Polytopes and Plane Graphs with no Long Monotone Paths

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joint work with

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Monotone Paths on Polytopes

**Conjecture:** Every 3D convex polytope with \( n \) vertices has a monotone path of length \( \Omega(\sqrt{n}) \) in some direction.

[Monotone Paths on Polytopes: Every 3D convex polytope with \( n \) vertices has a monotone path of length \( \Omega(\sqrt{n}) \) in some direction.]

\[ \langle \mathbf{u}, p_1 \rangle < \langle \mathbf{u}, p_2 \rangle < \langle \mathbf{u}, p_3 \rangle < \cdots \]

**THEOREM (2012-02-28).** There is a family of triangulated polytopes with \( n \) vertices, where the longest monotone path has length \( O(\log n) \).

(Motivation: Partial least-squares matching of point sets.)
Results on Polytopes

THEOREM (2012-02-28). There is a family of triangulated polytopes with $n$ vertices, where the longest monotone path has length $O(\log n)$. (L.B.: $\Omega(\log n / \log \log n)$)

THEOREM (2011). There is a family of triangulated polytopes with $n$ vertices and bounded degree $d$, where the longest monotone path has length $O(\log^2 n)$. (L.B.: $\Omega(\log n)$)

THEOREM (Chazelle, Edelsbrunner, Guibas 1989). Every polyhedral subdivision of the plane with $n$ vertices and degree $\leq d$ contains a monotone path with $\geq \Omega(\log_d n + \log n / \log \log n)$ edges. This is tight.
The characteristic region of a path

\[ \chi(P) = \text{the set of directions } (u, v, 1) \text{ for which } P \text{ or its inverse is a monotone path.} \]

= two intersections of half-planes
The $O(\log^2 n)$ construction

a hierarchical structure:
## The basic building block $\Delta$

<table>
<thead>
<tr>
<th>point</th>
<th>$(x, y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>$B, B'$</td>
<td>($\pm 1$, 1.5, 0)</td>
</tr>
<tr>
<td>$C$</td>
<td>(0, 1.4, 1)</td>
</tr>
<tr>
<td>$U, U'$</td>
<td>($\pm 0.1$, 0.8, 0.55)</td>
</tr>
<tr>
<td>$V, V'$</td>
<td>($\pm 0.25$, 0.6, 0.25)</td>
</tr>
<tr>
<td>$W, W'$</td>
<td>($\pm 0.25$, 0.8, 0.39)</td>
</tr>
</tbody>
</table>
The characteristic region of $\Delta$

- start in $A$, $B$, or $B'$
- visit at least two vertices of $UVW$ and at least two vertices of $U'V'W'$ (in either order)
- end in $A$, $B$, or $B'$
Placing the subcells
Placing the subcells
Inductive construction

Characteristic regions: • lie in $|v| \leq 2|u| + 1/2$
• have no triple intersections
• pairwise intersections lie within $\leq R = 2.5$ of the origin
Affine Transformations

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \varepsilon \cdot x \\ y \\ \varepsilon \cdot z \end{pmatrix}$$

squeeze

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ \varepsilon \cdot v \end{pmatrix}$$

scale

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \varepsilon \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

turn
Affine Transformations

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \mapsto \begin{pmatrix}
x \\
y \\
\varepsilon \cdot z
\end{pmatrix}
\]

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \mapsto \begin{pmatrix}
x \\
y \\
z + ax + by
\end{pmatrix}
\]

\text{turn}

\text{squash}

\text{tilt}

\text{push}
Affine Transformations

\begin{align*}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto \begin{pmatrix} x \\ y \\ \varepsilon \cdot z \end{pmatrix} \\
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\end{align*}

squash

tilt

push
The visited nodes

A monotone path $P$ in direction $c$ can visit both children of a node $\Delta$ only if

- $c$ lies in $\chi(\Delta)$, or
- $P$ starts or ends inside $\Delta$.

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$2k$ paths of length $k$
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$2k$ paths of length $k$

$\rightarrow O(k^2)$ nodes

$\rightarrow O(k^2) = O(\log^2 n)$ vertices
The Construction for $O(\log n)$
Results on Convex Planar Subdivisions

THEOREM. Let $v$ be a vertex in a convex subdivision of the plane with $n$ vertices and degree $\leq d$. There is path starting in $v$ with $\geq \Omega(\log_d n)$ edges that is monotone in some direction. (This is best possible; Chazelle, Edelsbrunner, Guibas 1989.)

THEOREM. Let $G$ be a convex subdivision of the plane with $n$ vertices and $k$ unbounded faces. Then $G$ contains a path with $\geq \Omega(\log \frac{n}{k} / \log \log \frac{n}{k})$ edges that is monotone in some direction. This bound is best possible.

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Polyhedral Subdivisions

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R(\varphi) := \text{the rightmost monotone path in direction } \varphi
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- \( R(\varphi) \) is still monotone in direction \( \varphi' \).

- The region between \( R(\varphi) \) and \( R(\varphi') \) can be connected to \( v \) by monotone paths (in direction \( \varphi' \)).
Monotone Paths in Convex Subdivisions

THEOREM. Let $v$ be a vertex in a convex subdivision of the plane with $n$ vertices and degree $\leq d$. There is path starting in $v$ with $\Omega(\log_d n)$ edges that is monotone in some direction.

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\( \rightarrow \) a directed graph in which \( v \) can reach every vertex by a monotone path.

\( R(\varphi') \)

\( R(\varphi'') \)

\( R(\varphi''') \)
Monotone Paths in Convex Subdivisions

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- The region between $R(\varphi)$ and $R(\varphi')$ can be connected to $v$ by monotone paths (in direction $\varphi'$).
  → a directed graph in which $v$ can reach every vertex by a monotone path.

- $\text{degree} \leq d \implies \text{longest path} \geq \log_d n$. QED
Degenerate Situations

Faces are not strictly convex.

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Weakly monotone paths work.
Tightness

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For $d \approx n$, this is tight, even for triangulations.
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8 edges.

What happens if the number of unbounded edges is bounded by a constant (say, 3)?
Few Unbounded Faces

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Upper-bound construction for $k$ constant. $m \colon= 2 \log n / \log \log n$, $m^m > n$.

Characteristic region $\chi$: can follow the zigzag
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Upper-bound construction for $k$ constant.

$m := 2 \log n / \log \log n$, $m^m > n$.

Characteristic region $\chi$: can follow the zigzag $m$ levels of fanout $m$.

Longest path $\leq m + m$. 
Monotone Face Chains

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(by duality)

THEOREM. Every polyhedral subdivision of the plane with $n$ vertices and face degree $\leq d$ contains a monotone face sequence with $\geq \Omega(\log_d n + \log n / \log \log n)$ faces. This is tight. The bound holds even for convex subdivisions.
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