

# On Primal-Dual Circle Packings



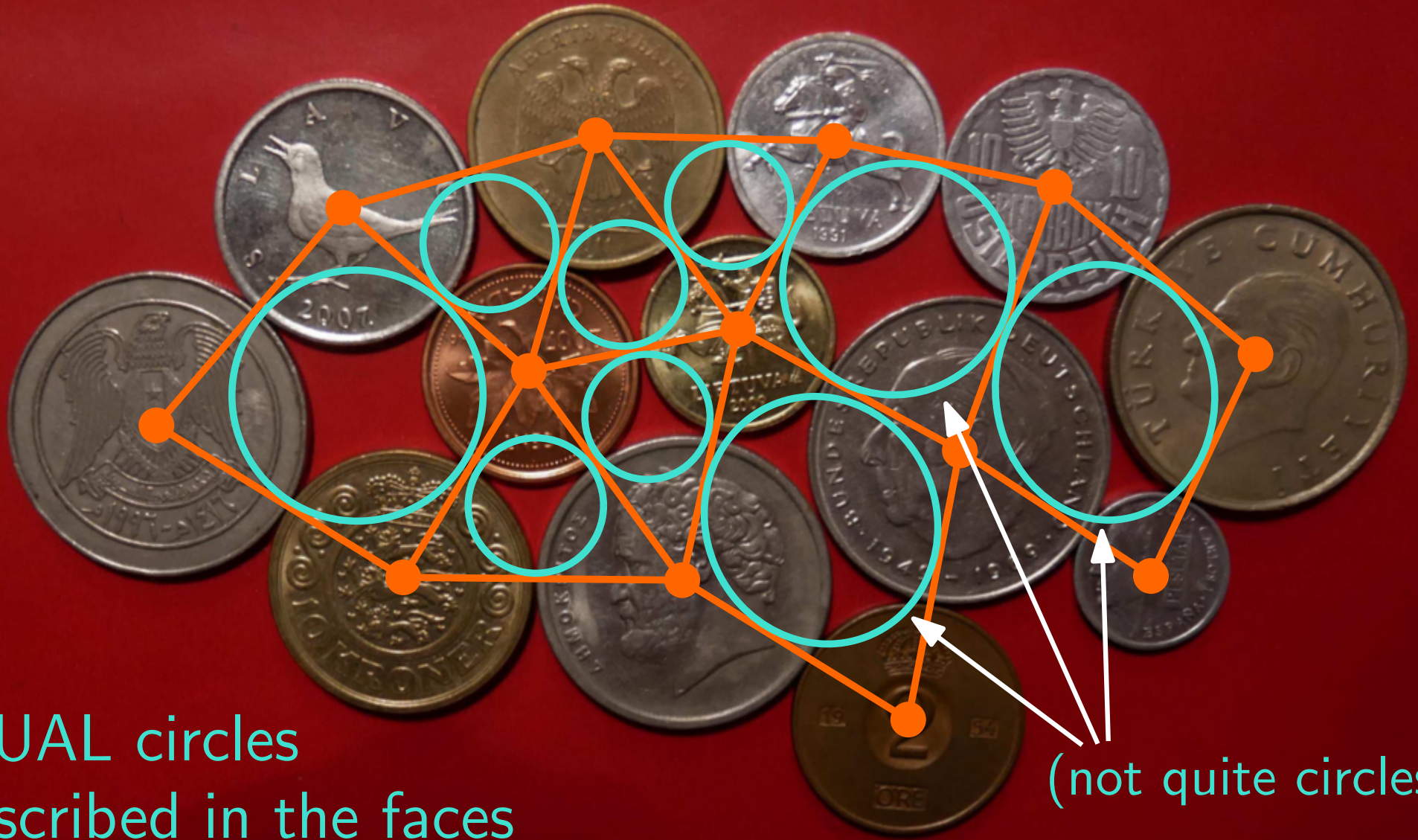
Stefan Felsner + Günter Rote

# On Primal-Dual Circle Packings



Stefan Felsner + Günter Rote

# On Primal-Dual Circle Packings

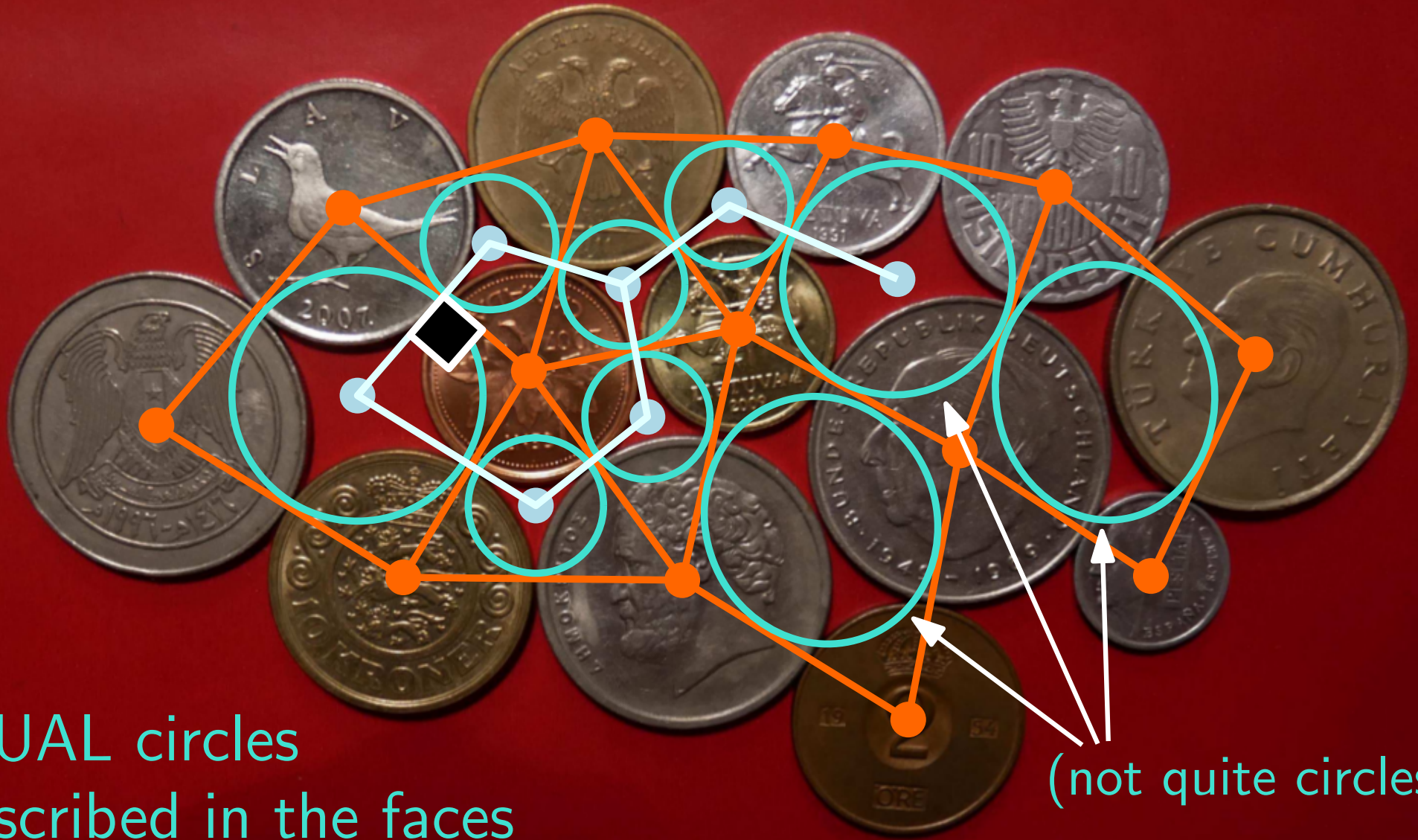


DUAL circles  
inscribed in the faces

(not quite circles)

Stefan Felsner + Günter Rote

# On Primal-Dual Circle Packings



DUAL circles  
inscribed in the faces

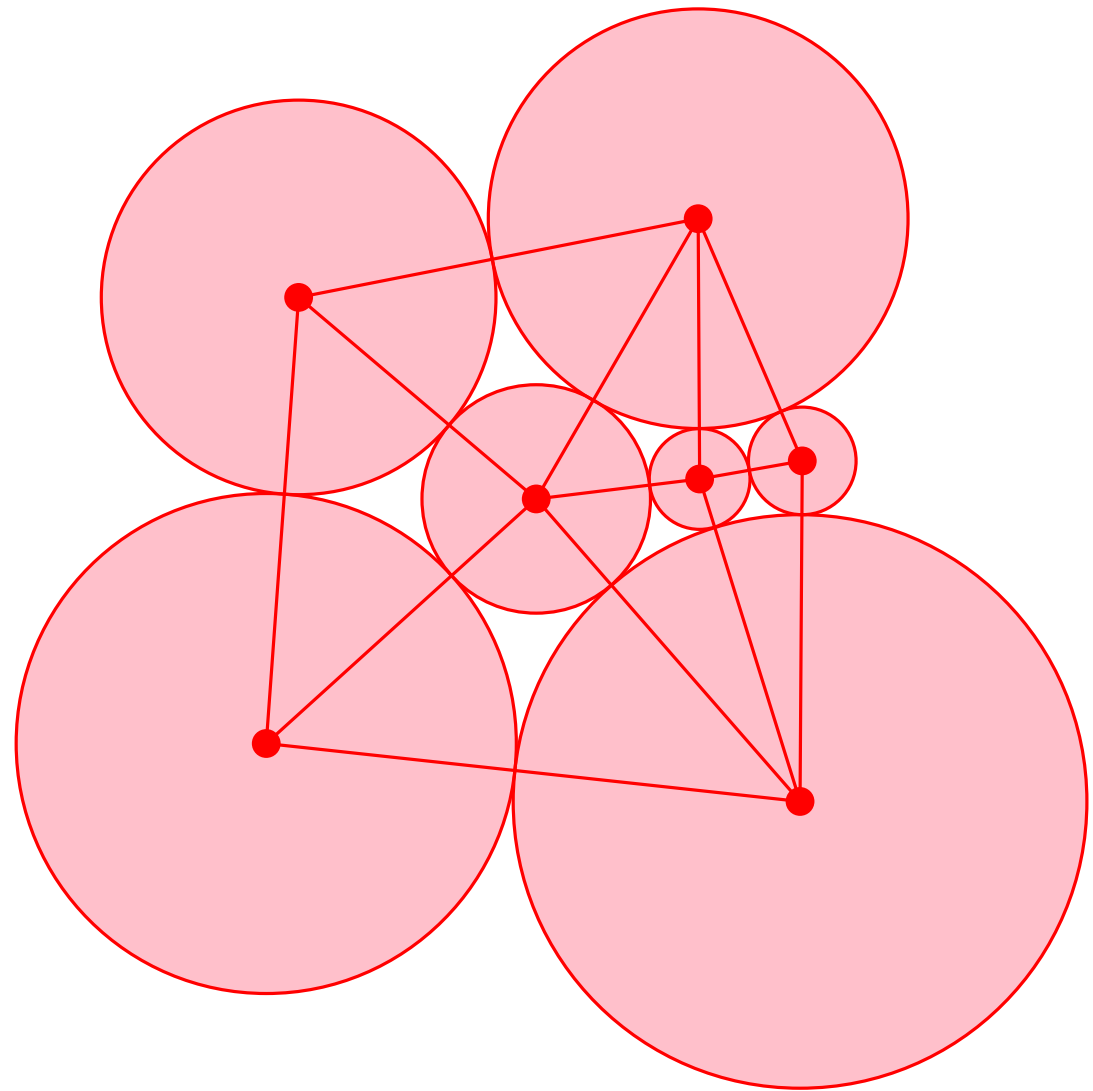
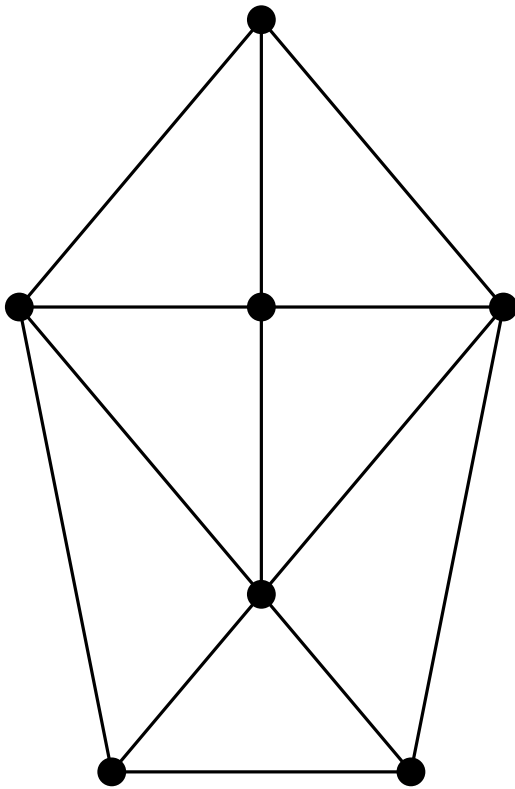
(not quite circles)

Stefan Felsner + Günter Rote

# Primal-Dual Circle Representation

Input:

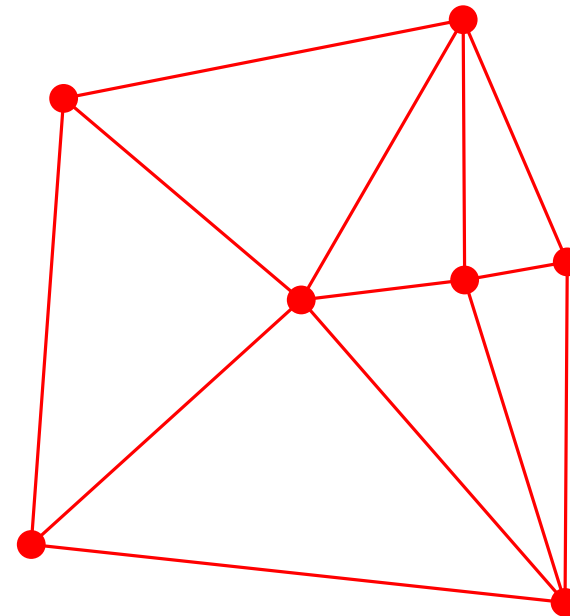
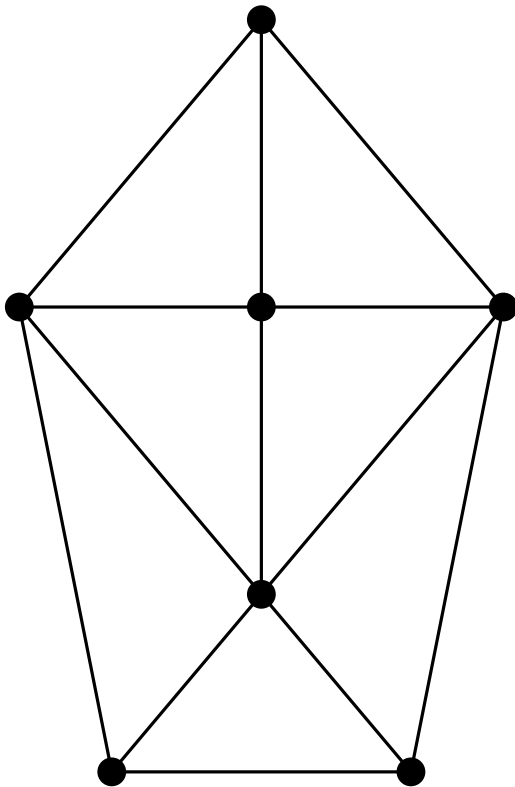
3-connected plane graph



# Primal-Dual Circle Representation

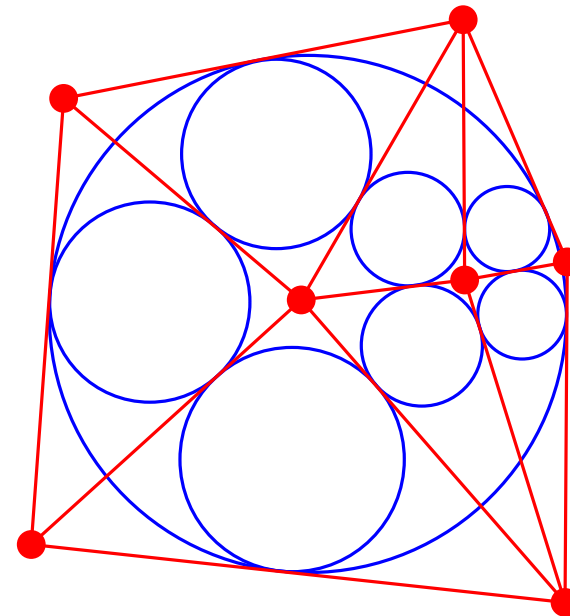
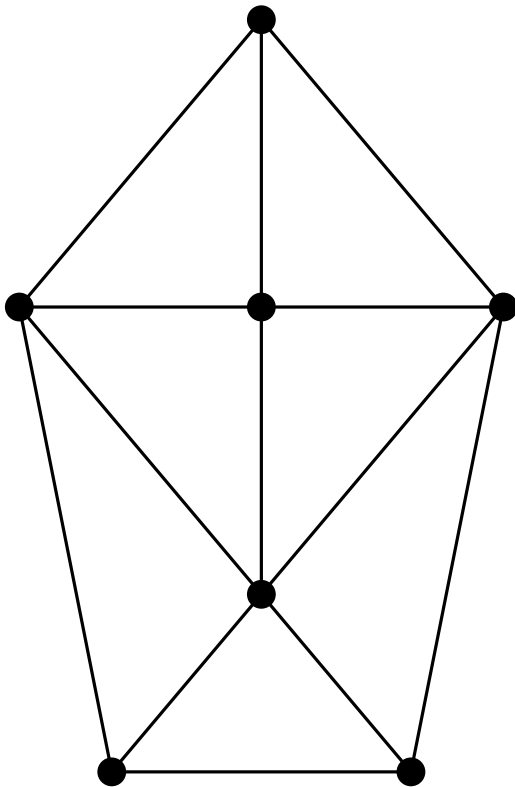
Input:

3-connected plane graph



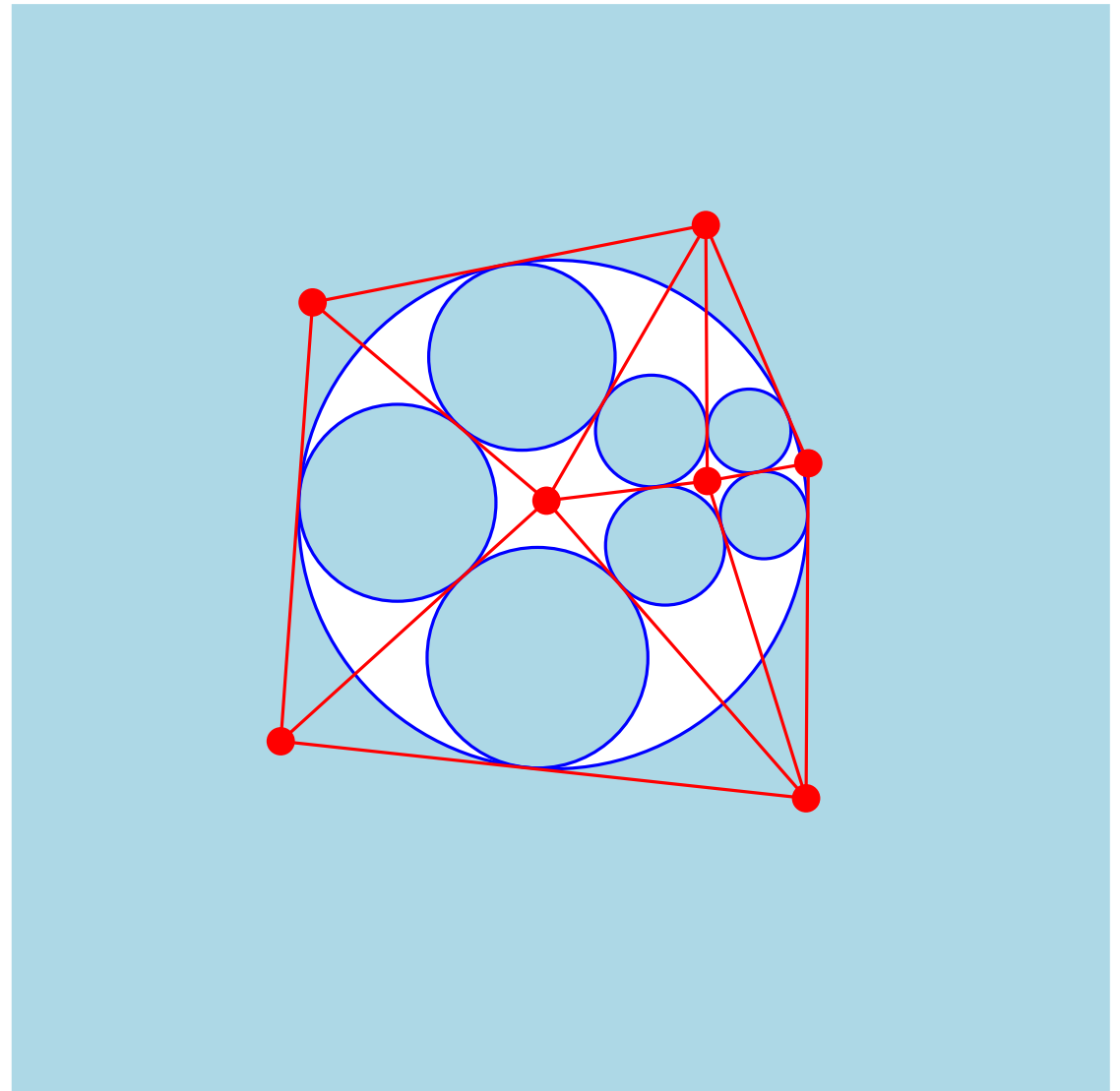
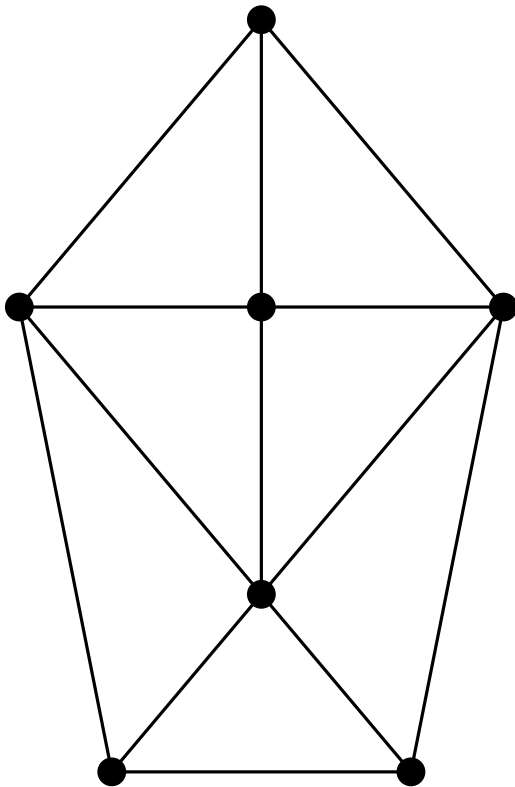
# Primal-Dual Circle Representation

Input:  
3-connected plane graph



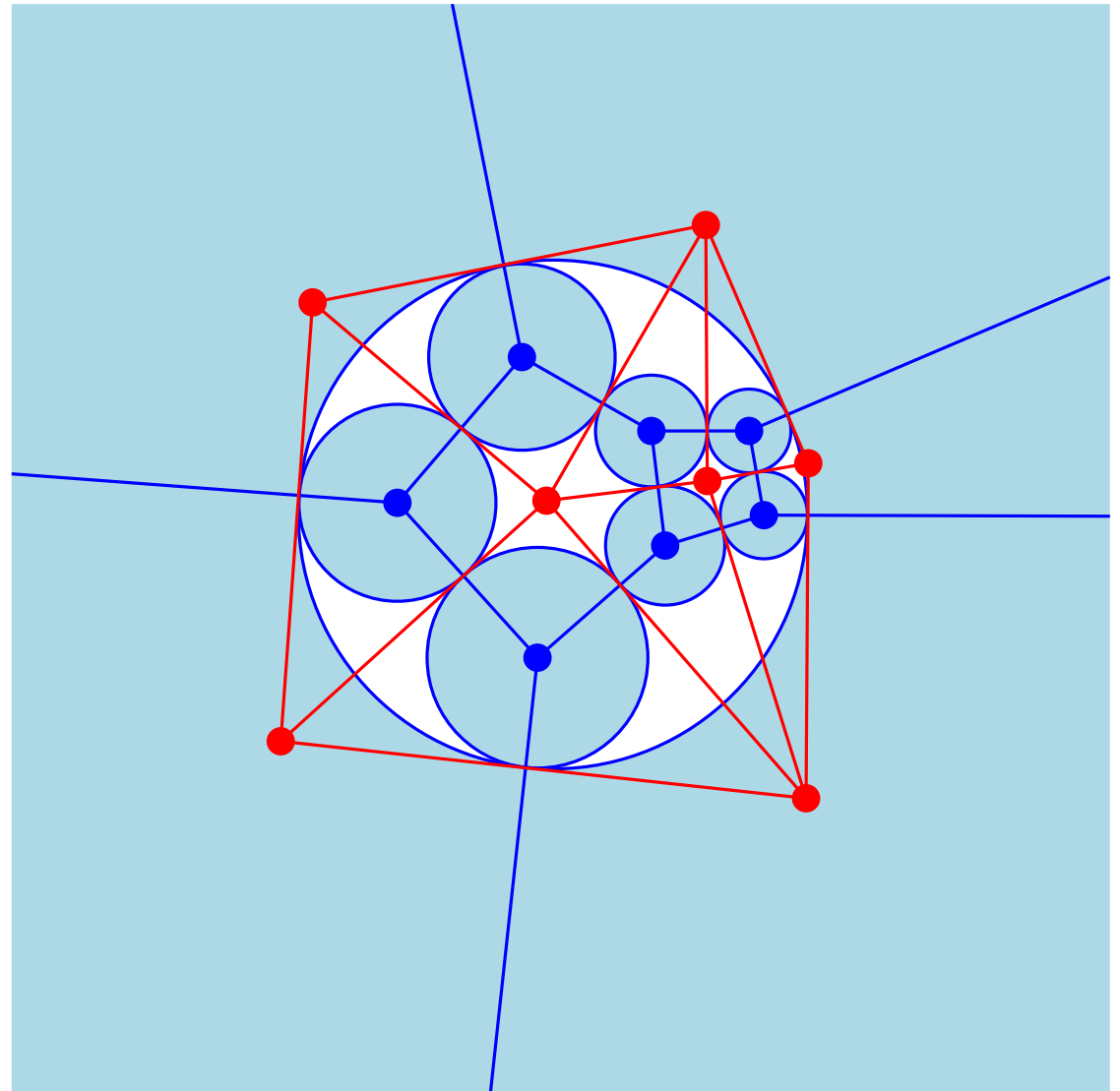
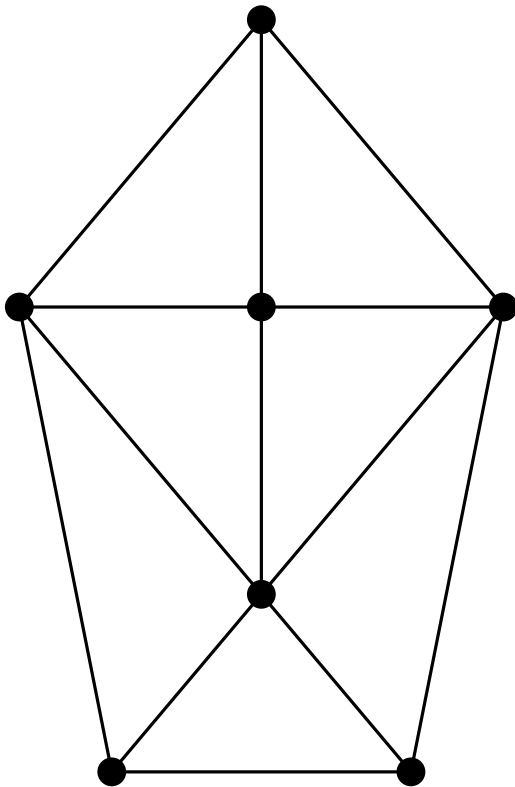
# Primal-Dual Circle Representation

Input:  
3-connected plane graph

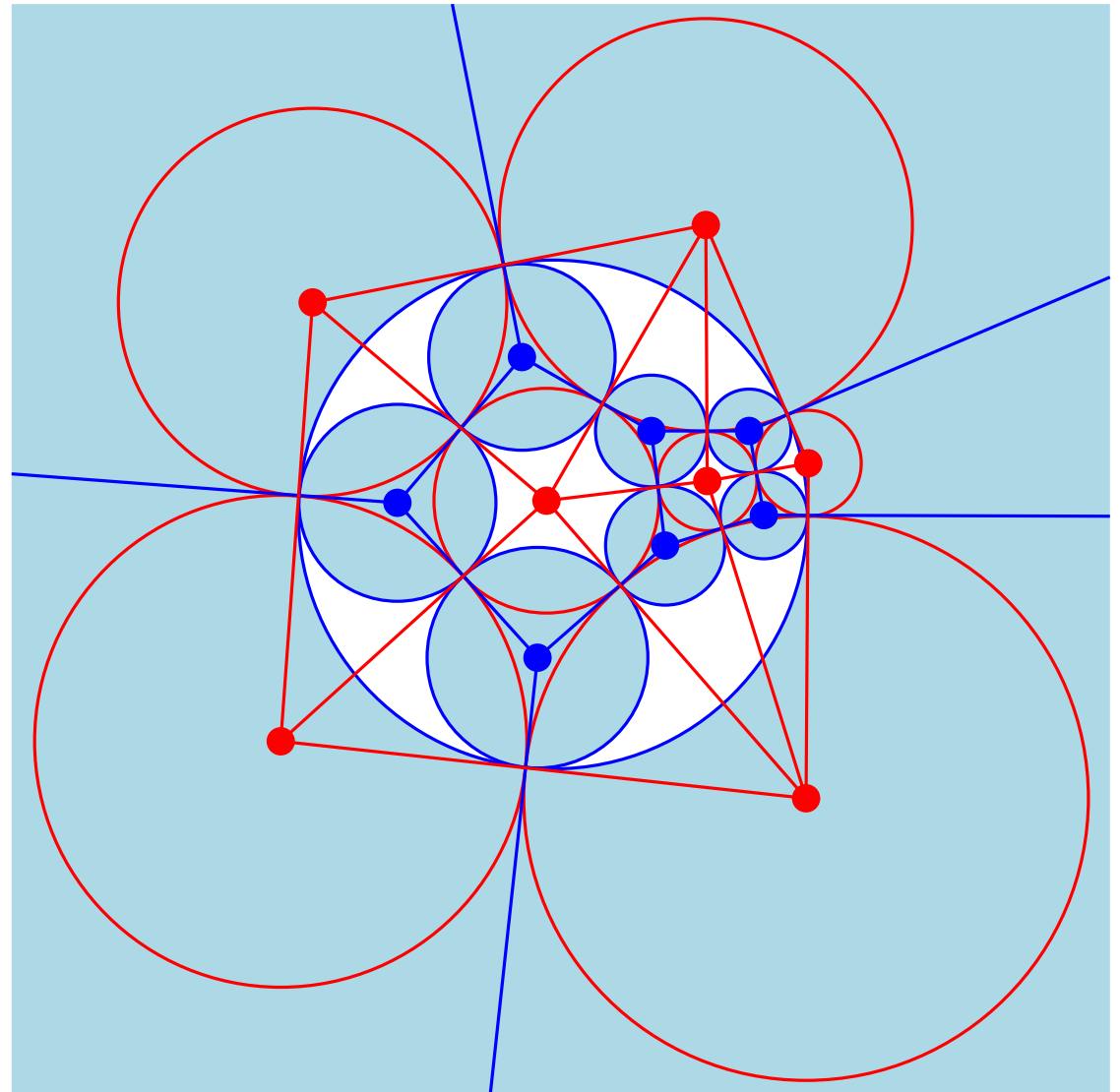
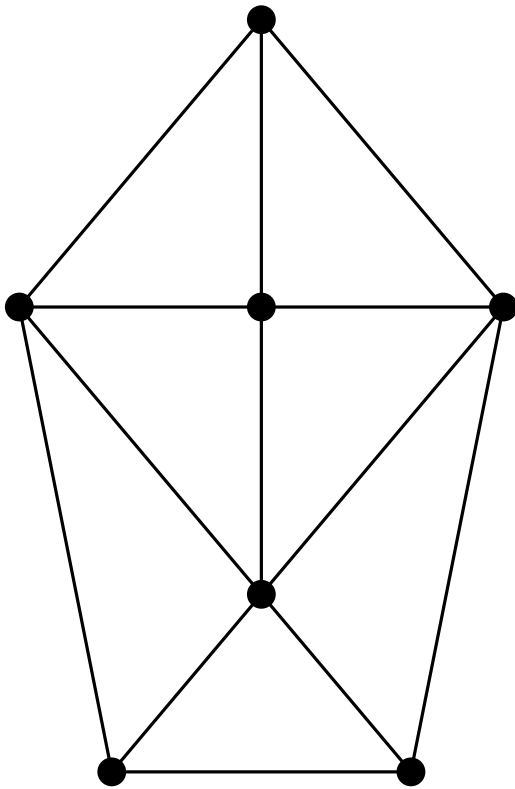


# Primal-Dual Circle Representation

Input:  
3-connected plane graph



Input:  
3-connected plane graph

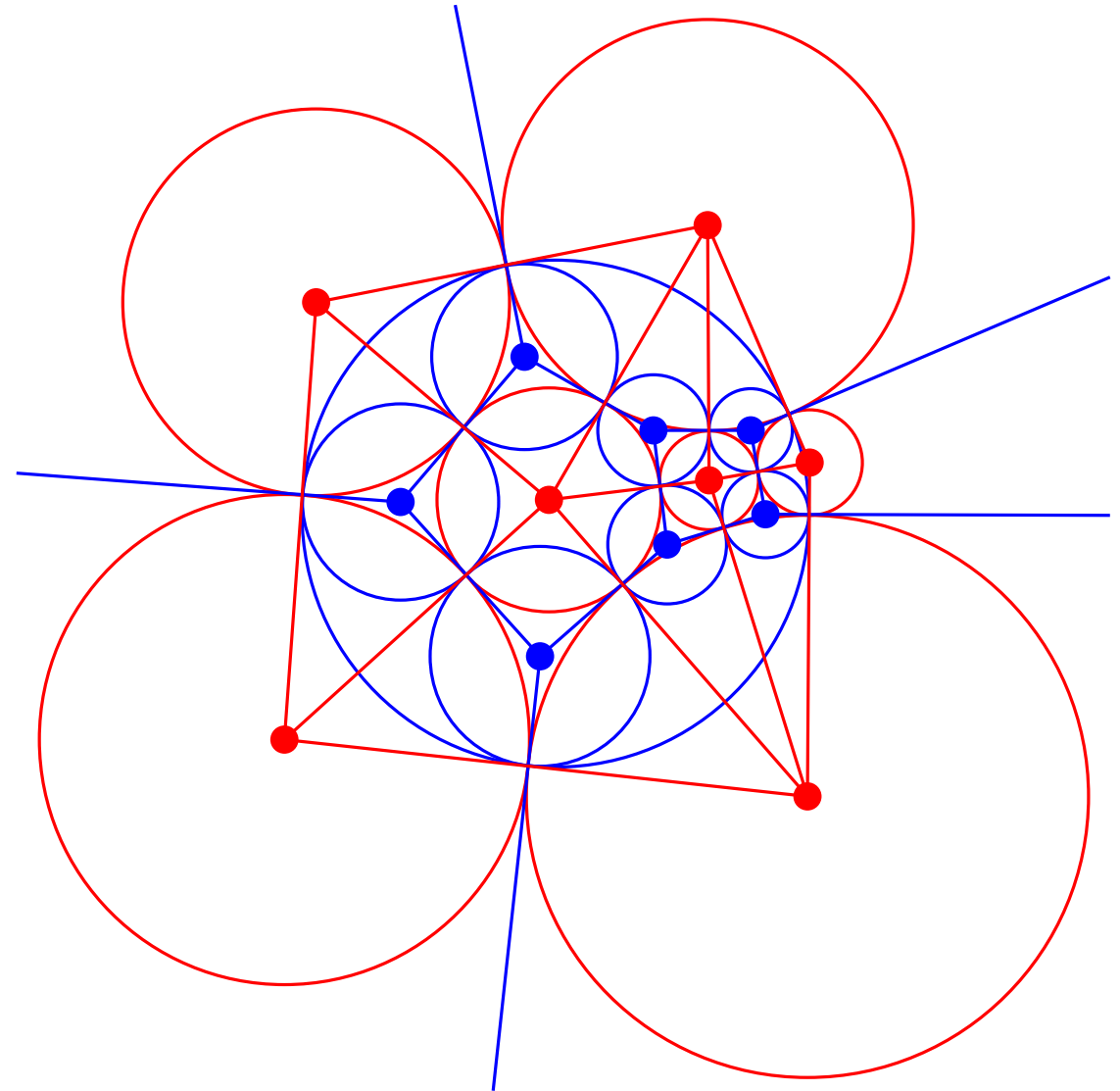
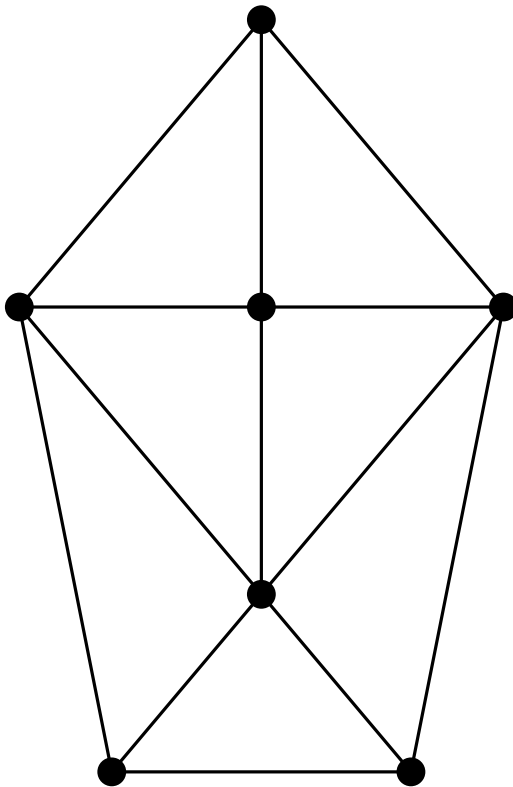


simultaneous “circle-packing” representation of the primal and the dual graph

# Primal-Dual Circle Representation

Input:

3-connected plane graph



simultaneous “circle-packing” representation of the primal and the dual graph

## THEOREM.

Every 3-connected planar graph has a primal-dual circle representation. It is unique up to Möbius transformations.

Primal-only: *Koebe–Andreyev–Thurston Theorem*  
1936      1971      ~1980

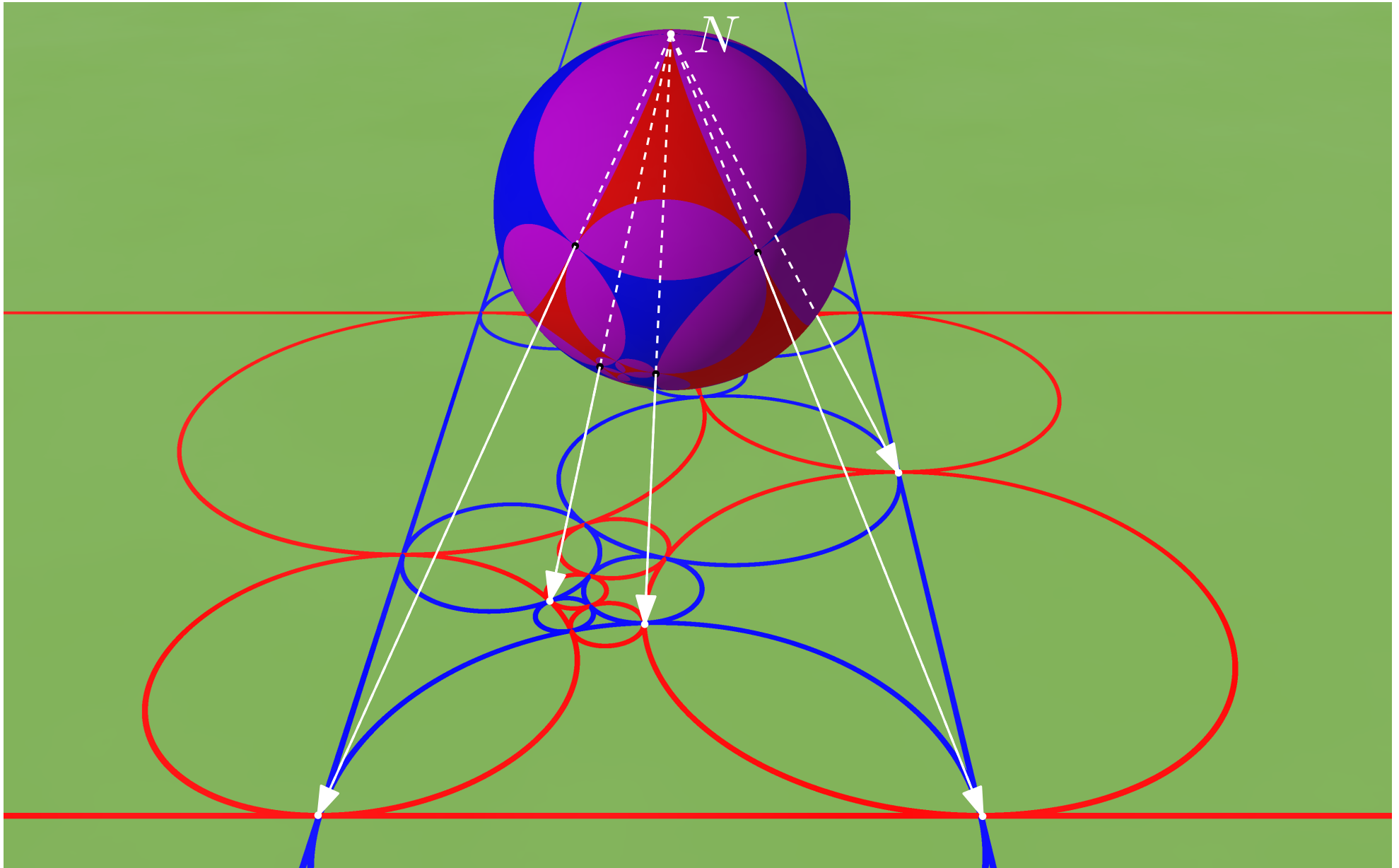
Primal-dual: Brightwell and Scheinerman 1993

Here: Streamlined complete and elementary proof.  
follows Brightwell & Scheinerman, Pulleyblank & Rote [unpubl.]

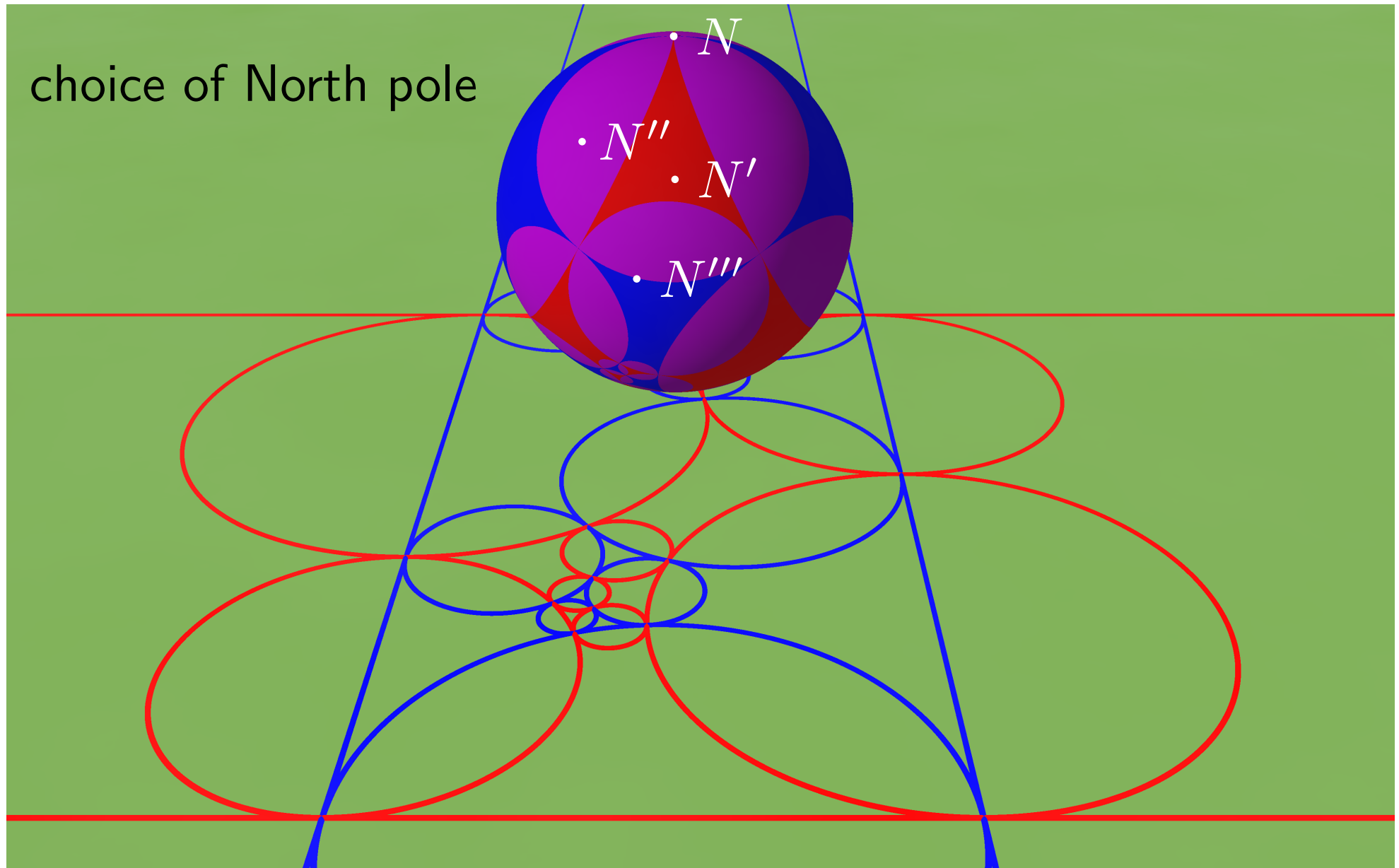
Other proofs: existential; minimum of a convex functional

Mohar [1997, 2000]: Polynomial-time approximation algorithm

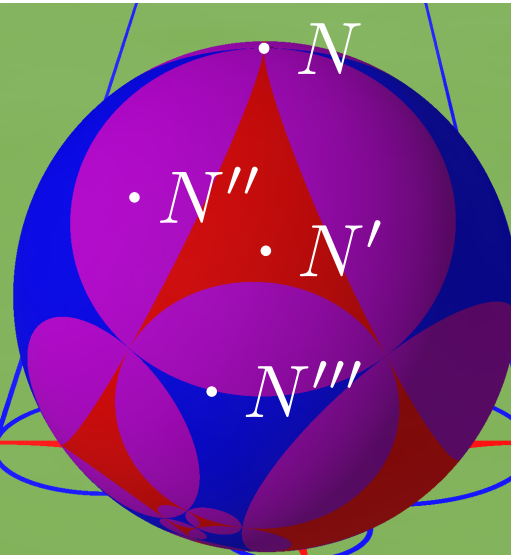
# Stereographic Projection to the Sphere



# Stereographic Projection to the Sphere



choice of North pole

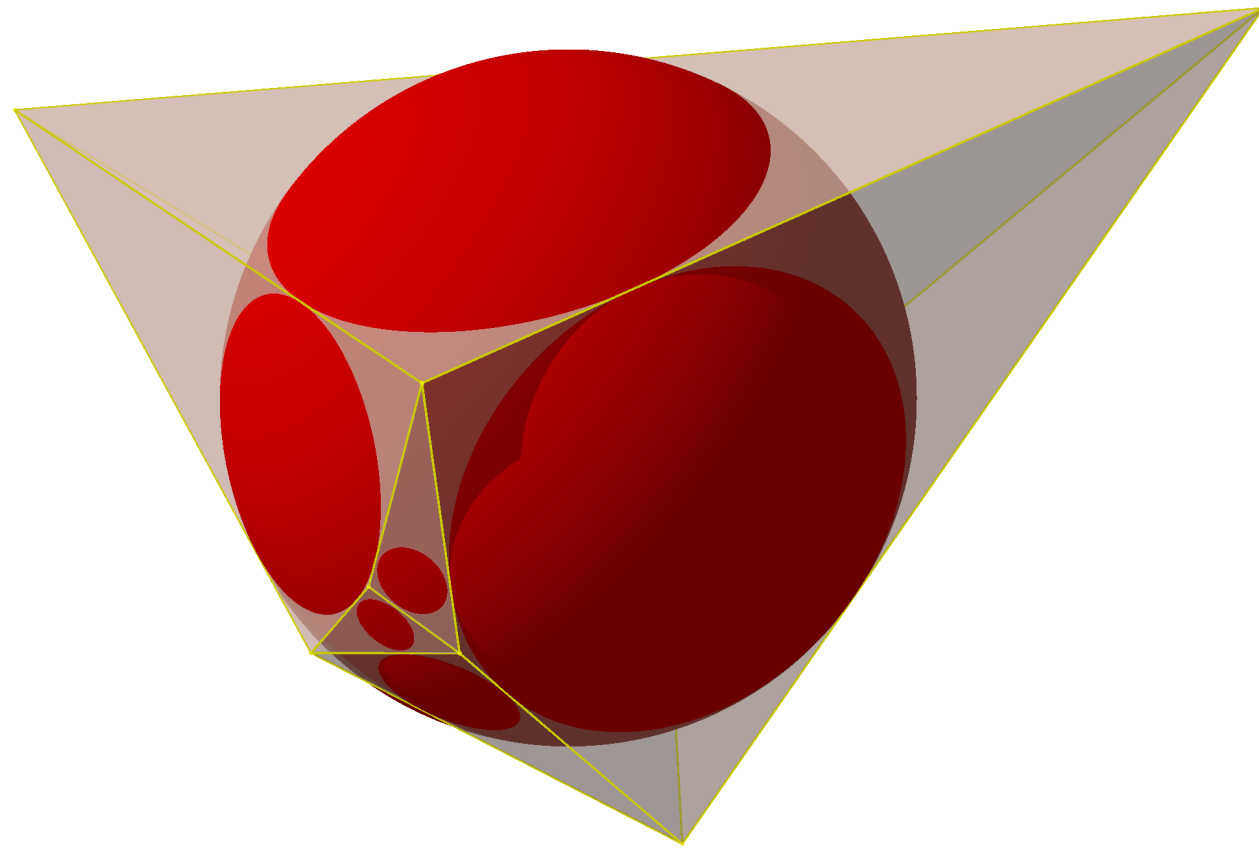
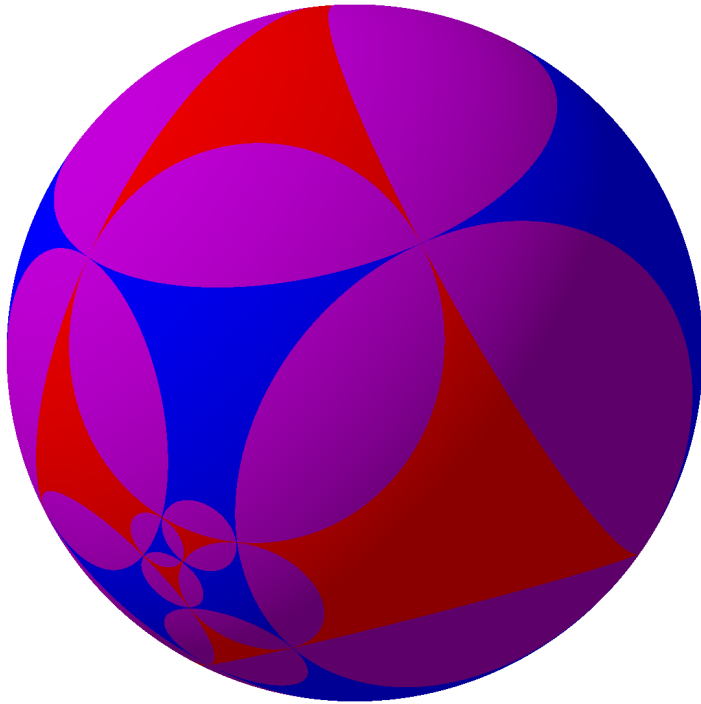


$N$  = special choice:  
where a primal and  
a dual circle cross

- Generalizations
  - *Contact representations*, other shapes
  - Other conditions than *touching*
- Applications
  - Planar separator theorem
  - *Midscribed* polytopes ←
  - Conformal mappings
  - Graph Drawing
- Proof overview
  - Kites
  - Iterative algorithm
  - Convergence
  - Layout

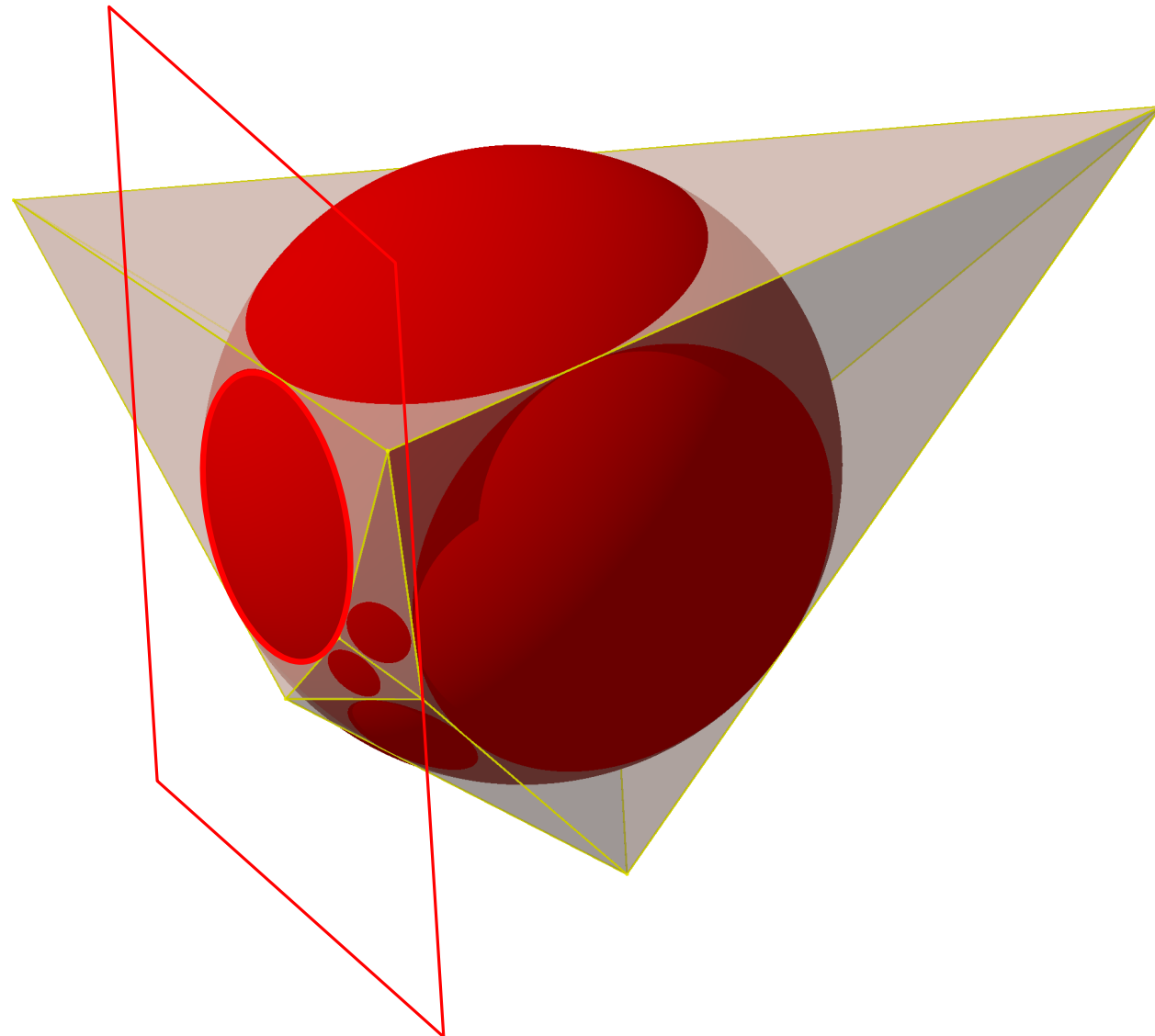
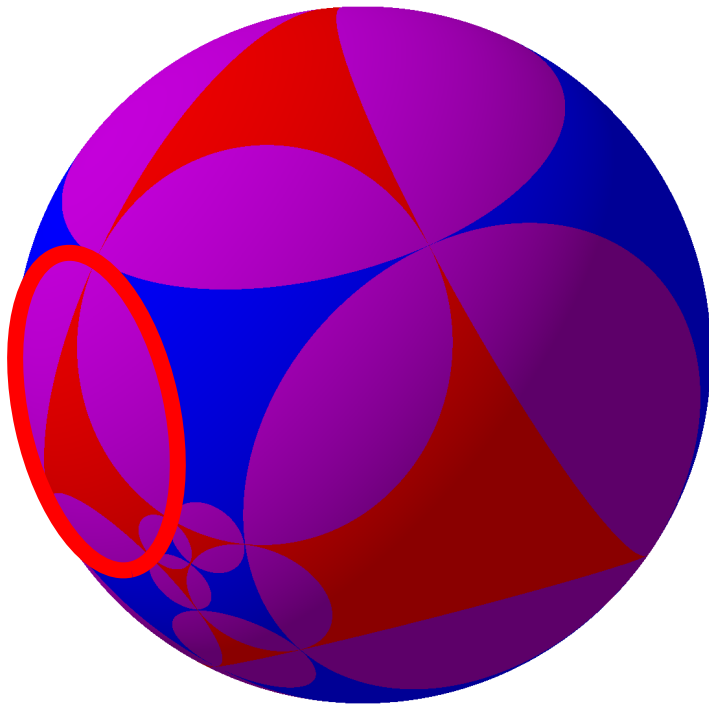
# Midscribed Polytopes

Every *edge* is tangent to the sphere.  
(As opposed to *circumscribed* or *inscribed*)



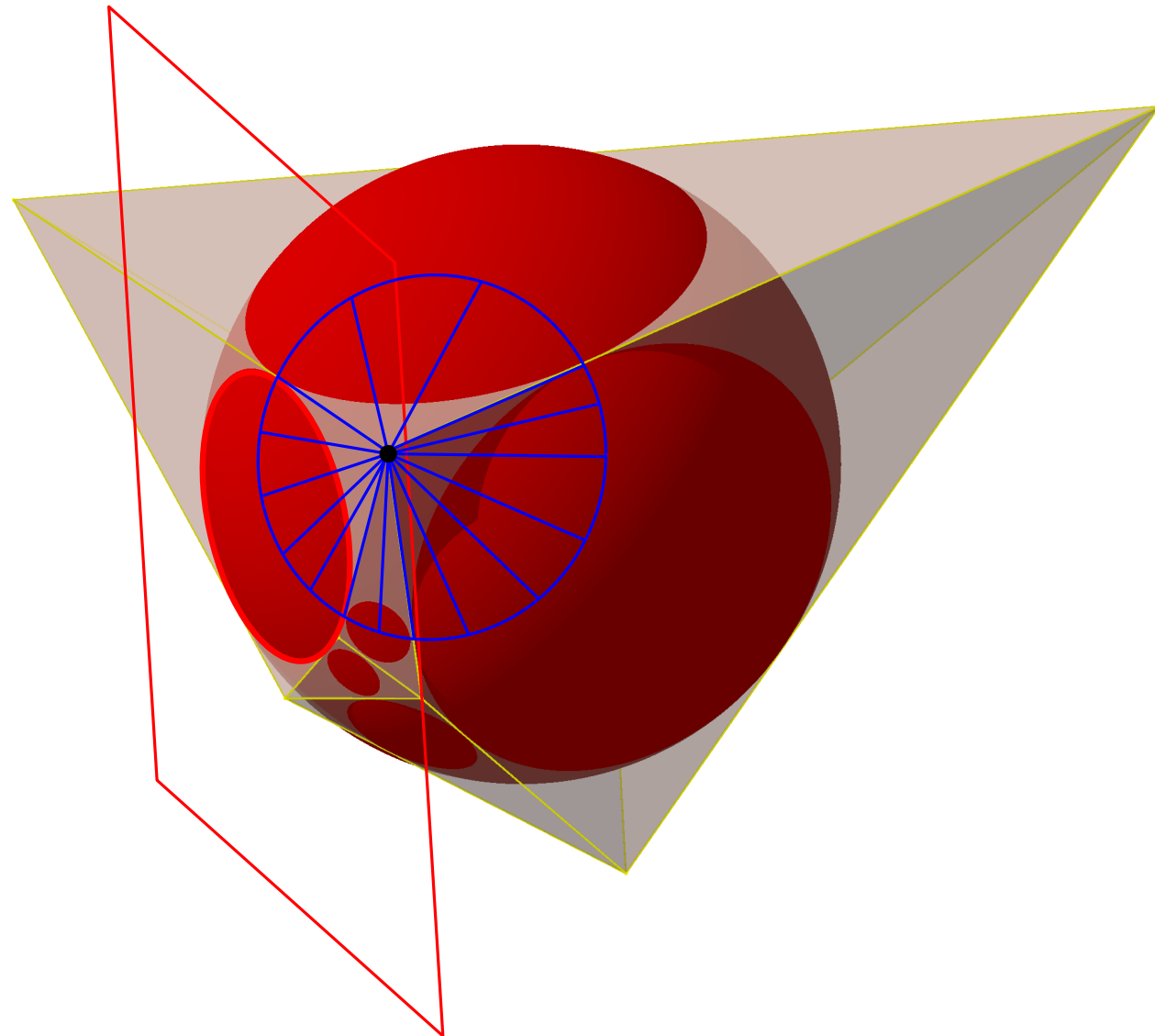
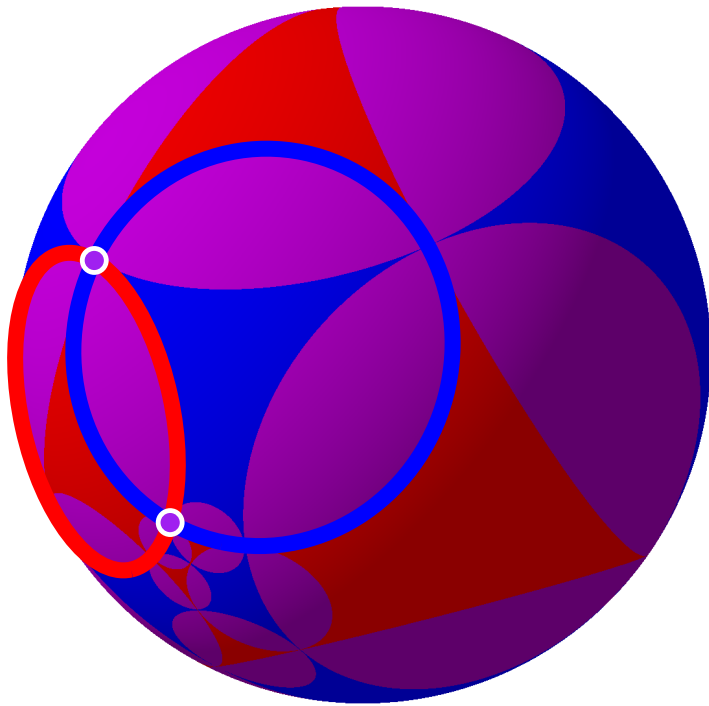
# Midscribed Polytopes

Every *edge* is tangent to the sphere.  
(As opposed to *circumscribed* or *inscribed*)

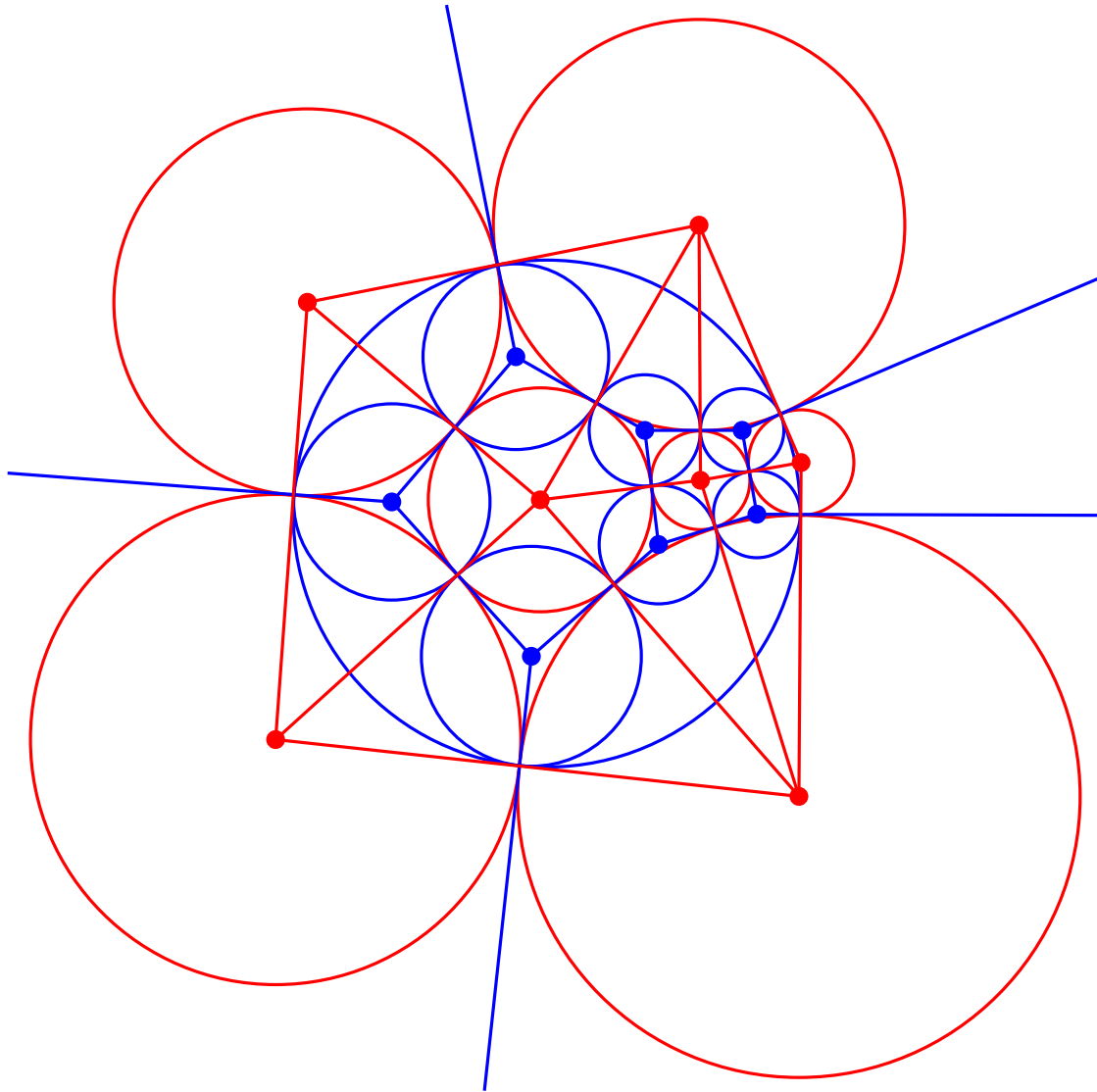


# Midscribed Polytopes

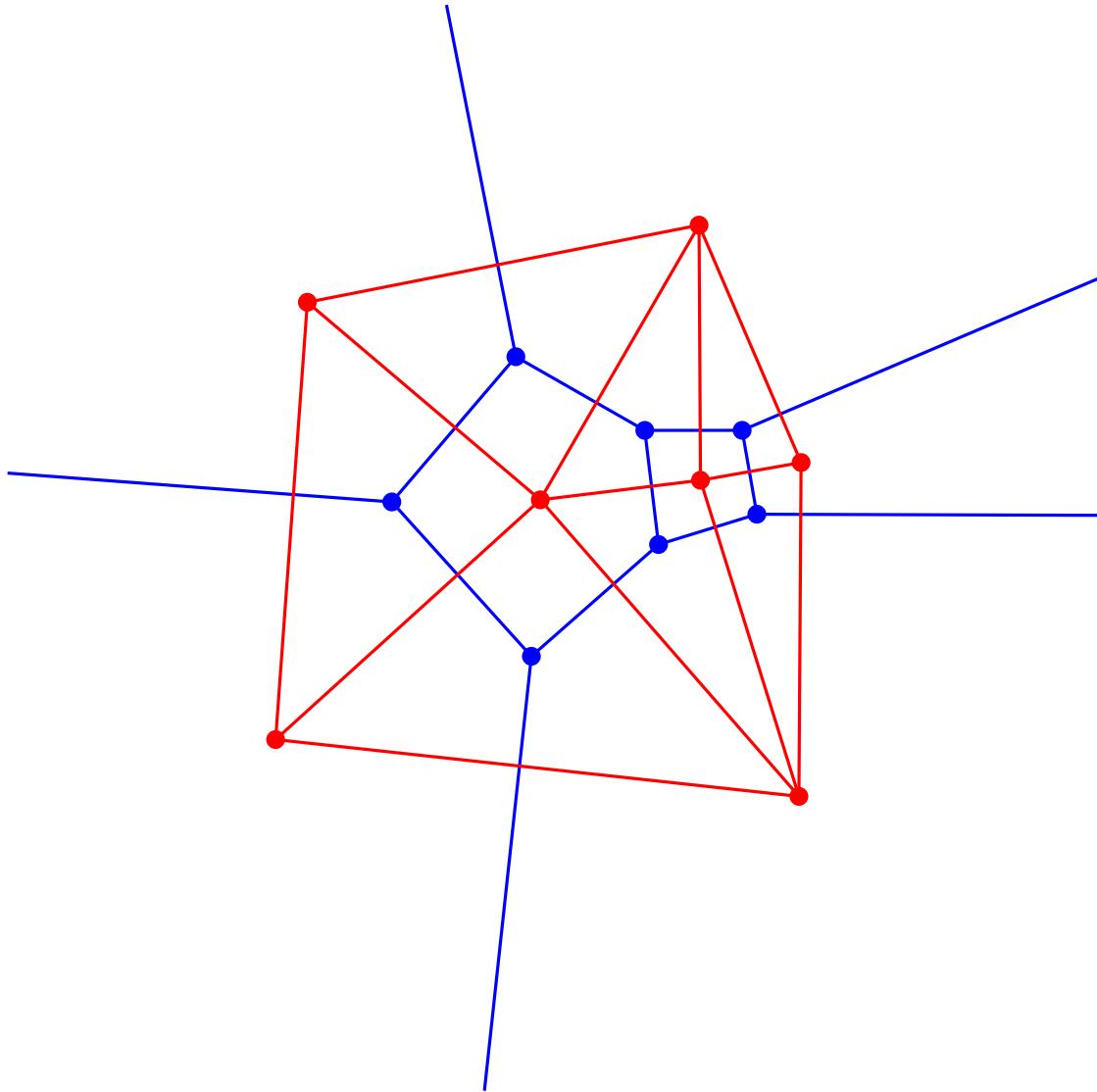
Every *edge* is tangent to the sphere.  
(As opposed to *circumscribed* or *inscribed*)



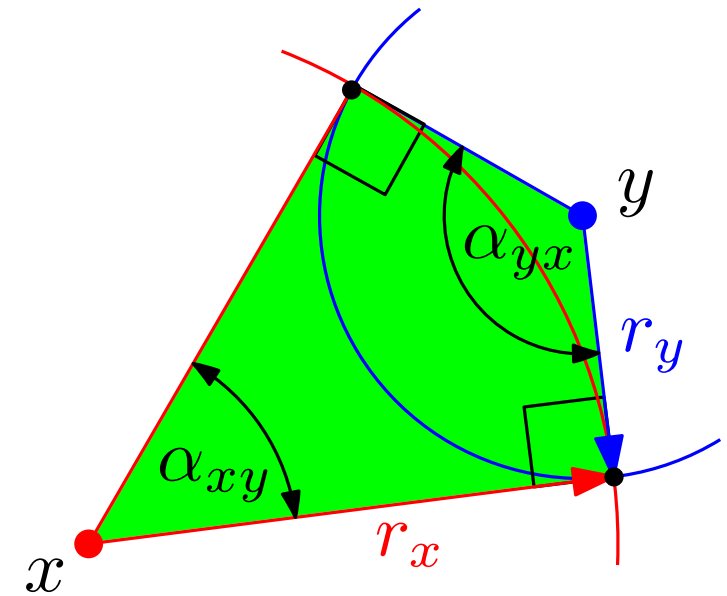
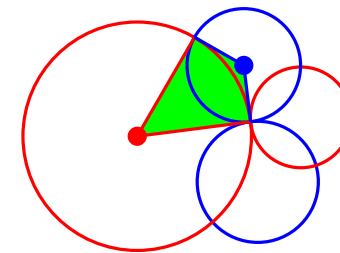
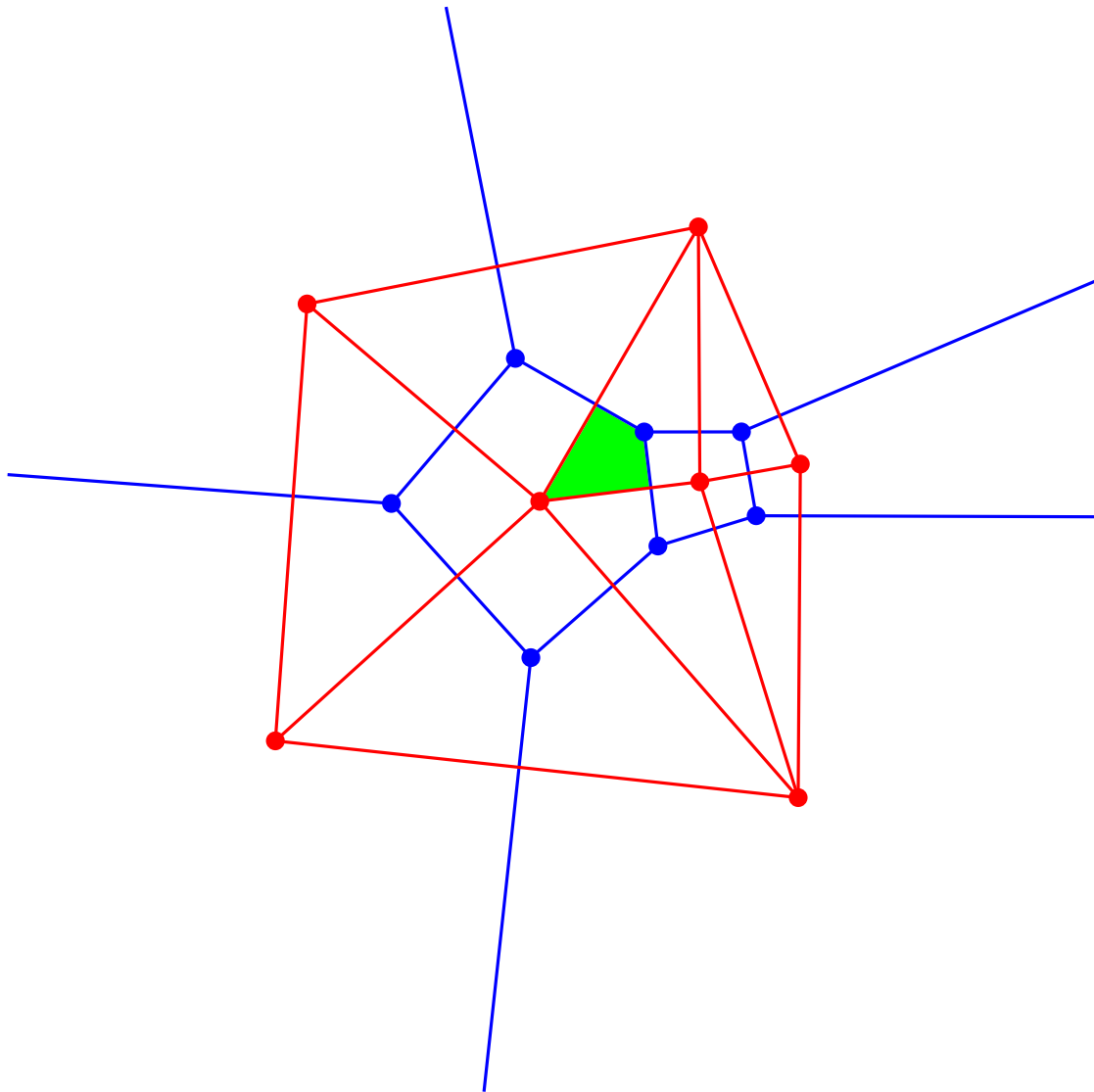
# Building block: a kite



# Building block: a kite



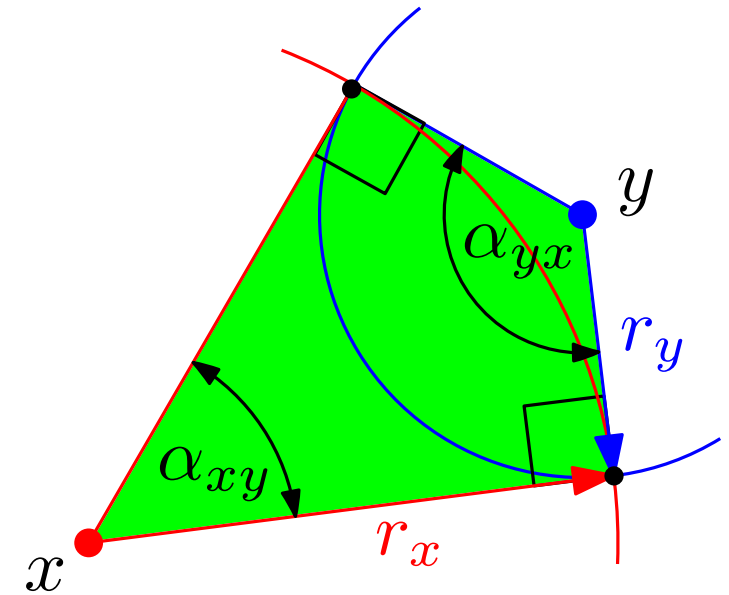
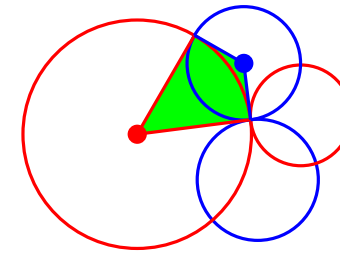
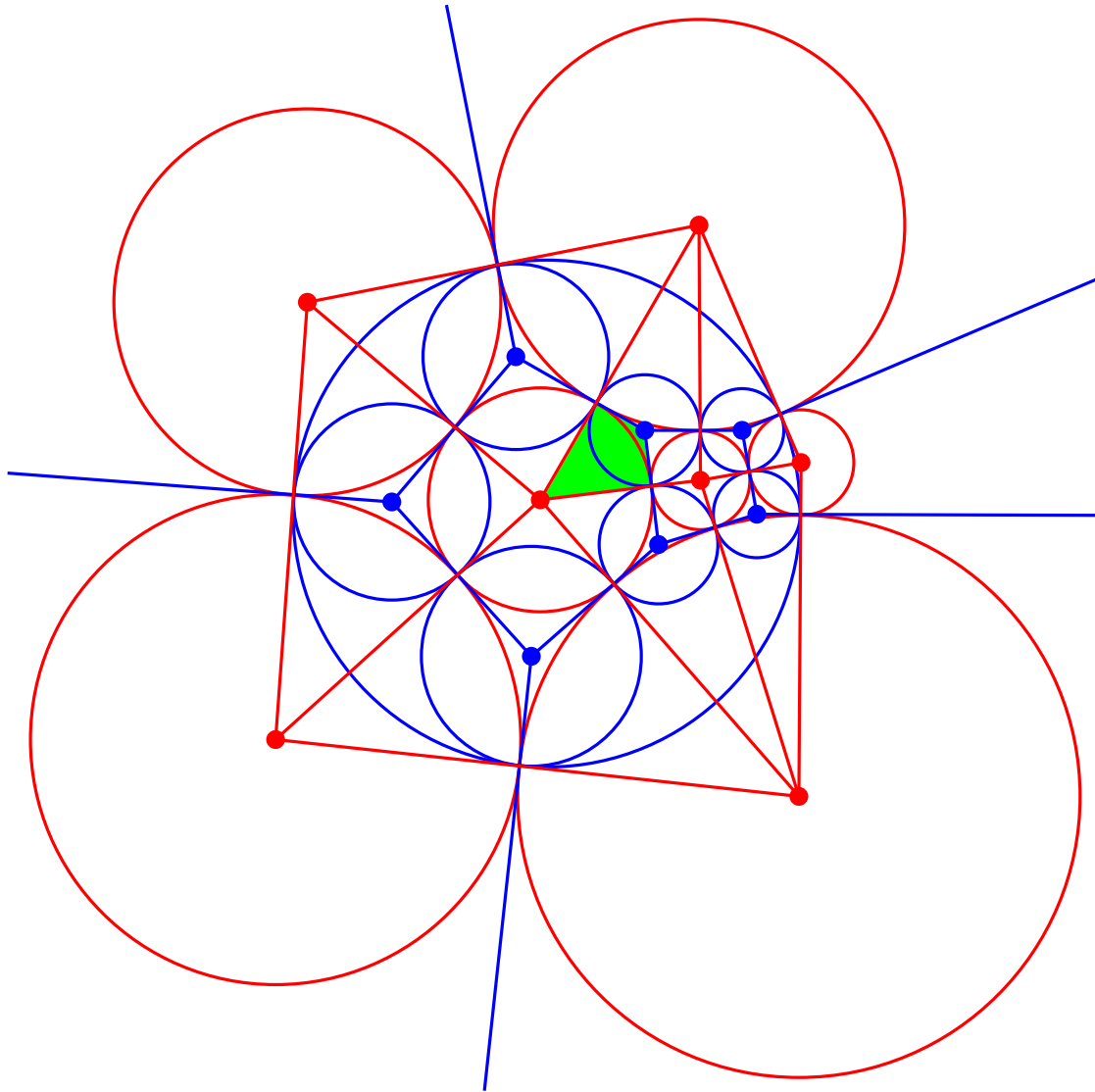
# Building block: a kite



$$\alpha_{xy} = 2 \arctan \frac{r_y}{r_x}$$

$$\alpha_{xy} + \alpha_{yx} = \pi$$

# Building block: a kite



$$\alpha_{xy} = 2 \arctan \frac{r_y}{r_x}$$

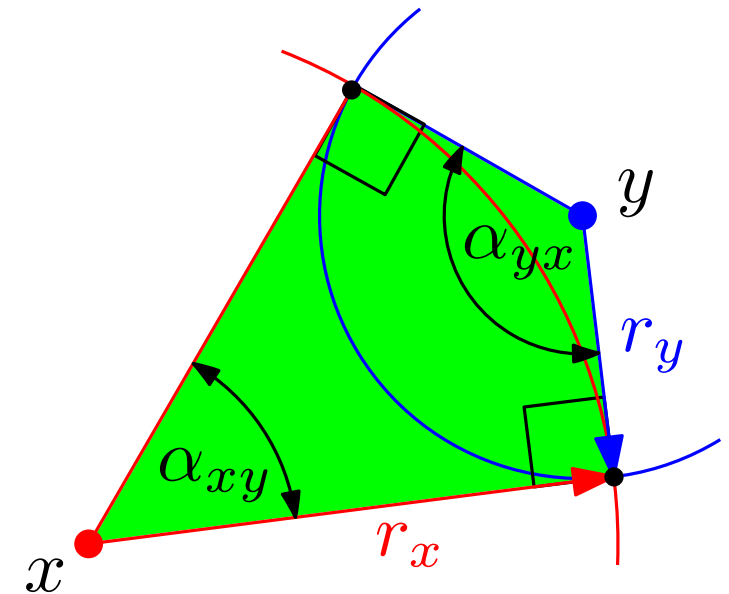
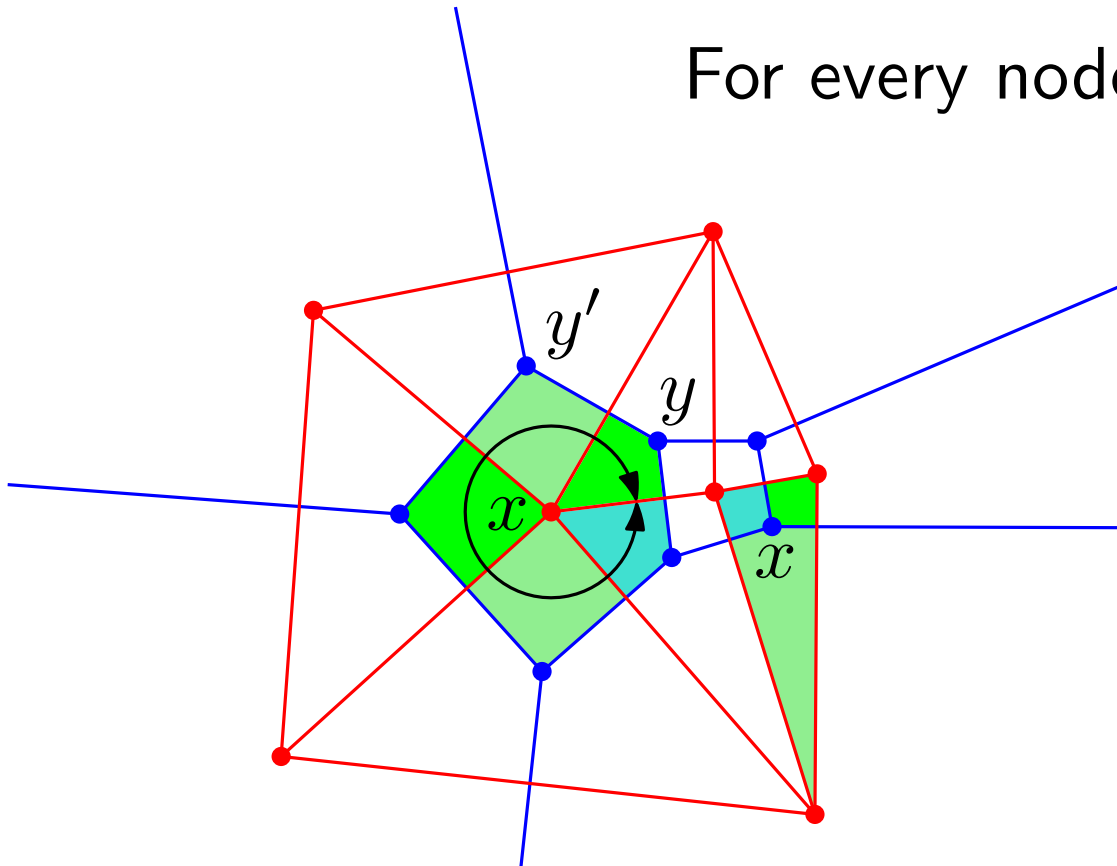
$$\alpha_{xy} + \alpha_{yx} = \pi$$

# Kites must close around every vertex

For every node  $x$ :

$$\sum_{y:y\sim x} \alpha_{xy} = 2\pi \quad (*)$$

$\underbrace{\hspace{10em}}_{\alpha_x}$



$$\alpha_{xy} = 2 \arctan \frac{r_y}{r_x}$$

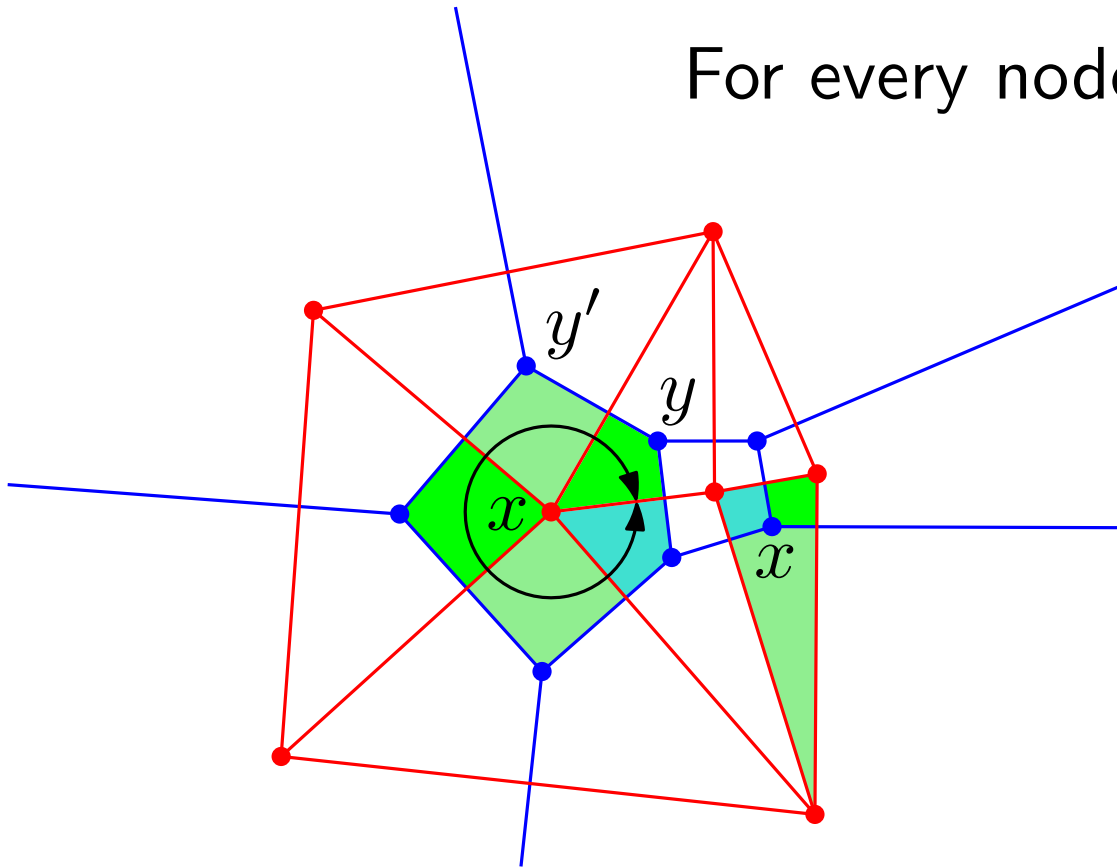
$$\alpha_{xy} + \alpha_{yx} = \pi$$

# Kites must close around every vertex

For every node  $x$ :

$$\sum_{y:y\sim x} \alpha_{xy} = 2\pi \quad (*)$$

$\underbrace{\hspace{10em}}_{\alpha_x}$

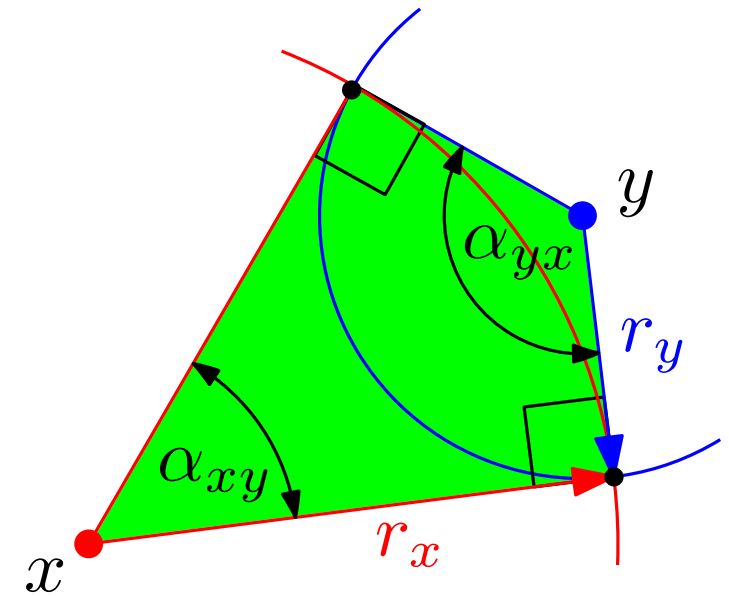


Thurston's Algorithm:

For each node  $x$ :

adjust  $r_x$  so that  $(*)$  is fulfilled

repeat till convergence

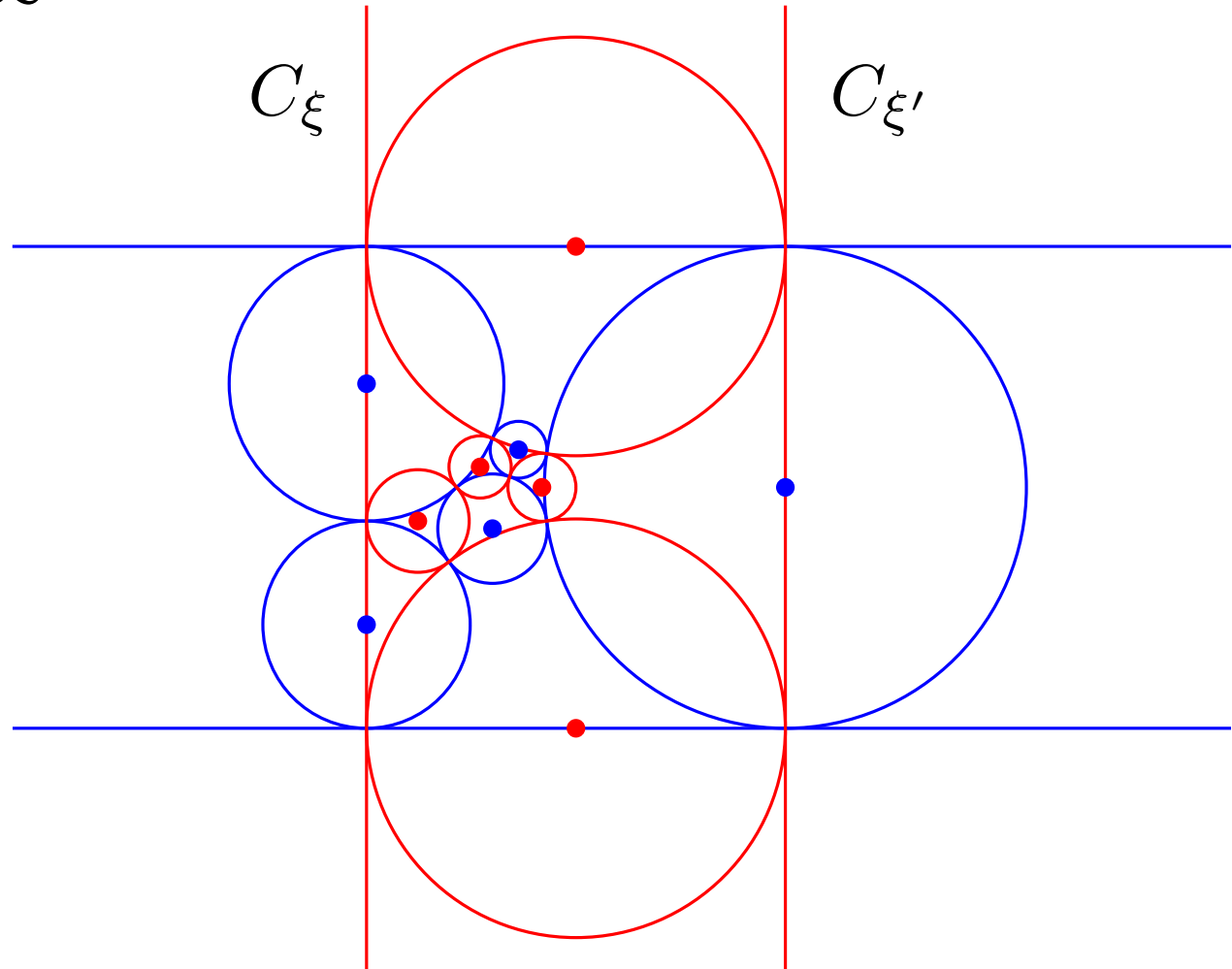
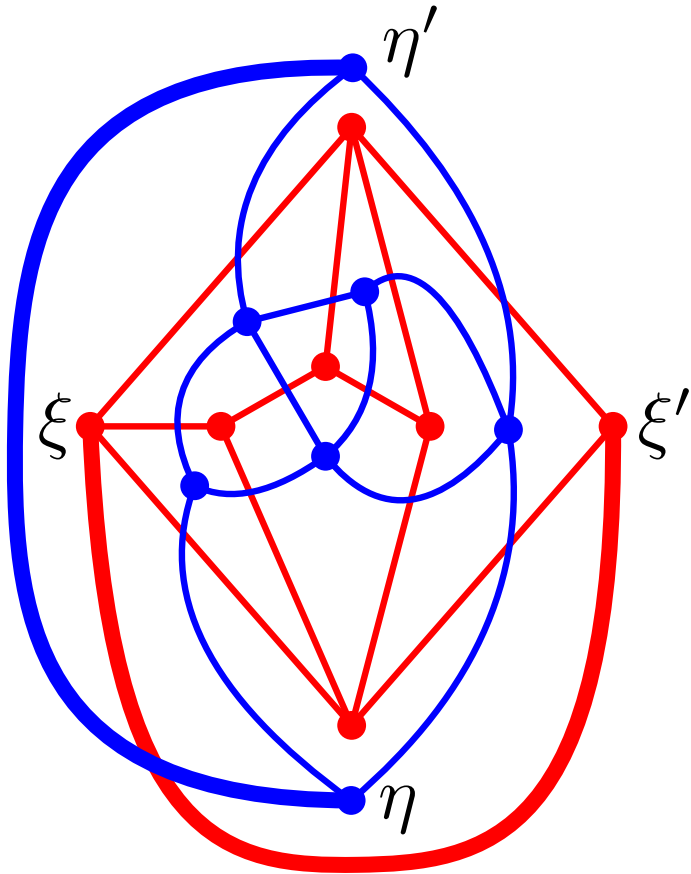


$$\alpha_{xy} = 2 \arctan \frac{r_y}{r_x}$$

$$\alpha_{xy} + \alpha_{yx} = \pi$$

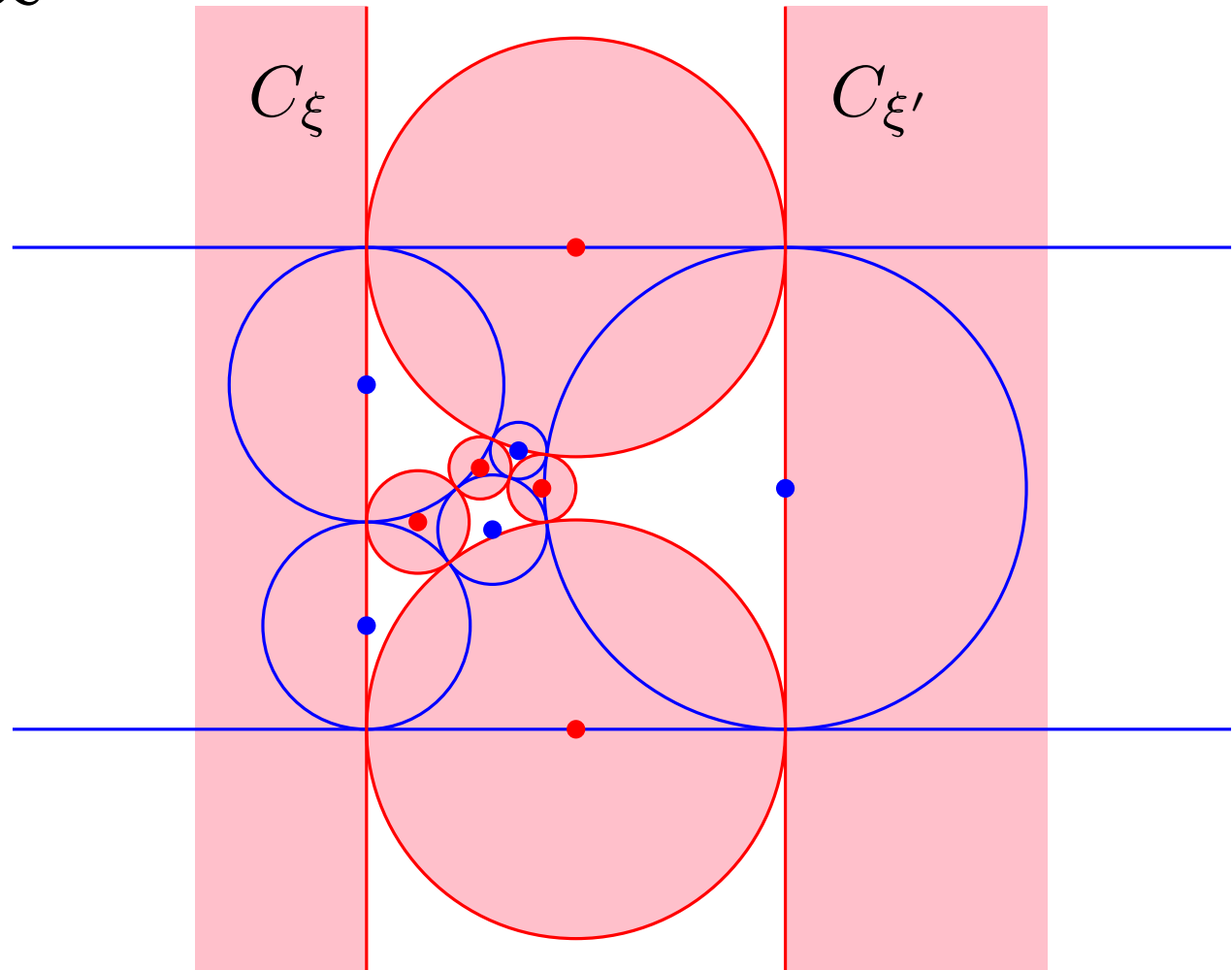
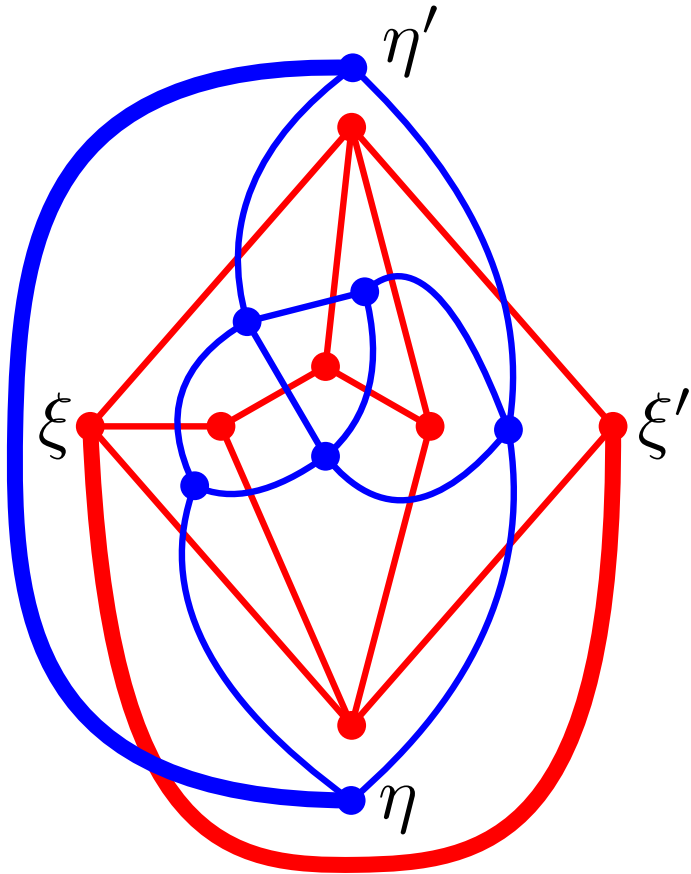
Select a primal-dual edge pair  $\xi\xi', \eta\eta'$ :

$$r_\xi = r_{\xi'} = r_\eta = r_{\eta'} = \infty$$



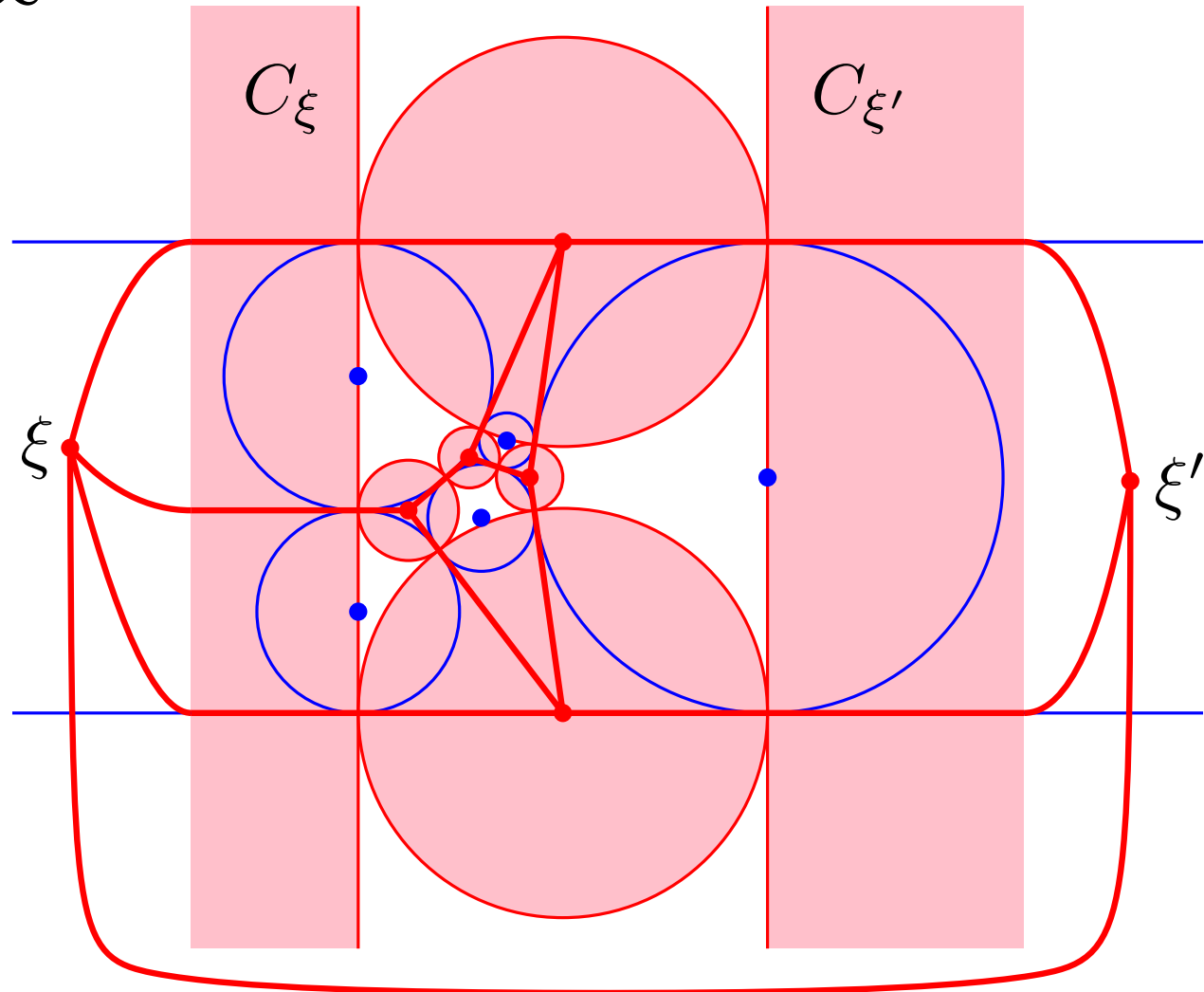
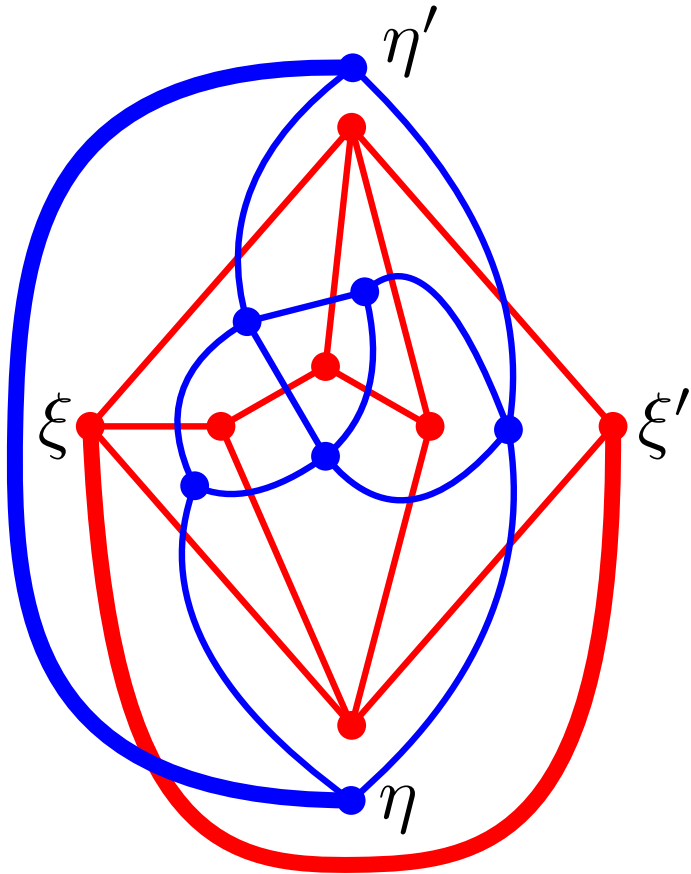
Select a primal-dual edge pair  $\xi\xi', \eta\eta'$ :

$$r_\xi = r_{\xi'} = r_\eta = r_{\eta'} = \infty$$



Select a primal-dual edge pair  $\xi\xi', \eta\eta'$ :

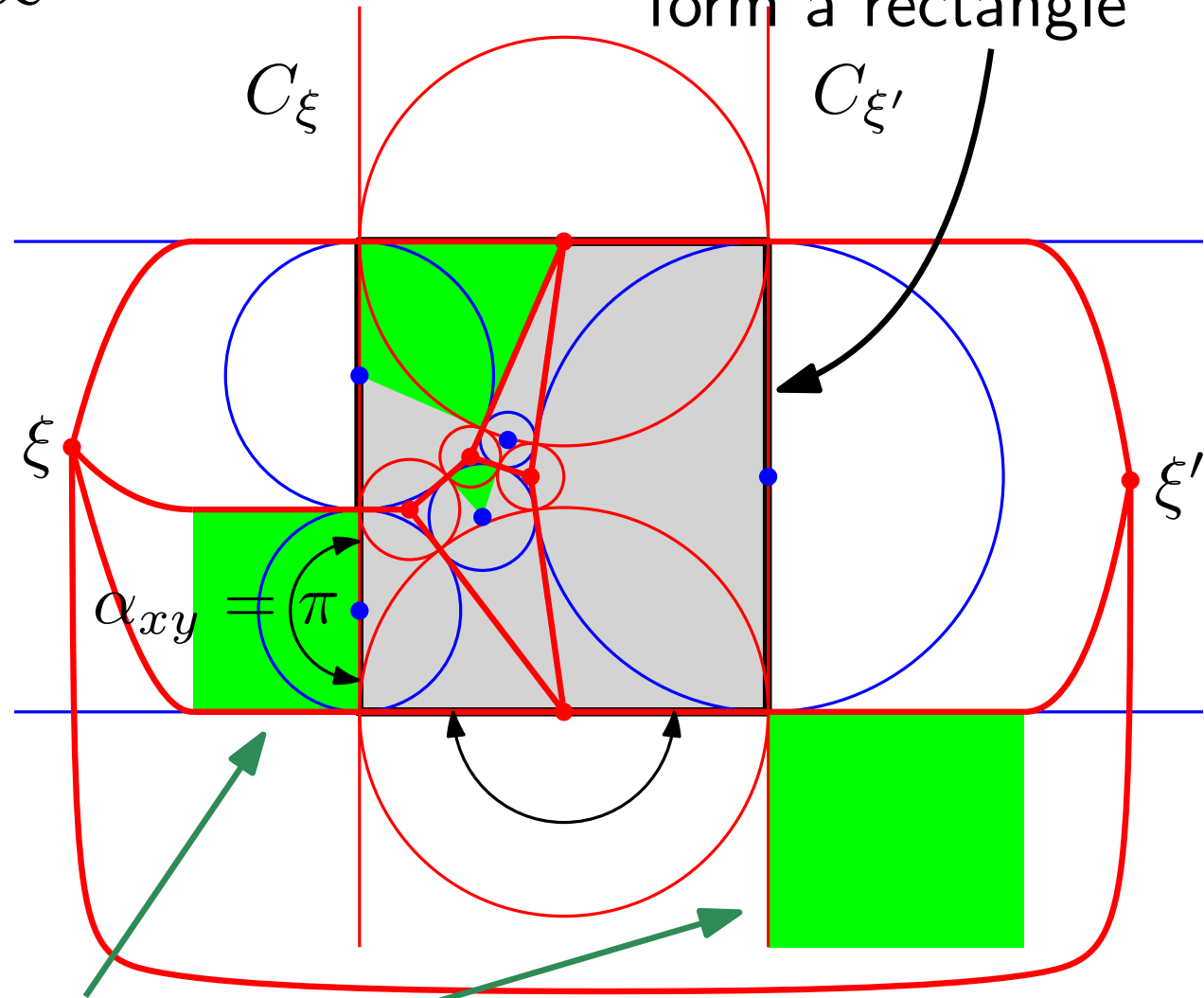
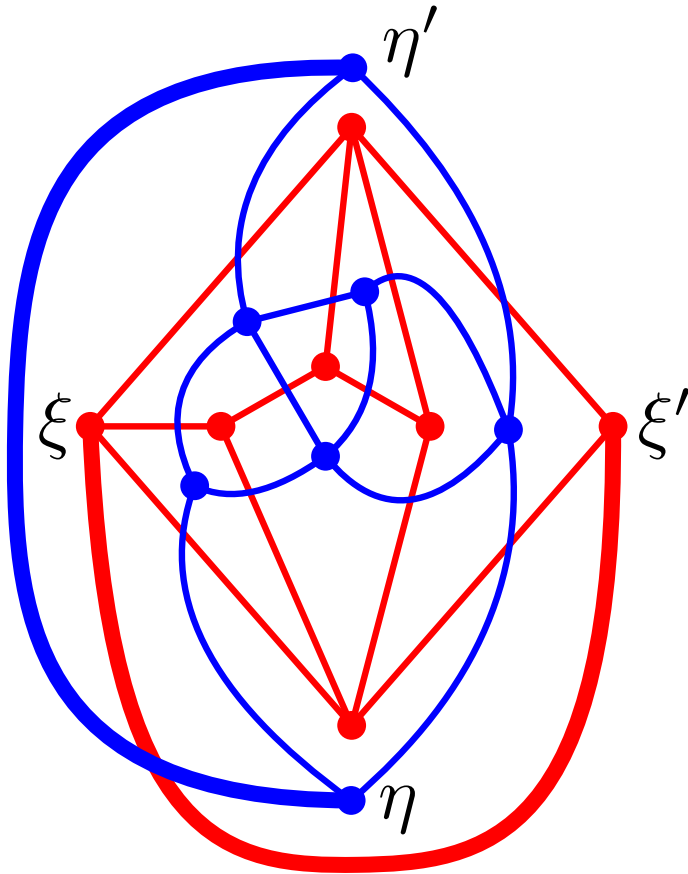
$$r_\xi = r_{\xi'} = r_\eta = r_{\eta'} = \infty$$



Select a primal-dual edge pair  $\xi\xi', \eta\eta'$ :

$$r_\xi = r_{\xi'} = r_\eta = r_{\eta'} = \infty$$

bounded kites  
form a rectangle



degenerate (unbounded) kites

Start with an arbitrary radius assignment, e.g.  $r_x \equiv 1$

Loop forever:

**for each** node  $x$  **do**

**if**  $\alpha_x > 2\pi$  **then** increase  $r_x$  until  $\alpha_x$  becomes  $2\pi$

LEMMA: The total angle sum is always correct:

$$\sum_x (\alpha_x - 2\pi) = 0$$

Proof:  $\alpha_{xy} + \alpha_{yx} = \pi$  and Euler's formula

CONSEQUENCE:

- If all  $\alpha_x$ 's converge, they converge to  $2\pi$ .

Start with an arbitrary radius assignment, e.g.  $r_x \equiv 1$

Loop forever:

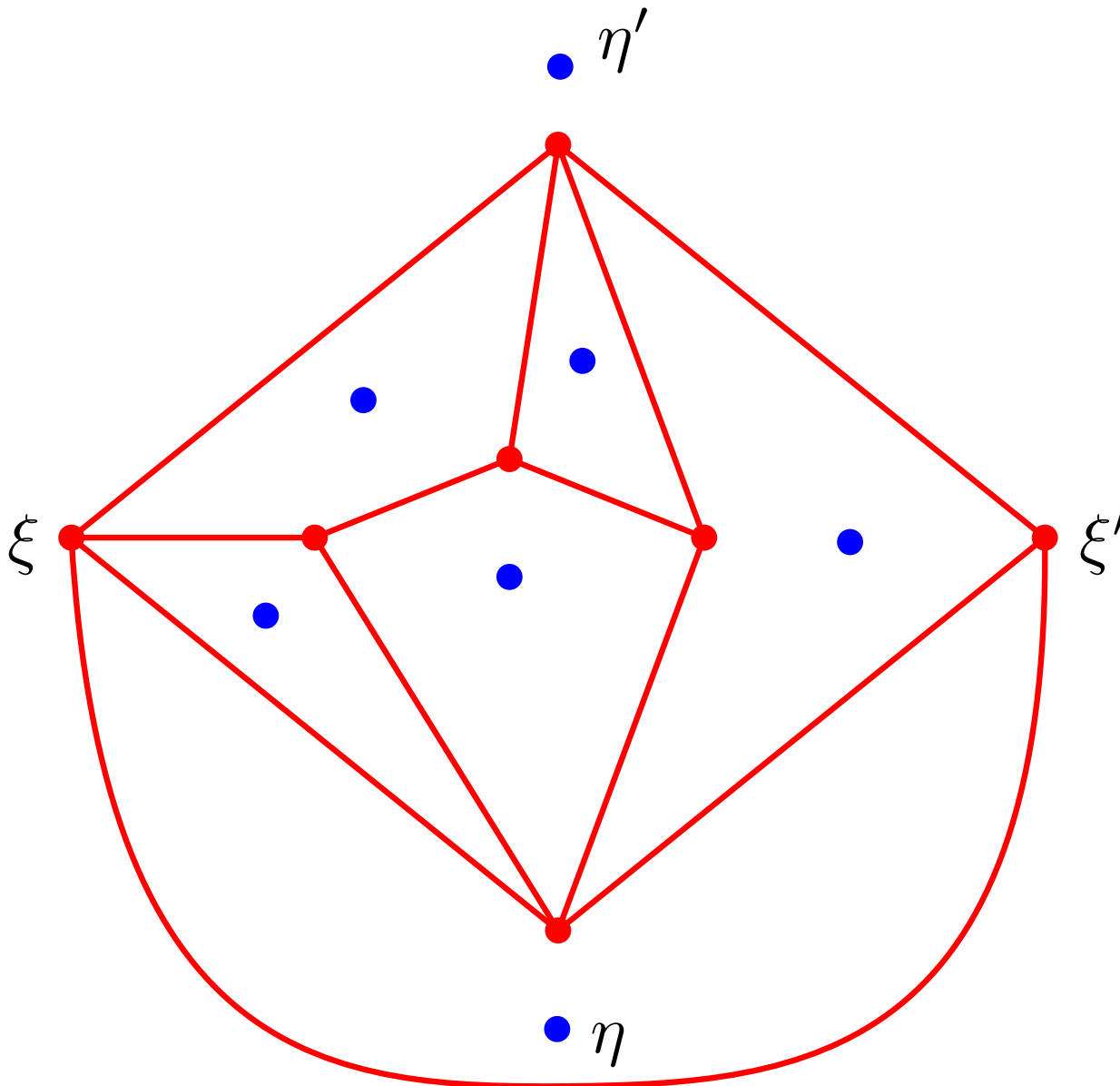
**for each** node  $x$  **do**

**if**  $\alpha_x > 2\pi$  **then** increase  $r_x$  until  $\alpha_x$  becomes  $2\pi$

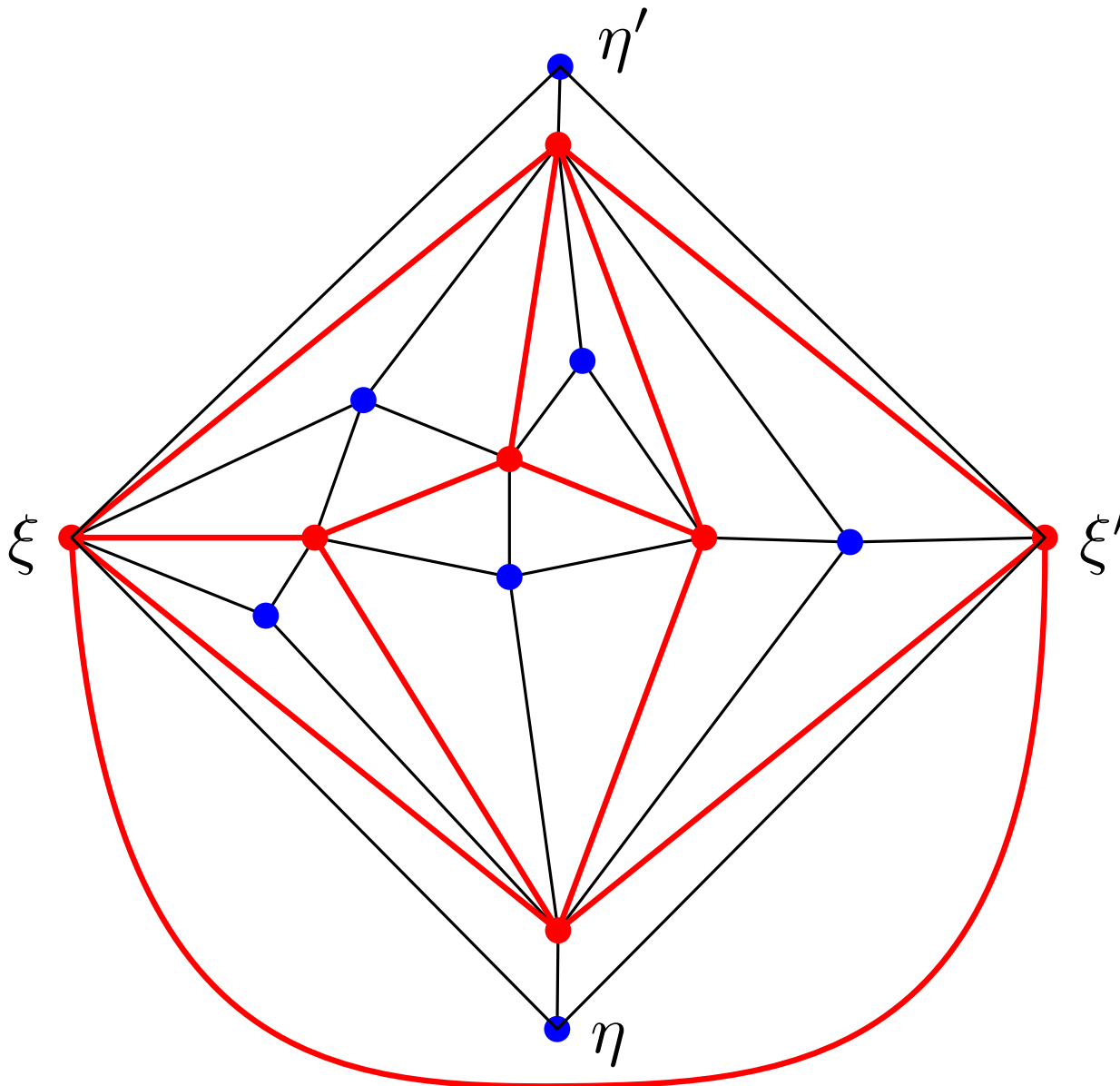
- If all  $\alpha_x$ 's converge, they converge to  $2\pi$ . (OUR GOAL)
- Increasing  $r_x$  will increase  $\alpha_y$  for  $y \sim x$ .
- Once  $\alpha_x \geq 2\pi$  for some  $x$ , this will stay so forever.
- $\exists x_0: \alpha_{x_0} \leq 2\pi$  holds forever, and  $r_{x_0}$  changes *never*.
- If all radii  $r_x$  converge, than all  $\alpha_x$ 's converge.

TASK: rule out the possibility that some radii diverge to  $\infty$ .

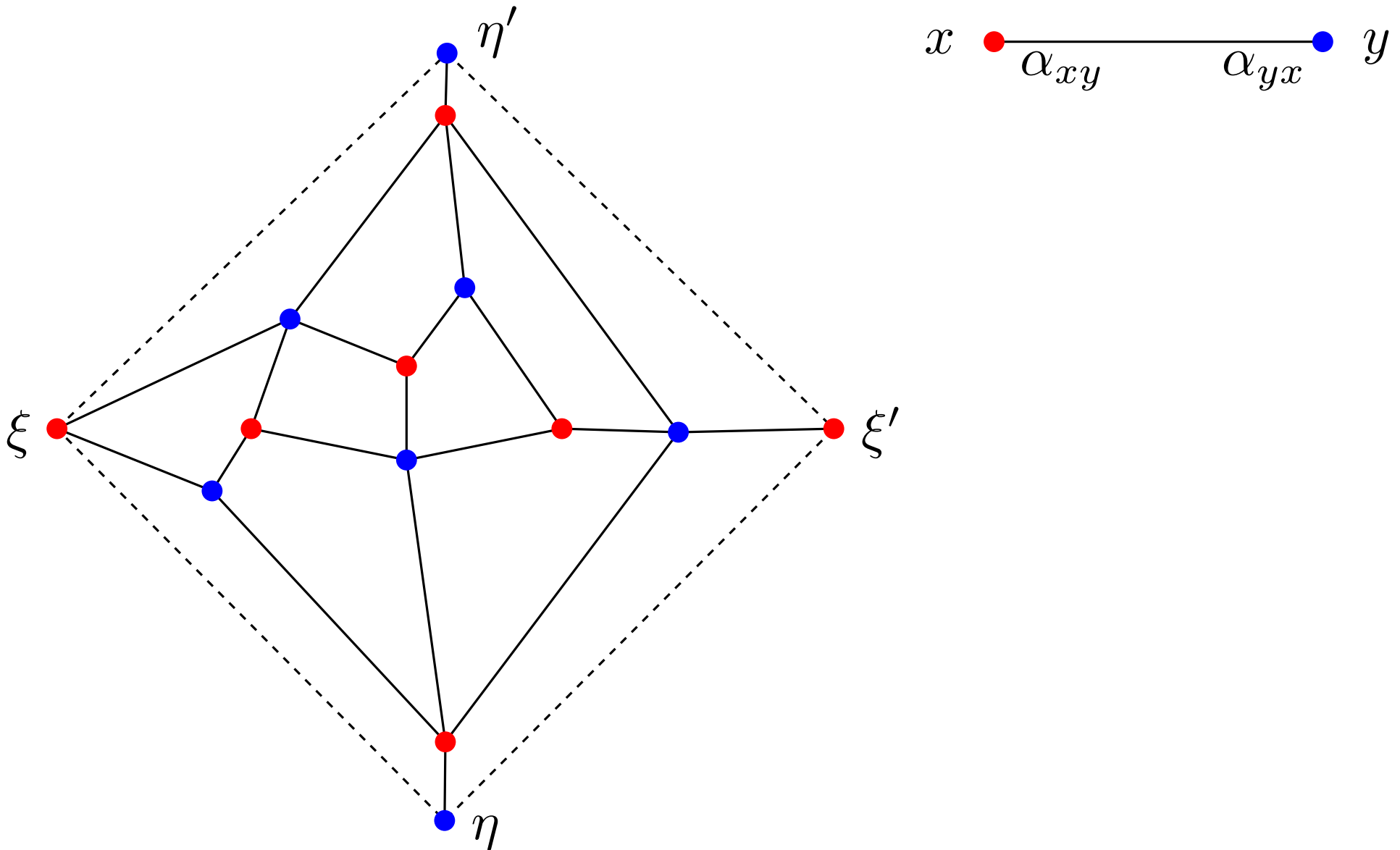
Vertex-face incidence graph (*angle graph*)  $G^\diamond$ : edges  $\equiv$  kites



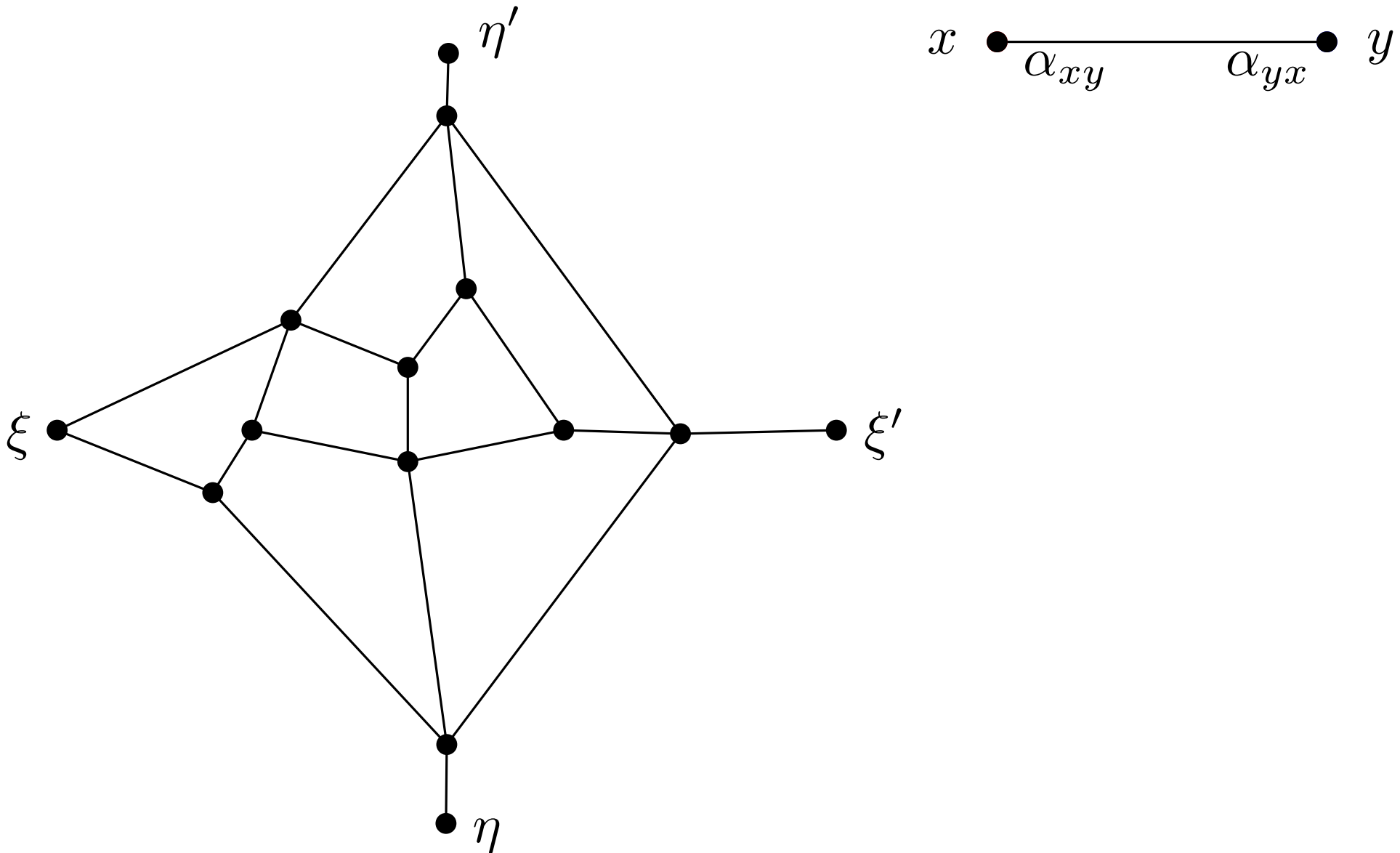
Vertex-face incidence graph (*angle graph*)  $G^\diamond$ : edges  $\equiv$  kites



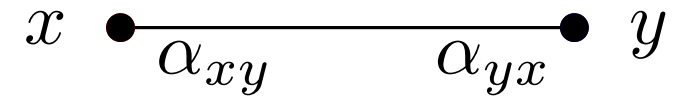
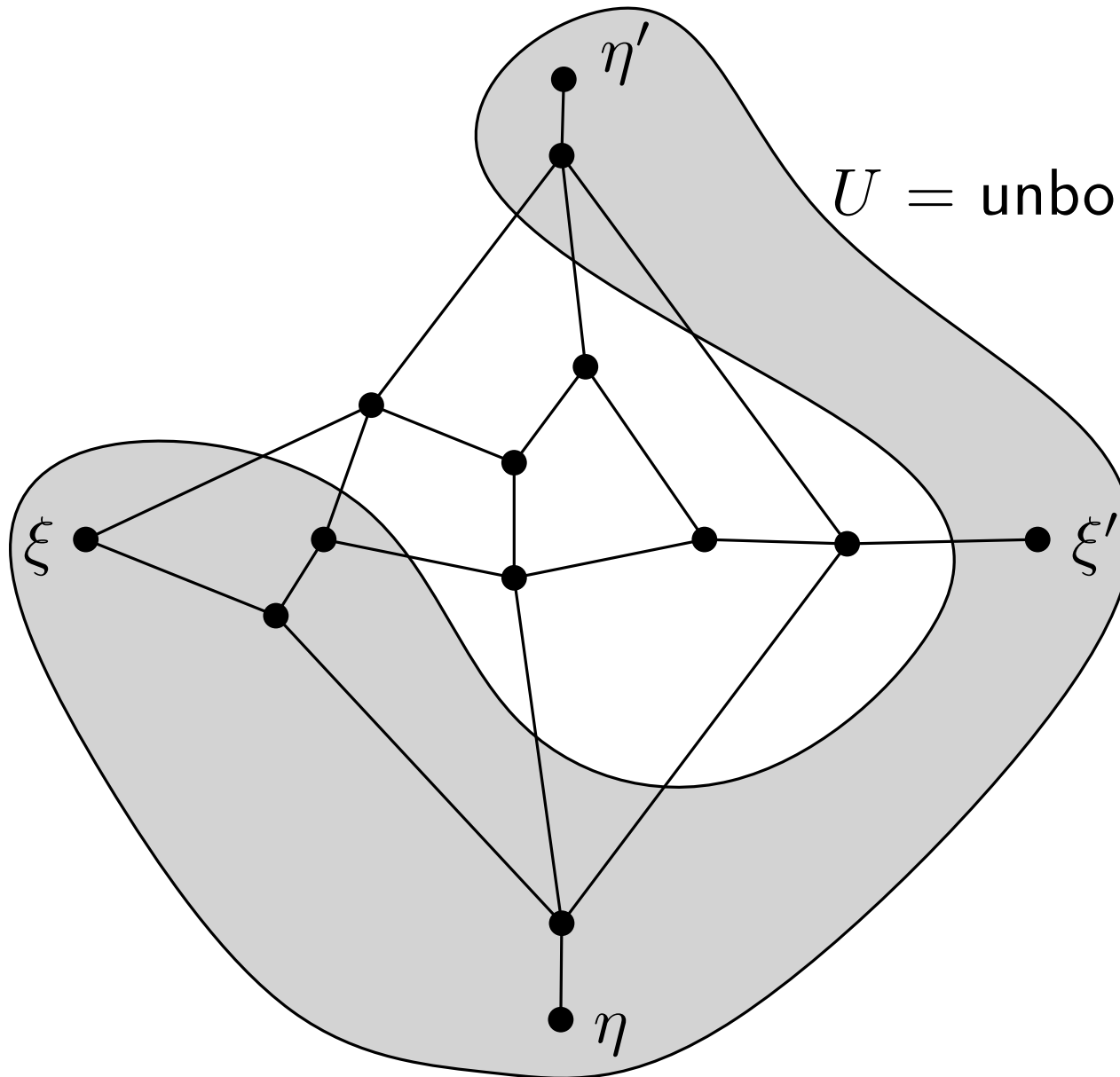
Vertex-face incidence graph (*angle graph*)  $G^\diamond$ : edges  $\equiv$  kites



Vertex-face incidence graph (*angle graph*)  $G^\diamond$ : edges  $\equiv$  kites

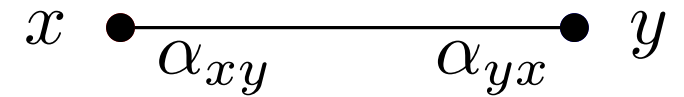
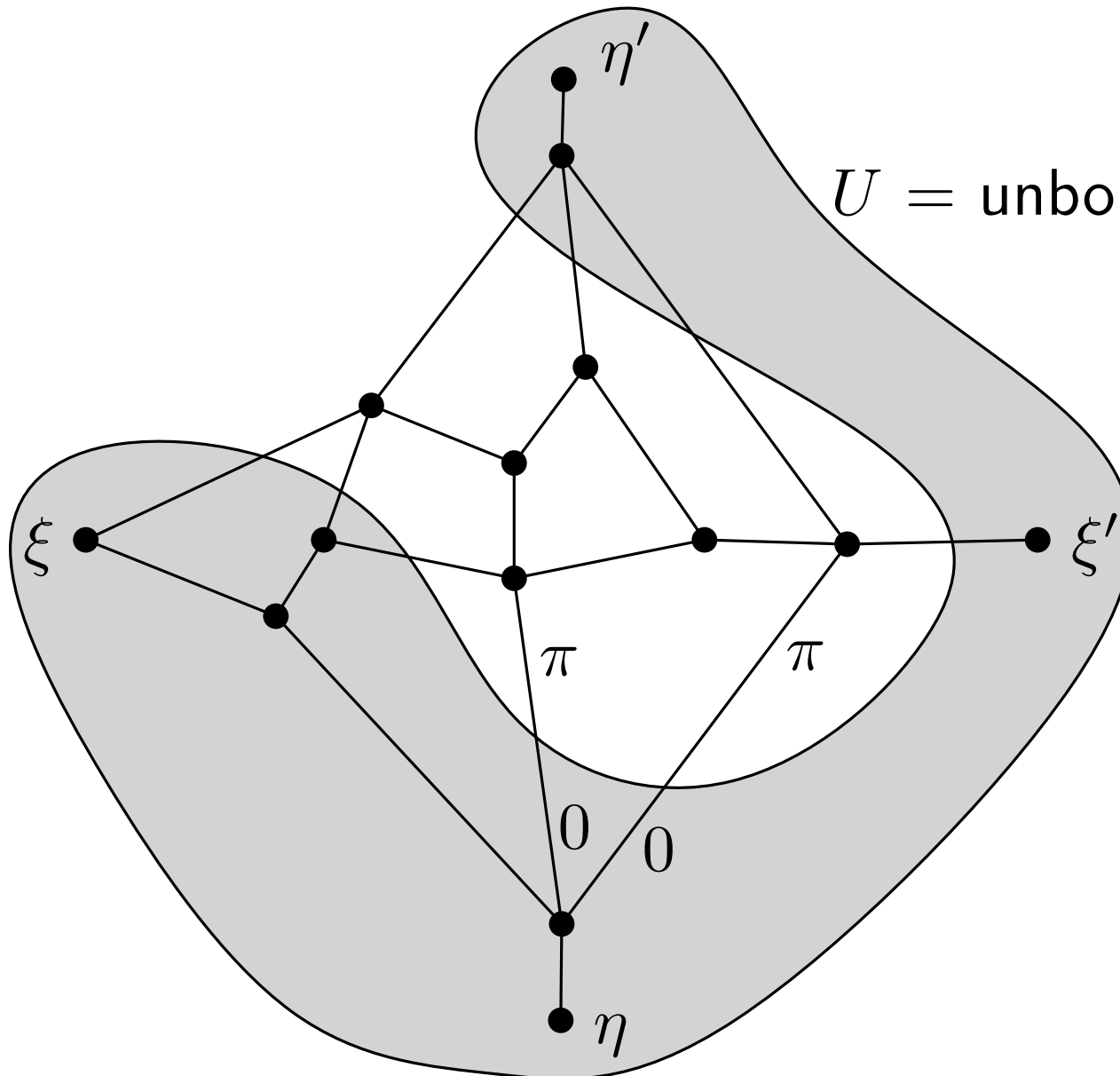


Vertex-face incidence graph (*angle graph*)  $G^\diamond$ : edges  $\equiv$  kites

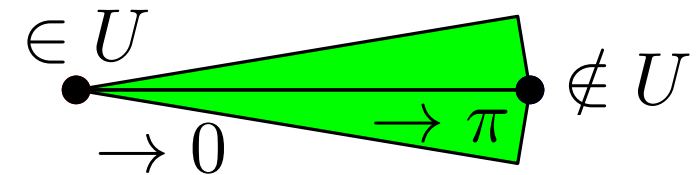


$U =$  unbounded nodes:  $r_x \rightarrow \infty$

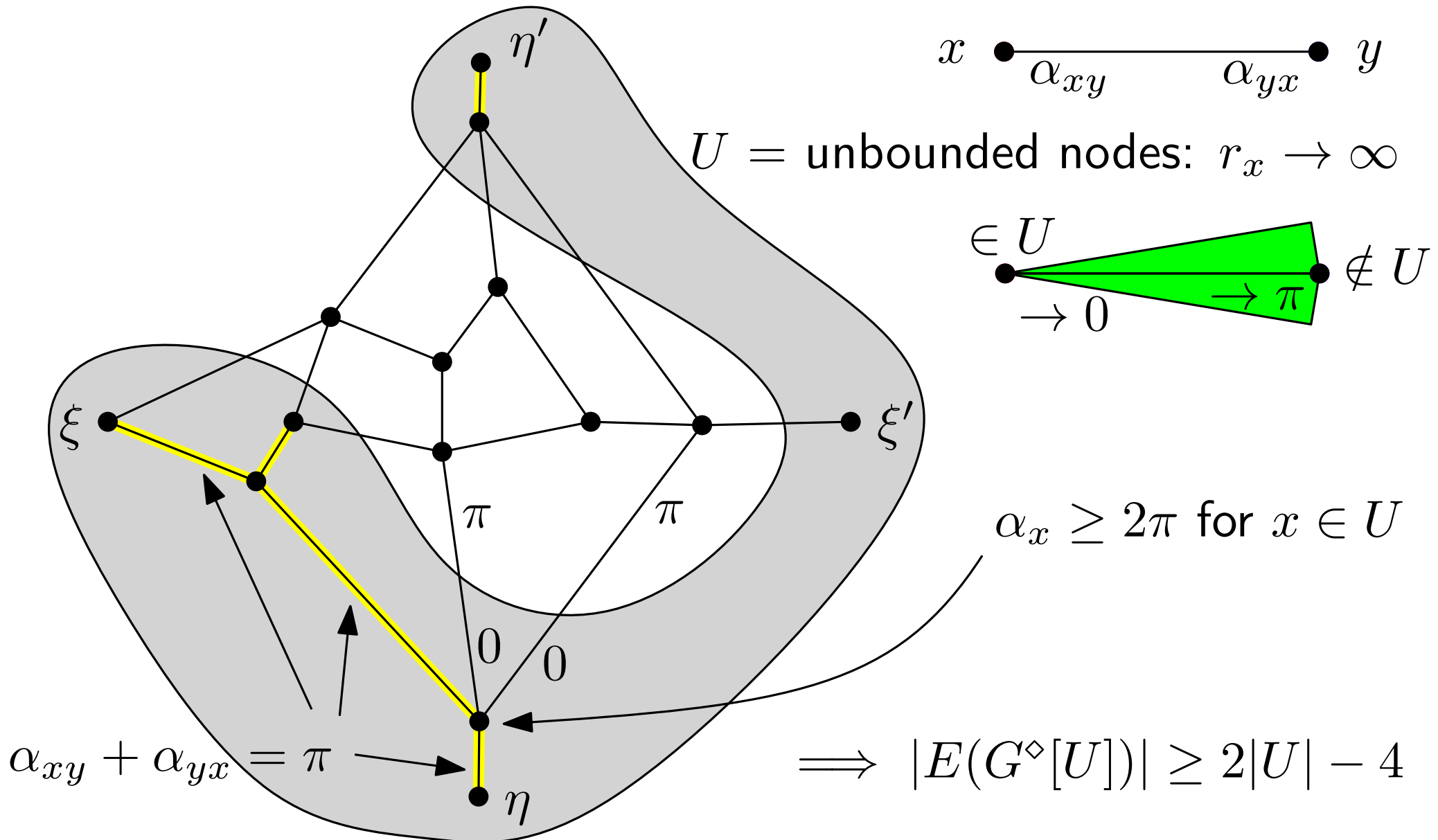
Vertex-face incidence graph (*angle graph*)  $G^\diamond$ : edges  $\equiv$  kites



$U =$  unbounded nodes:  $r_x \rightarrow \infty$



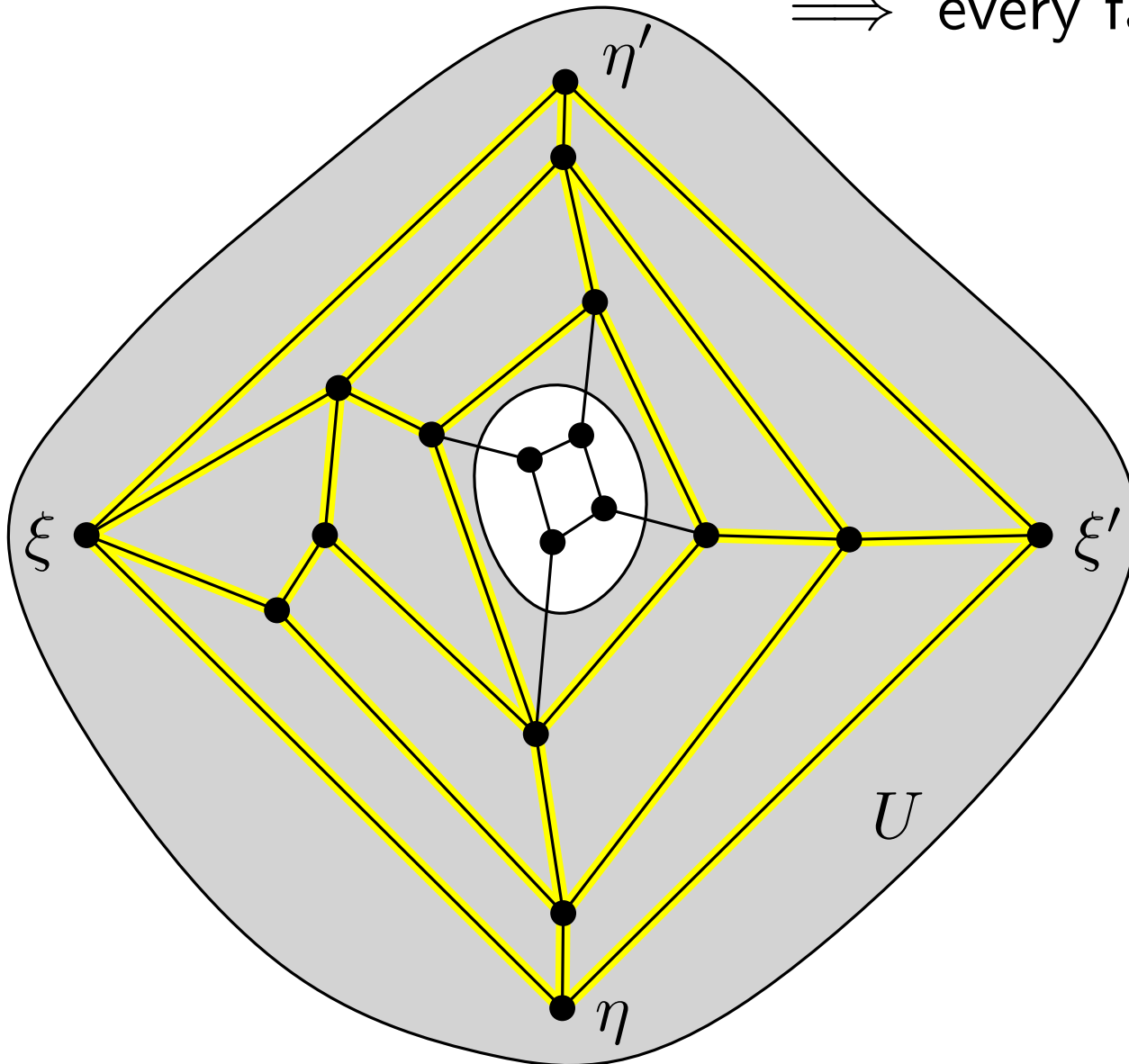
Vertex-face incidence graph (*angle graph*)  $G^\diamond$ : edges  $\equiv$  kites



Need to show:  $U = \{\xi, \xi', \eta, \eta'\}$

$|E(G^\diamond[U])| \geq 2|U| - 4$  and  $|U| \geq 4$  and bipartite

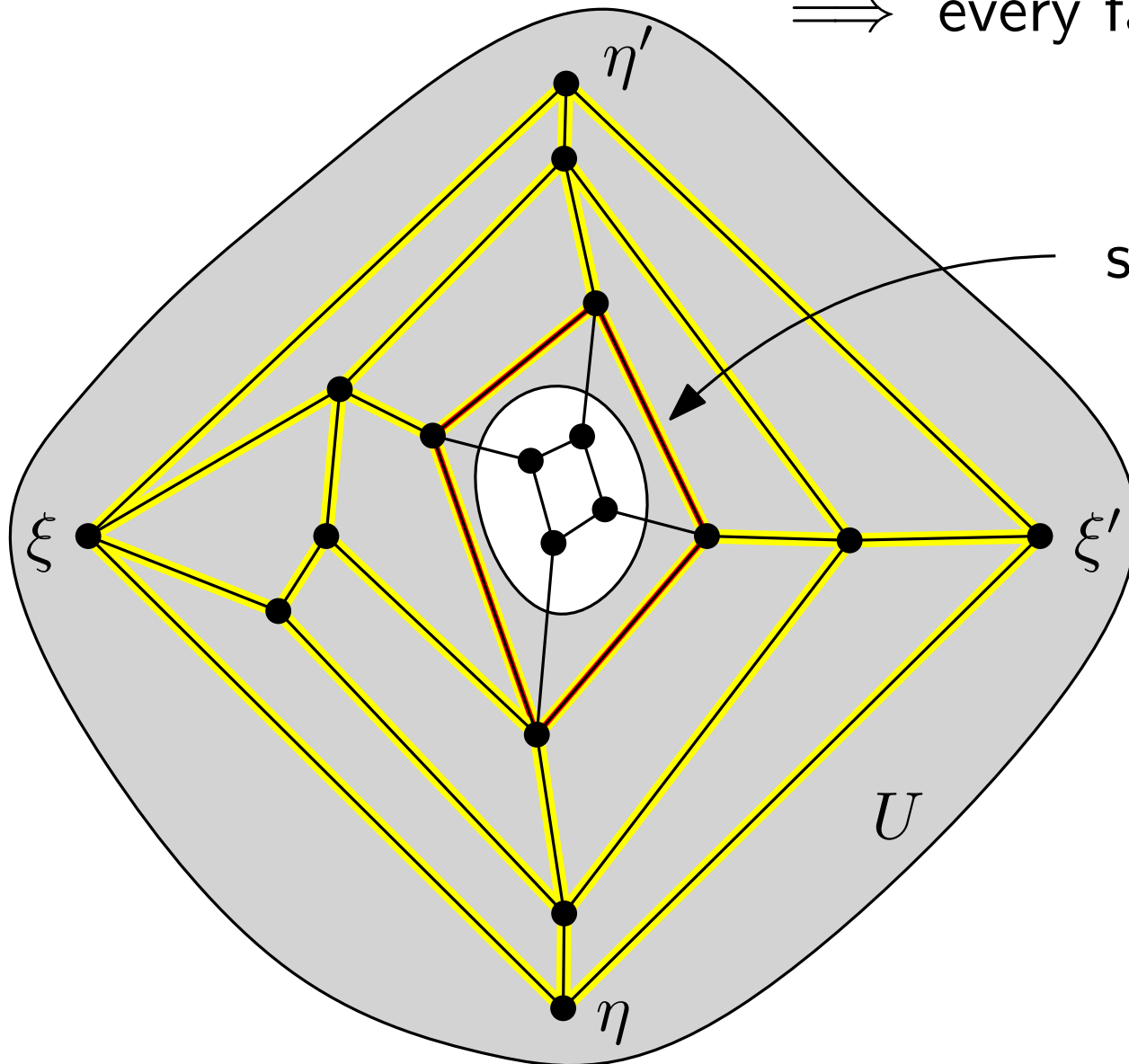
$\implies$  every face of  $G^\diamond[U]$  is a 4-gon



Need to show:  $U = \{\xi, \xi', \eta, \eta'\}$

$|E(G^\diamond[U])| \geq 2|U| - 4$  and  $|U| \geq 4$  and bipartite

$\implies$  every face of  $G^\diamond[U]$  is a 4-gon

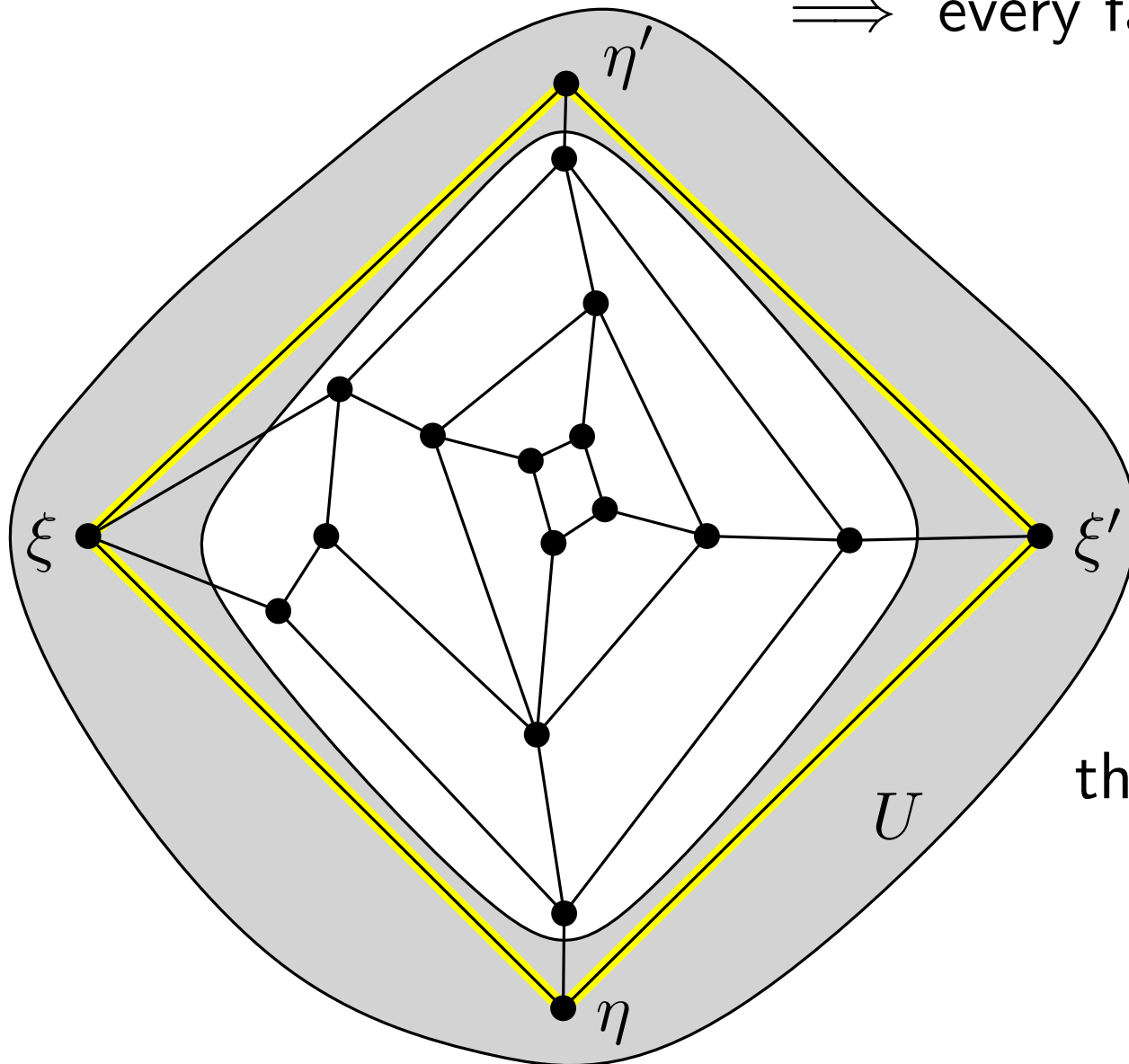


separating 4-cycle in  $G^\diamond$

Need to show:  $U = \{\xi, \xi', \eta, \eta'\}$

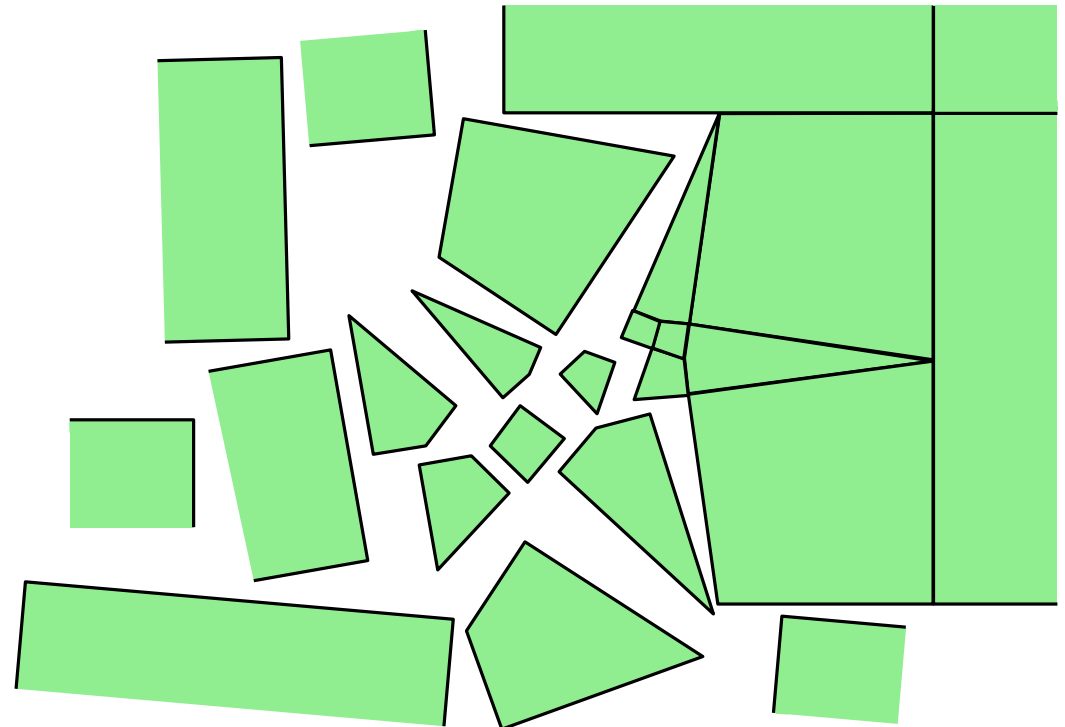
$|E(G^\diamond[U])| \geq 2|U| - 4$  and  $|U| \geq 4$  and bipartite

$\implies$  every face of  $G^\diamond[U]$  is a 4-gon



the only possibility

# Laying out the kites

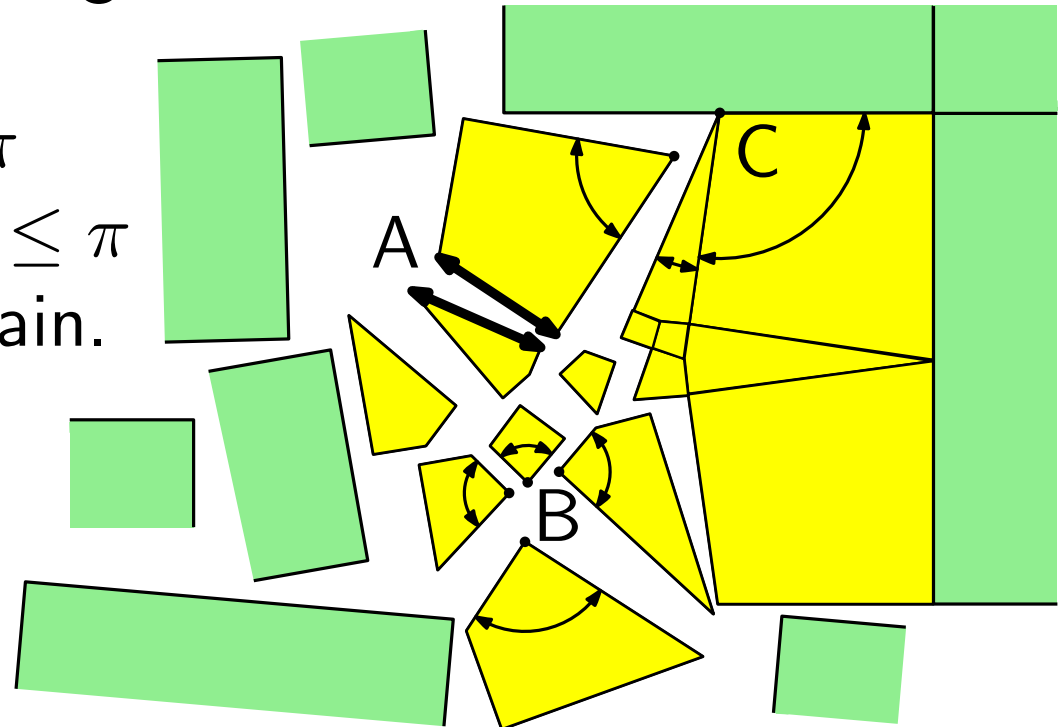


# Laying out the kites

LEMMA. If a set of tiles “fit together”:

- A) Edge lengths match
  - B) Interior angle sums are  $2\pi$
  - C) Boundary angle sums are  $\leq \pi$
- Then they tile a convex domain.

(Underlying structure:  
a 2-connected plane graph)

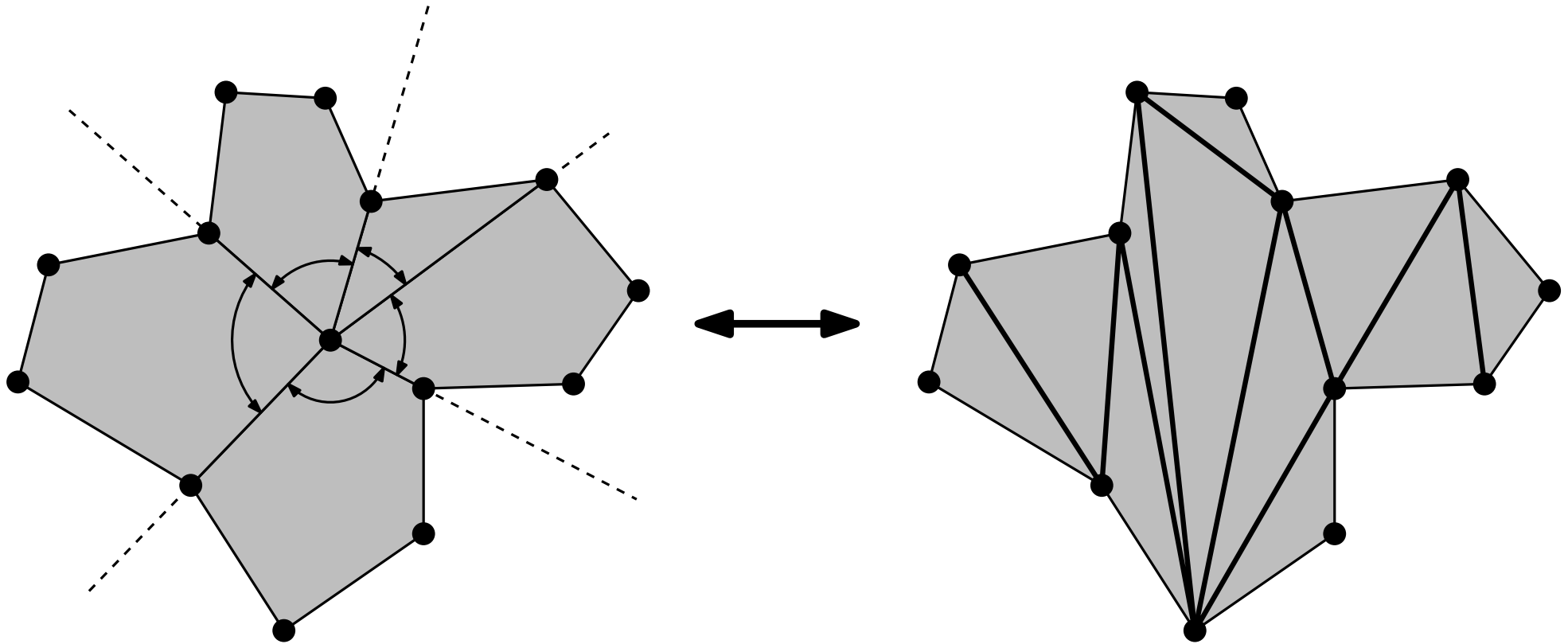


Two proofs in the paper:

- 1) geometric. Induction on the number of interior vertices
- 2) combinatorial. Put down tiles and show that all edges abut.  
Needs a separate argument that local consistency implies global consistency.

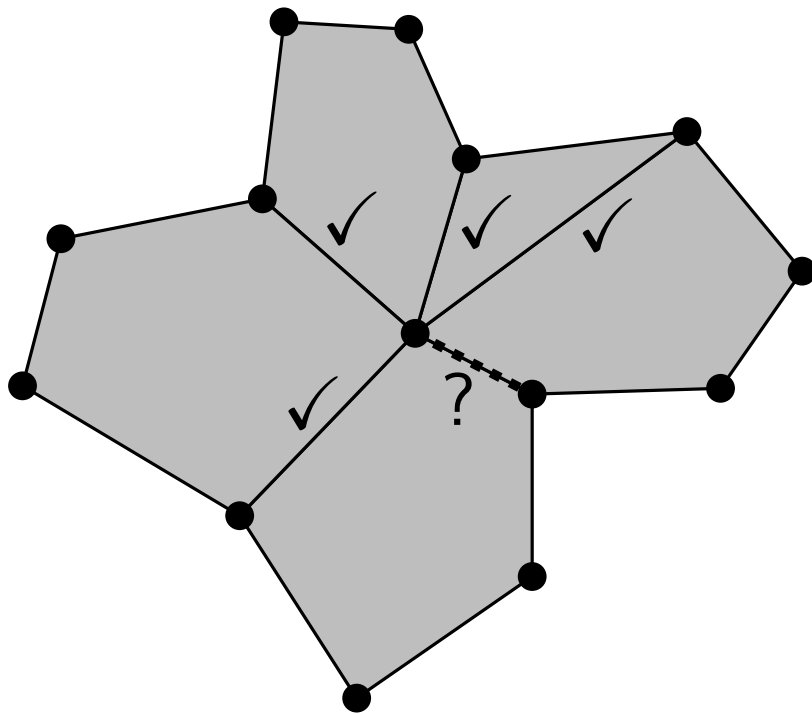
# 1) Geometric proof for tiling

Induction on the number of interior vertices  
(requires convex tiles)



## 2) Combinatorial proof for tiling

- glue along *some* edge pairs (spanning tree of the dual graph)
- show that the remaining edge pairs fit together

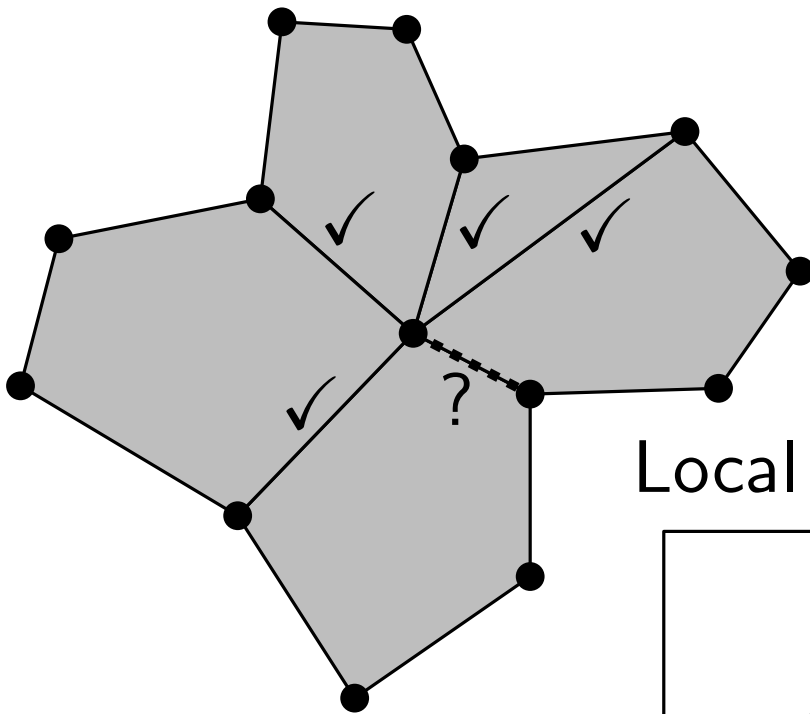


- If all but one of the incident edges are OK, the last edge is also OK.
- ... work along the spanning tree of the original graph ...

## 2) Combinatorial proof for tiling

- glue along *some* edge pairs (spanning tree of the dual graph)
- show that the remaining edge pairs fit together

- If all but one of the incident edges are OK, the last edge is also OK.
- ... work along the spanning tree of the original graph ...



Local consistency implies global consistency:

