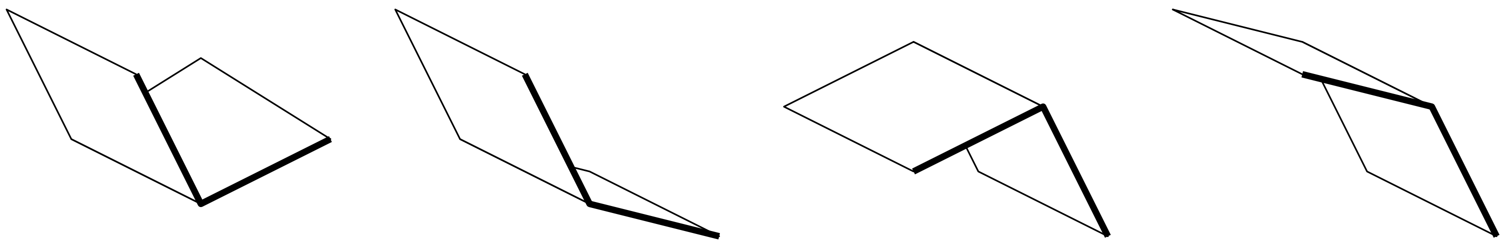
Tangent planes of lifted pseudotriangulations

For every plane which touches the peak from above, there is a unique parallel plane which cuts a vertex like a saddle (a "tangent plane").

Remember: In a pseudotriangle, for every direction, there is a unique line which is "tangent" at a reflex vertex or "cuts through" a corner.



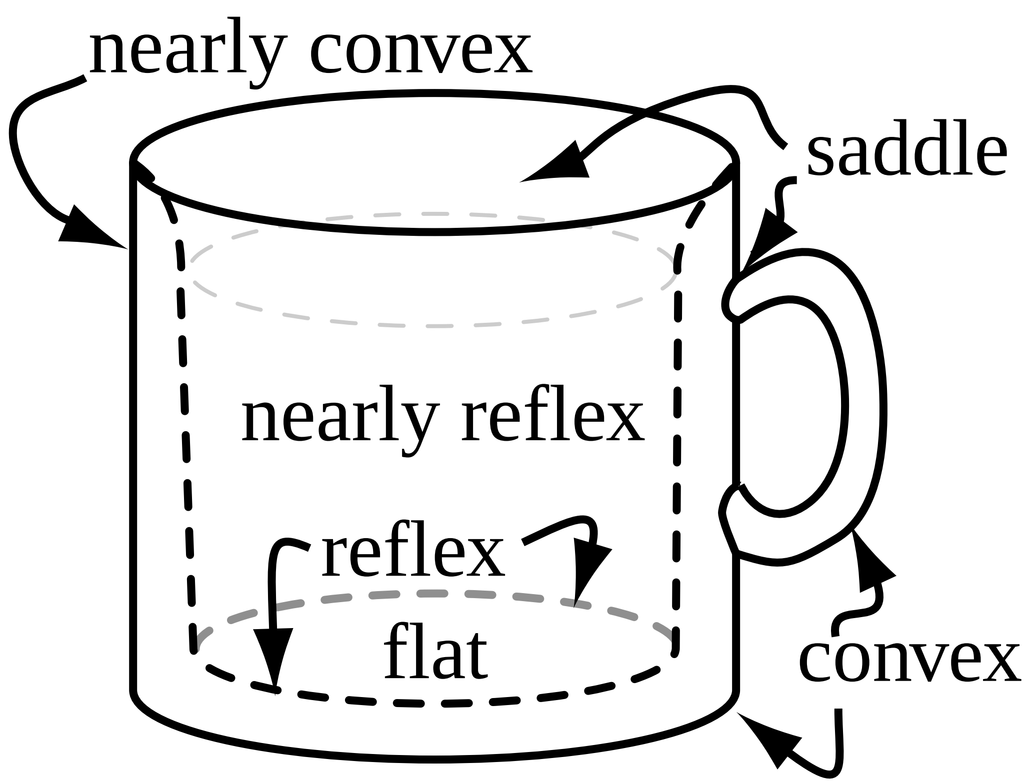
Valley and Mountain Folds



$\omega_{ij} > 0$ $\omega_{ij} < 0$

valley mountain

4b. LOCALLY CONVEX LIFTINGS
The reflex-free hull

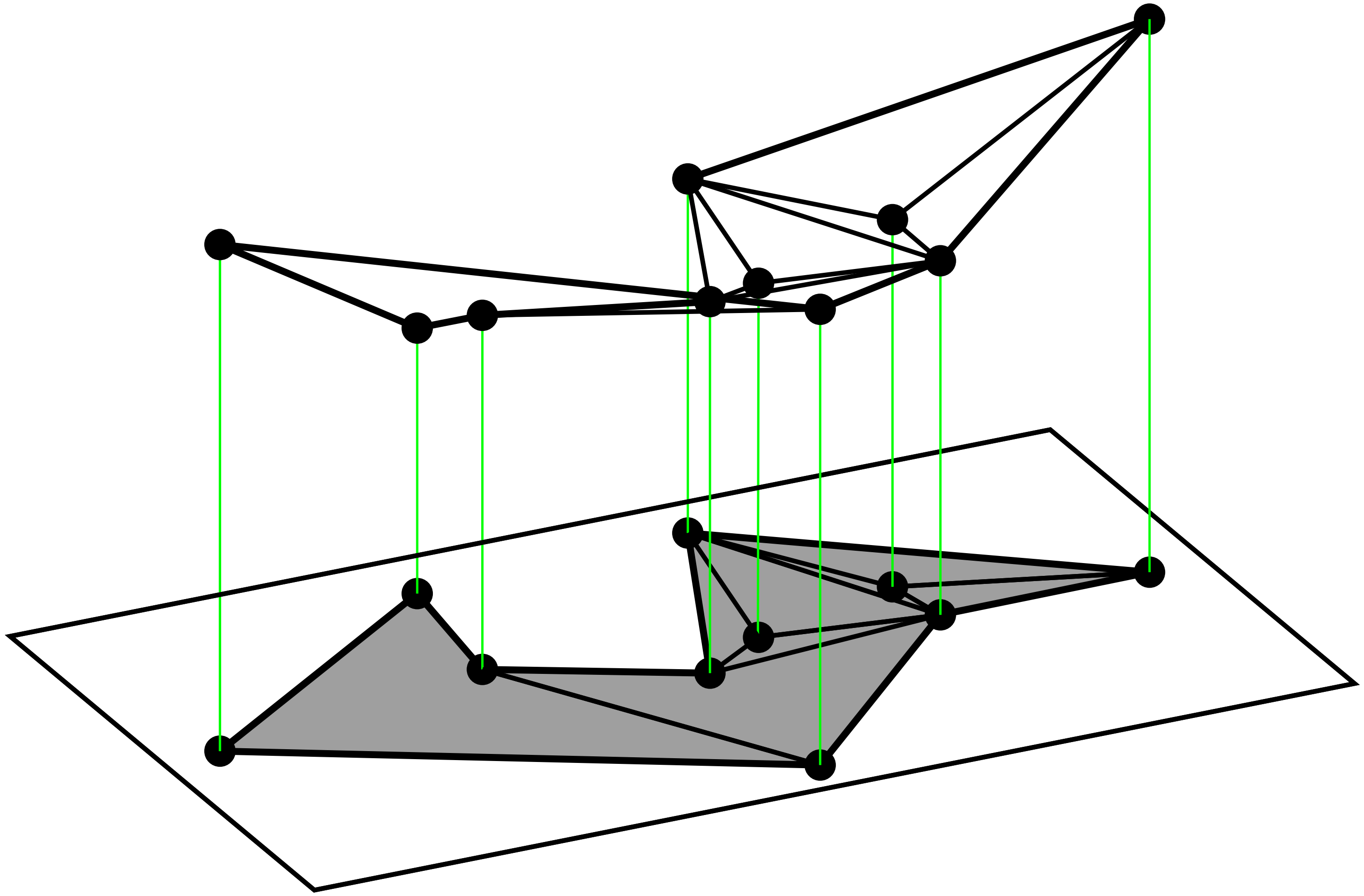


an approach for recognizing pockets in biomolecules

[Ahn, Cheng, Cheong, Snoeyink 2002]

Locally convex surfaces

A function over a polygonal domain P is *locally convex* if it is convex on every segment in P.



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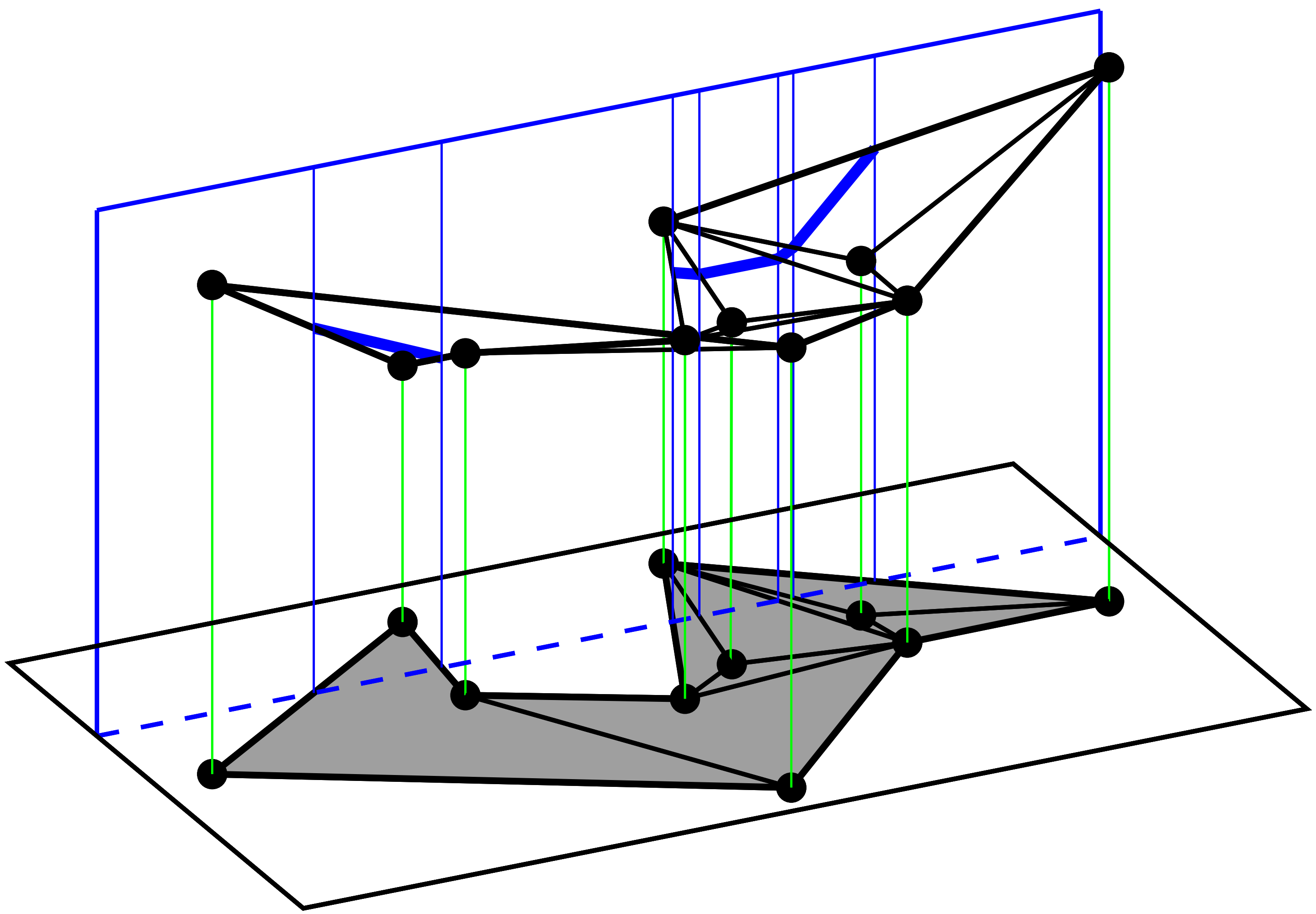


Locally convex functions on a poipogon

A *poipogon* (P, S) is a simple polygon P with some additional vertices inside.

Given a poipogon and a height value h_i for each $p_i \in S$, find the highest locally convex function $f\colon P \to \mathbb{R}$ with $f(p_i) \le h_i$.

If P is convex, this is the lower convex hull of the three-dimensional point set (p_i, h_i).

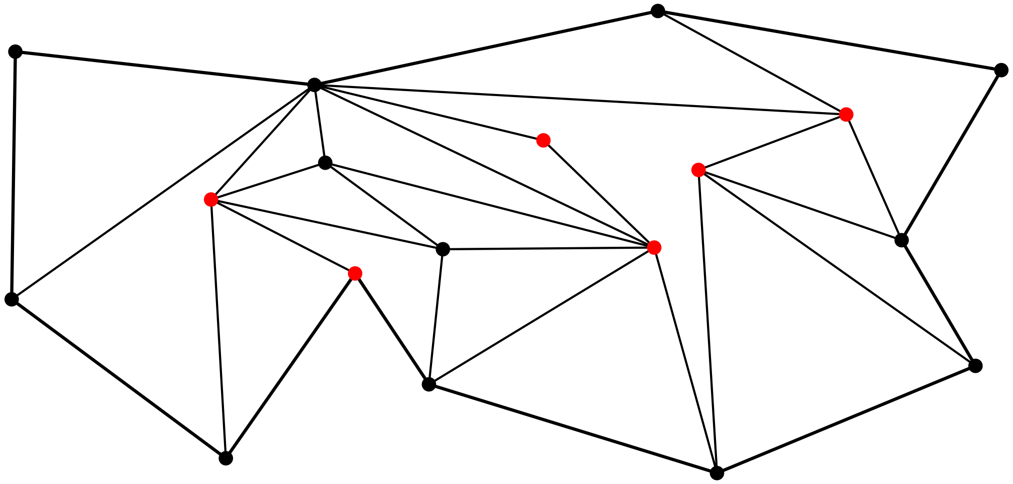
In general, the result is a piecewise linear function defined on a pseudotriangulation of (P, S). (Interior vertices may be missing.)

$\longrightarrow$ *regular pseudotriangulations*

[Aichholzer, Aurenhammer, Braß, Krasser 2003]

The surface theorem

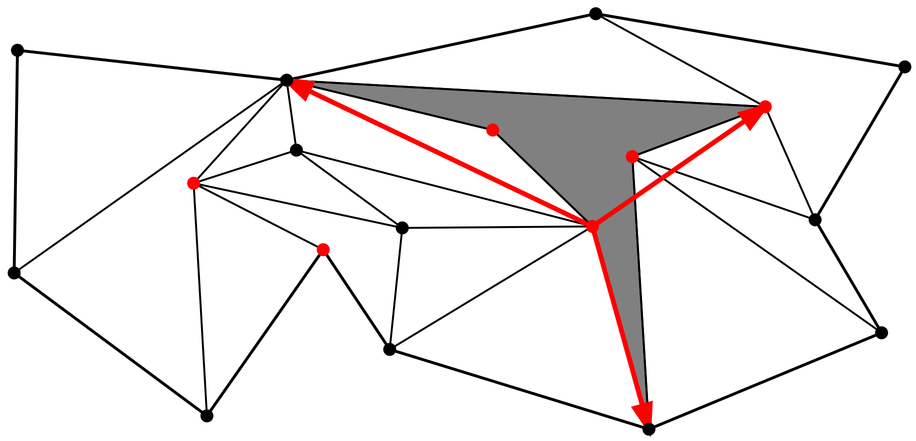
In a pseudotriangulation T of (P, S), a vertex is *complete* if it is a corner in all pseudotriangulations to which it belongs.



Theorem. *For any given set of heights h_i for the complete vertices, there is a unique piecewise linear function on the pseudotriangulation with the complete vertices. The function depends monotonically on the given heights.*

In a triangulation, all vertices are complete.

Proof of the surface theorem



Each incomplete vertex p_i is a convex combination of the three corners of the pseudotriangle in which its large angle lies:

$$p_i = \alpha p_j + \beta p_k + \gamma p_l, \text{ with } \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma > 0.$$

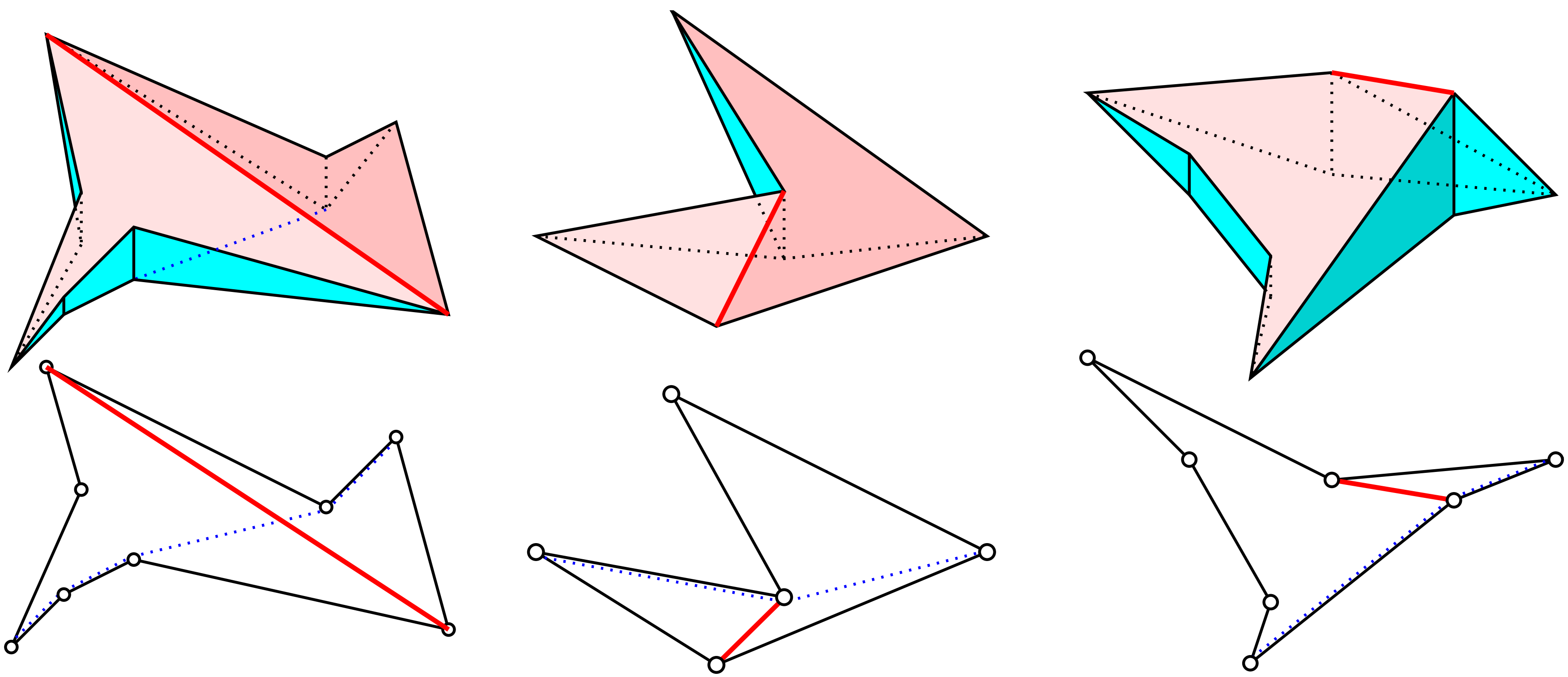
$$\rightarrow h_i = \alpha h_j + \beta h_k + \gamma h_l$$

The coefficient matrix of this mapping $F\colon (h_1, \ldots, h_n) \mapsto (h'_1, \ldots, h'_n)$ is a stochastic matrix. F is a monotone function, and $F^{(n)}$ is a contraction.

$\rightarrow$ there is always a unique solution.

Flipping to optimality

Find an edge where convexity is violated, and flip it.



convexifying flips a planarizing flip

A flip has a non-local effect on the whole surface.

The surface moves down monotonically.

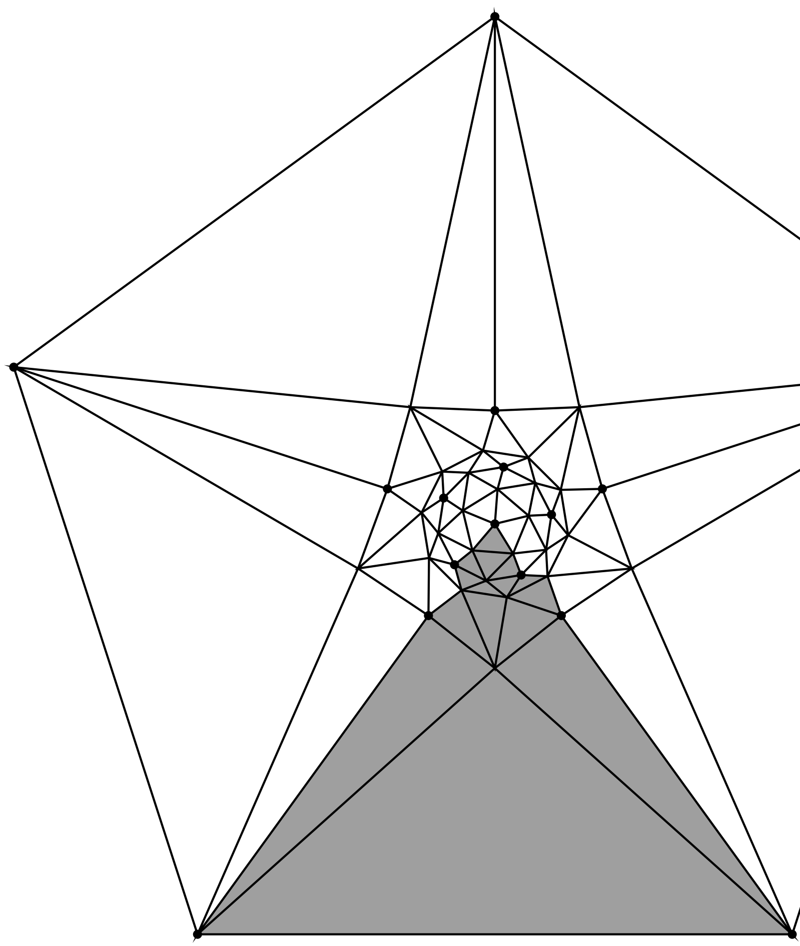
Realization as a polytope

There exists a convex polytope whose vertices are in one-to-one correspondence with the regular pseudotriangulations of a poipogon, and whose edges represent flips.

For a simple polygon (without interior points), all pseudotriangulations are regular.

5. Minimal pseudotriangulations

Minimal pseudotriangulations (w.r.t. $\subseteq$) are not necessarily minimum-cardinality pseudotriangulations.



A minimal pseudotriangulation has at most $3n - 8$ edges, and this is tight for infinitely many values of n.

[Rote, C. A. Wang, L. Wang, Xu 2003]

Pseudotriangulations/ Geodesic Triangulations

Other applications:

- data structures for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, and Snoeyink 1994] and visibility [Pocchiola and Vegter 1996]
- kinetic collision detection [Agarwal, Basch, Erickson, Guibas, Hershberger, Zhang 1999–2001] [Kirkpatrick, Snoeyink, and Speckmann 2000] [Kirkpatrick & Speckmann 2002]
- art gallery problems [Pocchiola and Vegter 1996b], [Speckmann and Tóth 2001]

Open Questions

1. Pseudotriangulations on a small grid. $O(n) \times O(n)$?

2. Pseudotriangulations in 3-space

   (a) locally convex functions

   (b) the expansion cone