Fall School on Computational Geometry
Pseudotriangulations

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1. (4 points) Consider the following conditions for a straight-line graph G in the plane on a set V of n vertices, with a subset $V_p \subseteq V$ with $|V_p| = y$.

(X)	The edges are non-crossing.
($\geq$)	There are at least $3n - y - 3$ edges.
($\leq$)	There are at most $3n - y - 3$ edges.
(P)	The vertices in V_p are pointed.
(NP)	The vertices in $V - V_p$ are nonpointed.
(Δ)	G is non-crossing and decomposes the convex hull into pseudotriangles.

   Show that the following sets of conditions are equivalent: (X)$\wedge$(P)$\wedge$($\geq$), (Δ)$\wedge$(P)$\wedge$(NP), (Δ)$\wedge$(P)$\wedge$($\geq$), (Δ)$\wedge$(NP)$\wedge$($\leq$).

   (Are there other subsets of the conditions which are equivalent to these?)

2. (3 points) Find an efficient algorithm to test whether a given polygon is a pseudotriangle [a convex polygon, a pseudoquadrangle].

3. (6 points) Suppose you have a pseudotriangulation of points which are moving. Which conditions would you check to ensure that the graph remains a valid pseudotriangulation?

   Compare the number of conditions that have to be monitored for the case of a pointed pseudotriangulation and for the case of a triangulation.

   If the graph stops being a pseudotriangulation, which updates would you make in order to restore the pseudotriangulation?

4. (4 points) Find an efficient algorithm to triangulate a pseudotriangle.

5. (*5 points) Let T be a triangulation of a point set S and let $P \subseteq T$ be a pseudotriangulation of S.

   Show that, for every interior pointed vertex v of P, one can select an edge of $T - P$ which is incident to v and lies in the reflex angle at v, in such a way that each edge of $T - P$ is selected once. (In other words, we have a *matching* between pointed vertices of P and edges of $T - P$.)

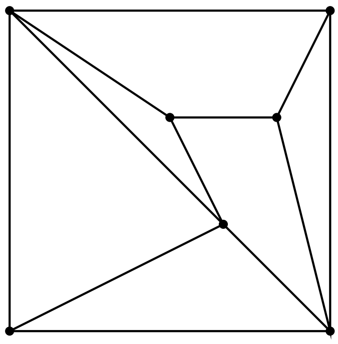
   (Is this matching unique? How many possibilities are there?)

6. (8 points) Show that a planar framework with two linearly independent self-stresses always has a self-stress for which the reciprocal is self-crossing.

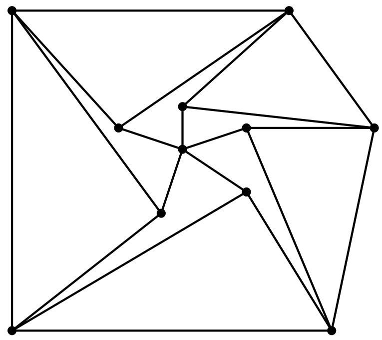
7. (4 points) A *minimal* pseudotriangulation is a pseudotriangulation for which no proper subset of the edges forms a pseudotriangulation of the same point set. Show that a minimal pseudotriangulation of $n \geq 4$ points contains at most $3n - 7$ edges.

   (Can you show an upper bound of $3n - 8$?)

8. (3 points) Construct the (essentially unique) reciprocal diagram of the following planar frameworks with a ruler alone (or with some geometry software like Cinderella).



(a)



(b)

Does the framework in (b) contain a pointed pseudotriangulation?

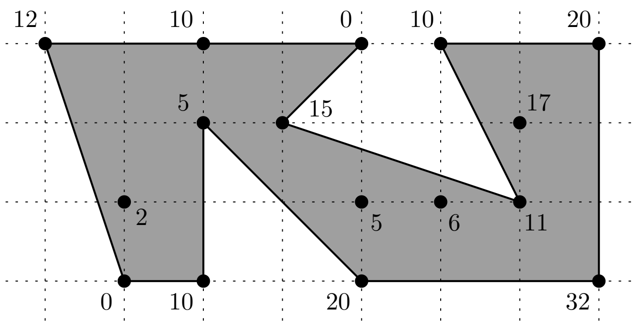
9. (12 points) Can a reciprocal of a given planar framework always be constructed with a ruler alone, i. e., by drawing parallels and by intersecting lines.

10. (5 points) Let P be a simple polygon with k convex vertices. Let Q be a convex k-gon whose vertices correspond to the convex vertices of P in cyclic order.

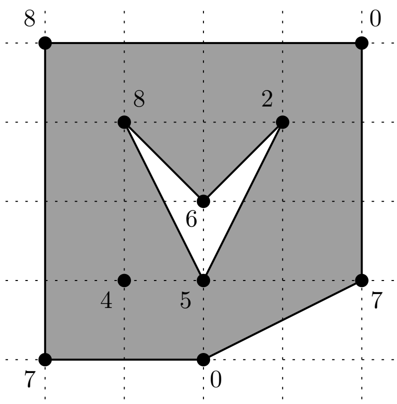
    Consider a triangulation T_Q of Q. For every interior edge of Q consider the geodesic path (the shortest path inside P) between the corresponding vertices of P. Show that the union of these geodesic paths generates a pointed pseudotriangulation inside P.

    (Can every pointed pseudotriangulation of P be generated in this way?)

11. (3 points) Find the highest locally convex functions over the following polygonal regions which remain below the given values at the marked points.



(a)



(b)

12. (10 points) Formulate the problem of finding the highest locally convex function over a polygonal domain, subject to upper bounds on the values at certain points, as a linear programming problem.

13. (12 points) Set up the dual linear programming problem and find a probabilistic, geometric, mechanical, or other intuitive interpretation for it.

14. (4 points) A line ℓ through a vertex v of a polygon t is called *tangent* at v if

    (a) v is a corner of t and ℓ crosses the boundary of t at v from the interior to the exterior, or

    (b) v is a reflex vertex of t and ℓ does not cross the boundary of t at v, or

(c) ℓ goes through one of the edges incident to v. (This is the limit case of the other two cases.)

Show that, for a pseudotriangle t,

(a) through every interior point of t, there are exactly three tangent lines, and

(b) through every point exterior to t, there is exactly one tangent line.

What are corresponding statements for pseudoquadrangles?

15. (*3 points) A *bitangent* of a polygon t is a line ℓ which is tangent to t at two positions, in the sense of the previous exercise (where the two adjacent vertices in case (14c) count only as one tangency.)

    Show that a pseudoquadrangle has always exactly two bitangents.

16. (For the mathematically inclined) Consider the system of equations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} M\binom{x}{y} \\ b \end{pmatrix} \tag{1}$$

for a vector $\binom{x}{y} \in \mathbb{R}^{n+m}$, where M is a nonnegative $n \times (n+m)$ matrix with row sums 1, and $b \in \mathbb{R}^m$. The solution vector $\binom{x}{y}$ is called a (discrete) harmonic function. We can use the fixed-point iteration

$$x^{(i+1)} := M \begin{pmatrix} x^{(i)} \\ b \end{pmatrix} \tag{2}$$

to solve the system or to prove properties of the solution.

(a) (5 points) Show that the system (1) always has a solution.

(b) (6 points) Find conditions for M under which the iteration (2) converges. (What can be said about the speed of convergence?)

(c) (7 points) Under which conditions on M is the solution of (1) unique? (Hint: Consider M as the adjacency matrix of a directed graph.)

(d) (7 points) Show that the iteration (2) converges if (1) has a unique solution.

(e) (*6 points) The maximum principle for harmonic functions.
Suppose that (1) has a unique solution. Show that no entry of the vector x is larger than the largest entry of y.

(f) (5 points) Regard the entries of M as probabilities and find an interpretation of the problem (1) in terms of a random walk.

(g) (*4 points) Show that, when the solution x of (1) is unique, it depends monotonically on the data b. (You may use part (d).)

(h) (*4 points) Suppose that (1) has a unique solution x^*, and let $x^{(0)}$ be a vector which fulfills (2) as an inequality:

$$x^{(0)} \geq M \begin{pmatrix} x^{(0)} \\ b \end{pmatrix}$$

Show that $x^* \leq x^{(0)}$.

17. (3 points) Show that the piecewise maximum of two locally convex functions over the same domain is locally convex.