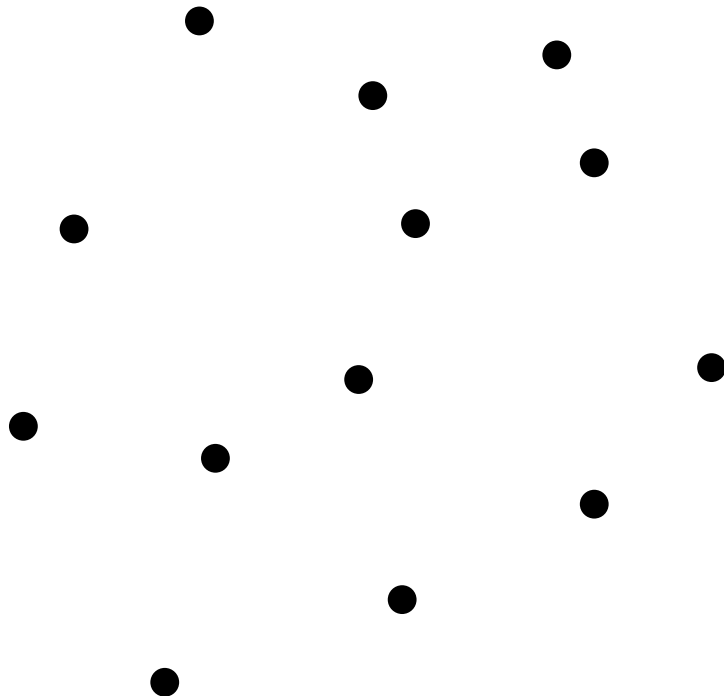


Most General Position

joint work with

Jan Kynčl, Alexander Pilz, André Schulz

supported by the EuroGIGA project of the European Science Foundation

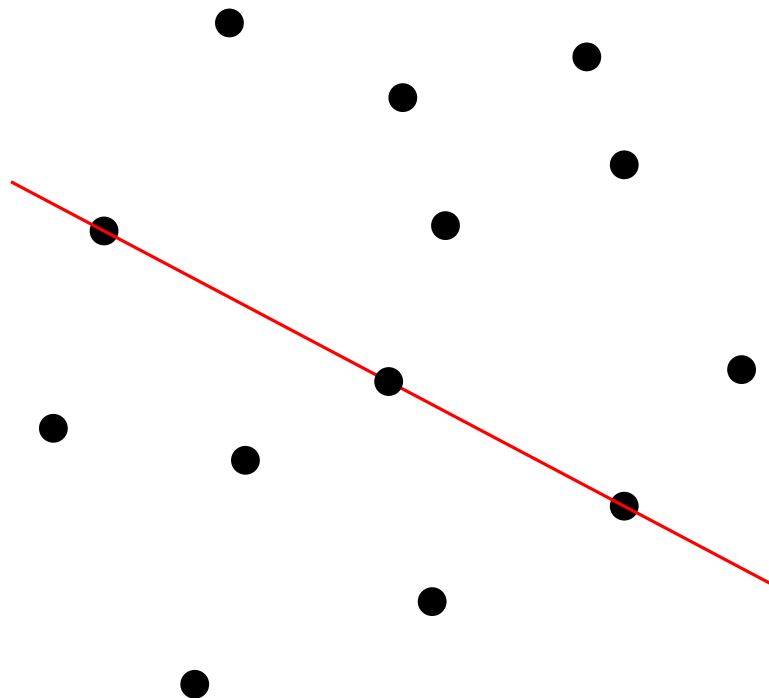


Most General Position

joint work with

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avoid

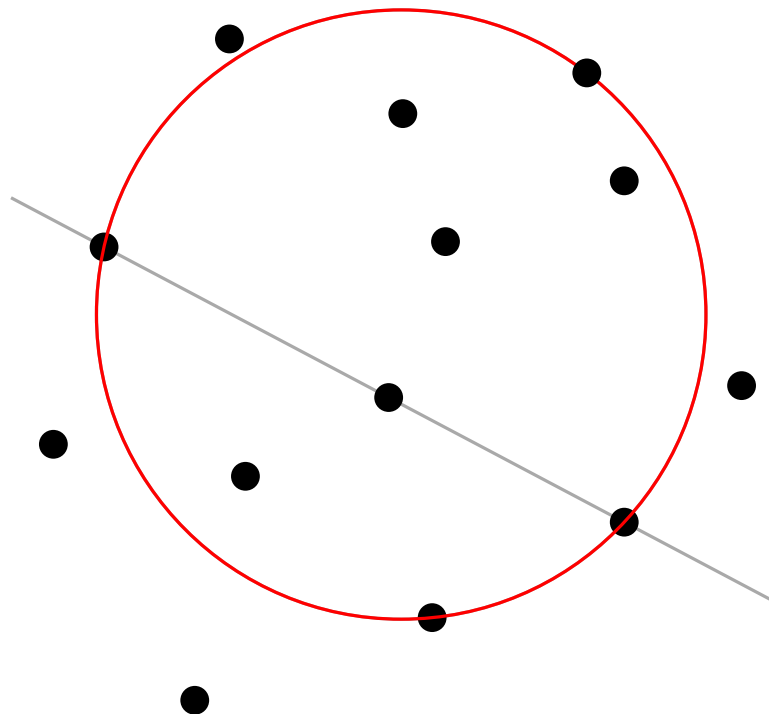
- collinear points

Most General Position

joint work with

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avoid

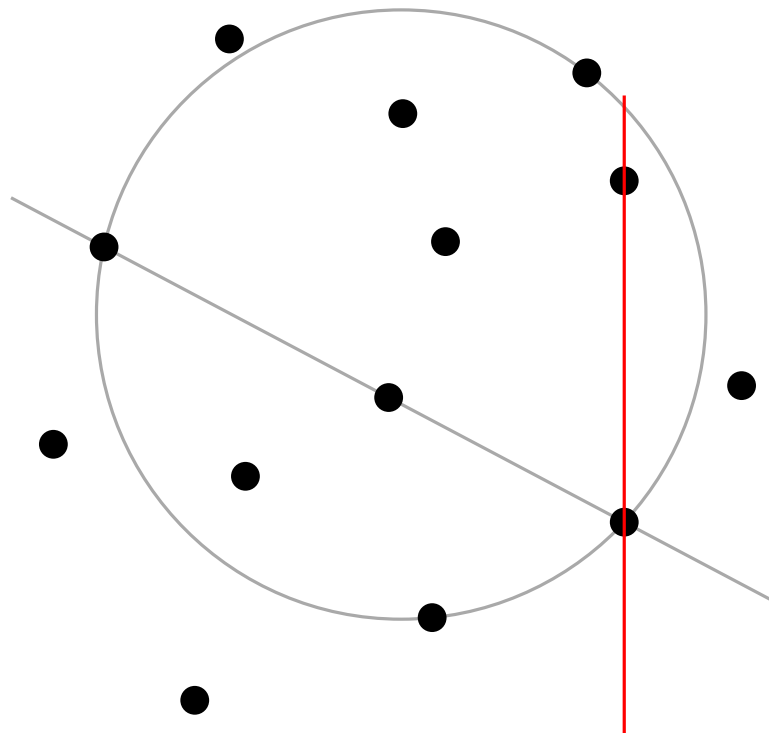
- collinear points
- cocircular points

Most General Position

joint work with

Jan Kynčl, Alexander Pilz, André Schulz

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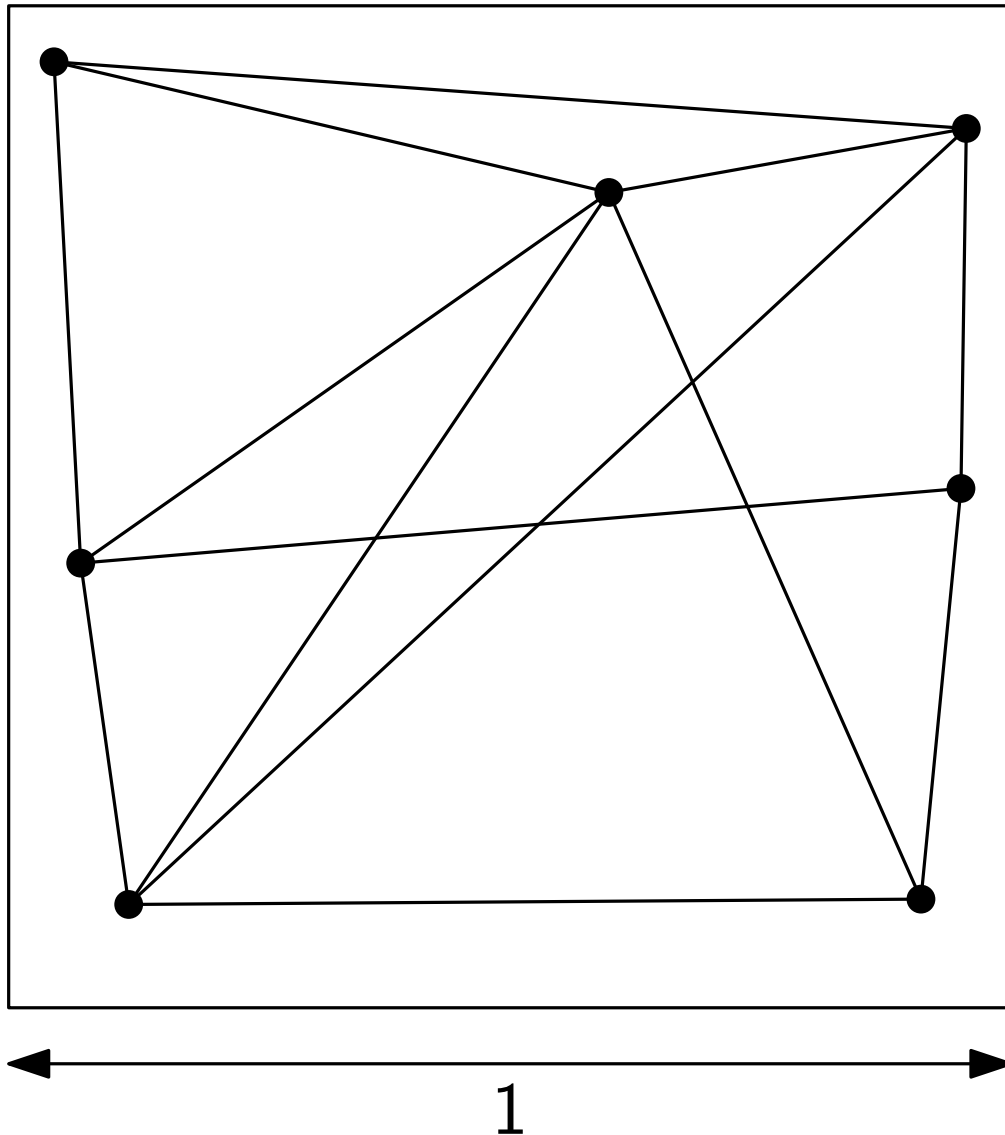


avoid

- collinear points
- cocircular points
- points with the same x -coordinate

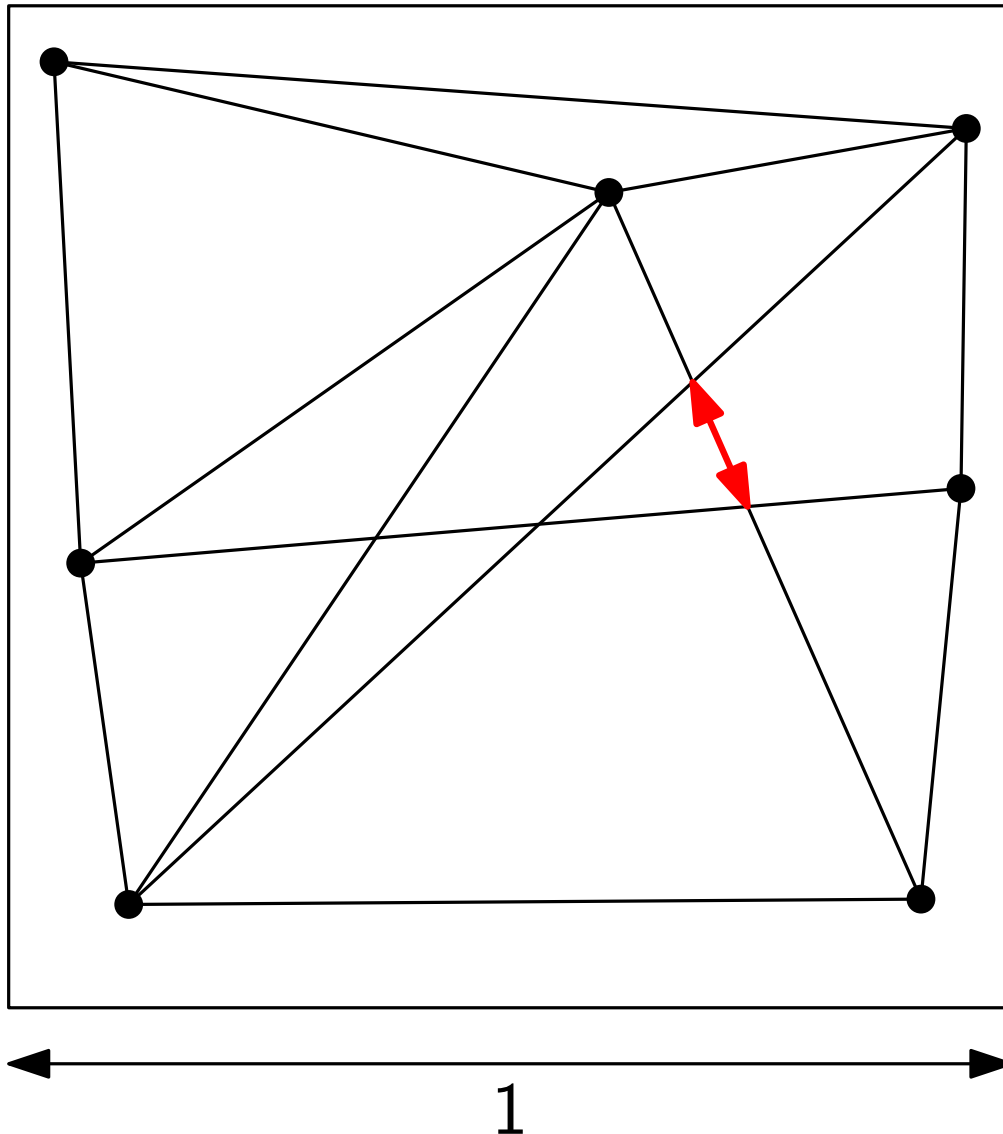
Good Drawings

Find a (nonplanar) straight-line drawing of a given graph.



Good Drawings

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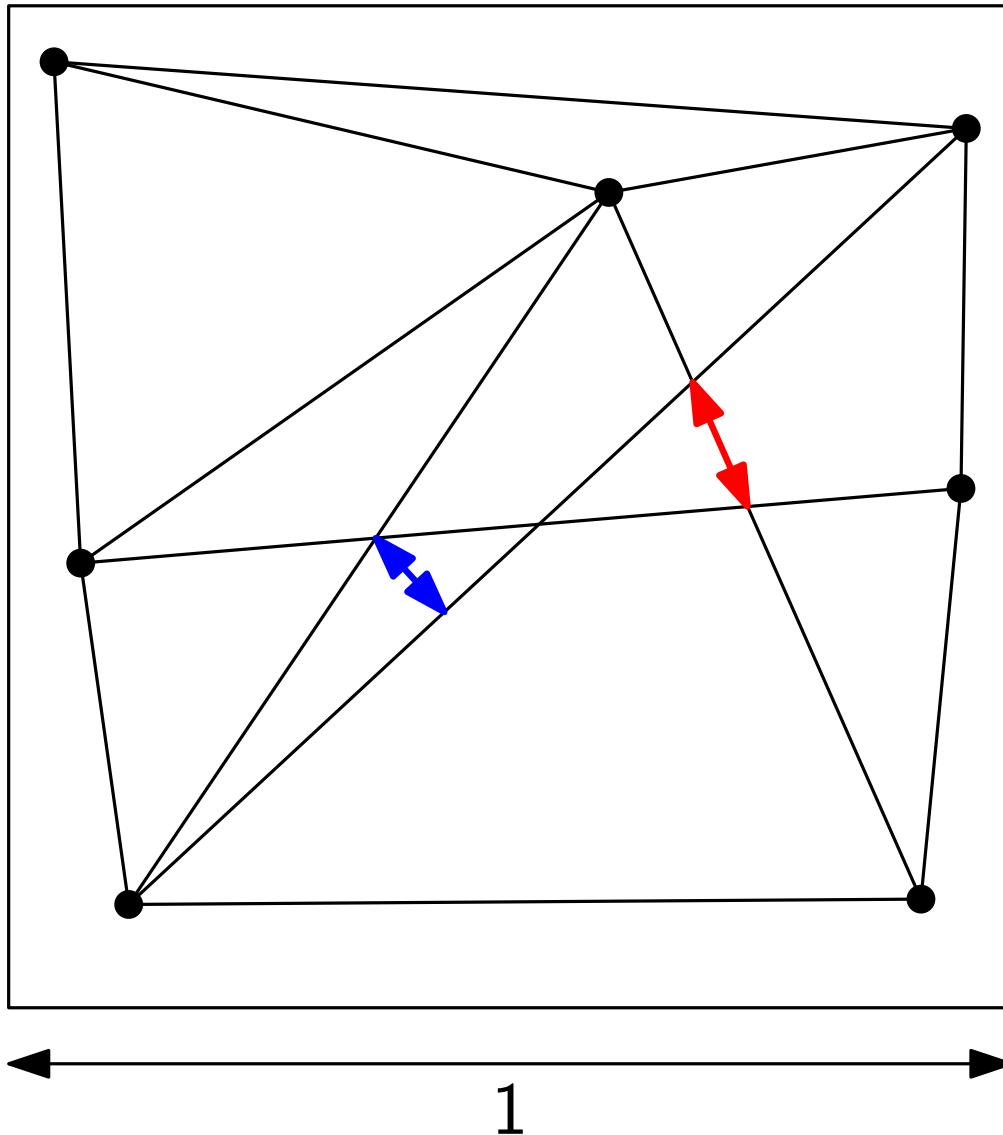


Maximize $\varepsilon =$

- the shortest edge

Good Drawings

Find a (nonplanar) straight-line drawing of a given graph.

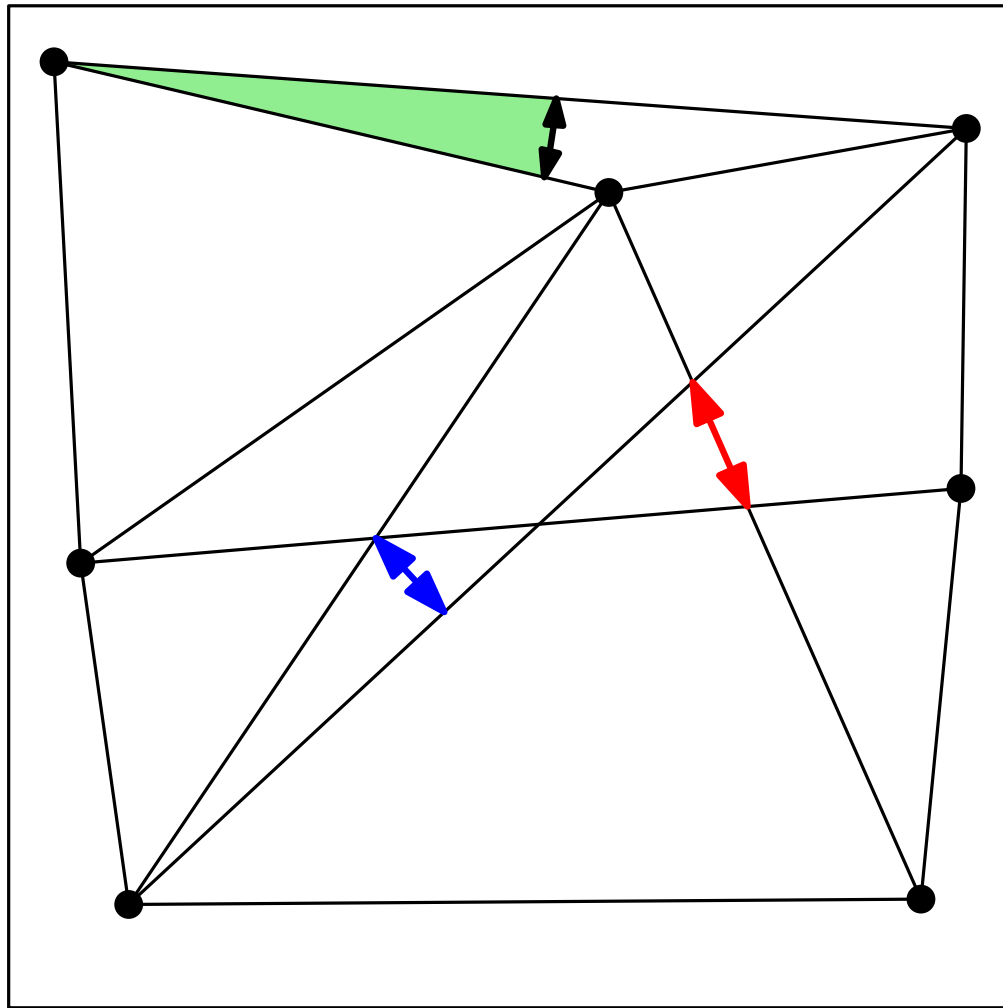


Maximize $\varepsilon =$

- the shortest edge
- the smallest feature distance

Good Drawings

Find a (nonplanar) straight-line drawing of a given graph.



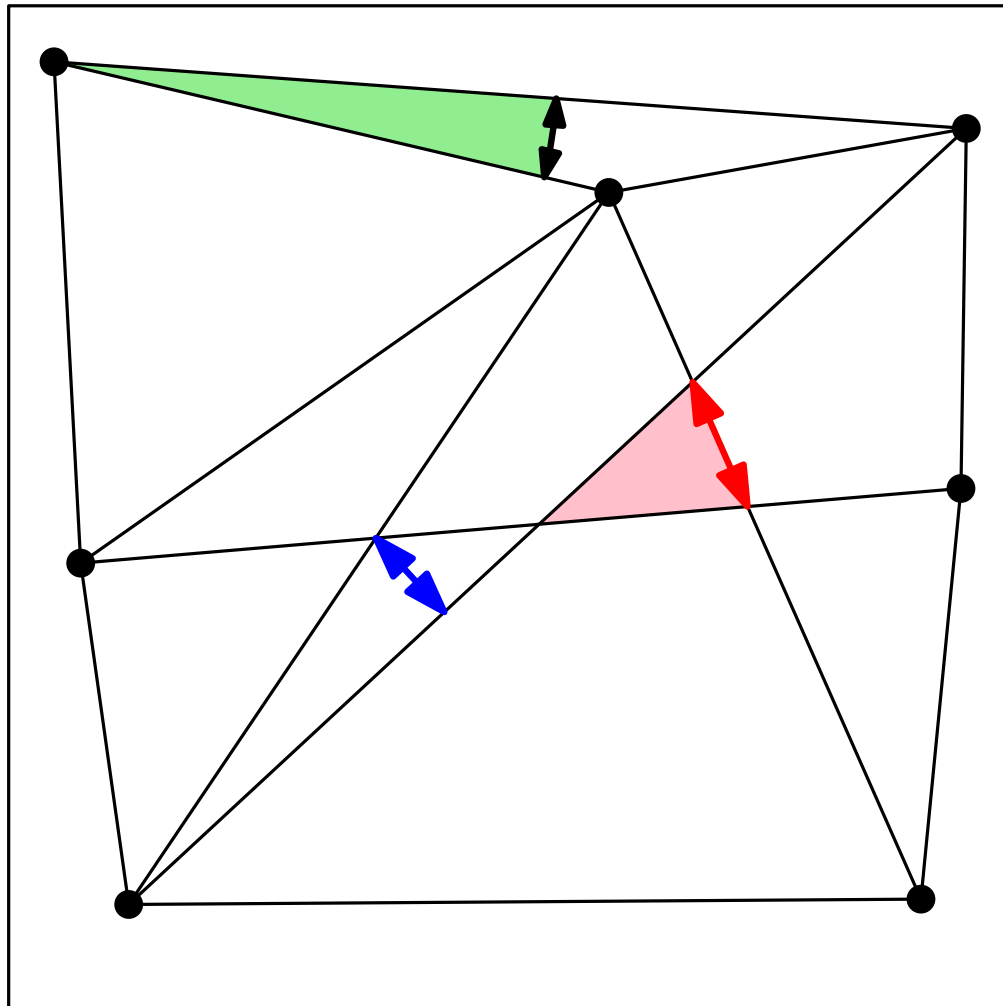
Maximize $\varepsilon =$

- the shortest edge
- the smallest feature distance
- the smallest angle

<http://userpage.fu-berlin.de/~uibyo/geometrica/>

Good Drawings

Find a (nonplanar) straight-line drawing of a given graph.



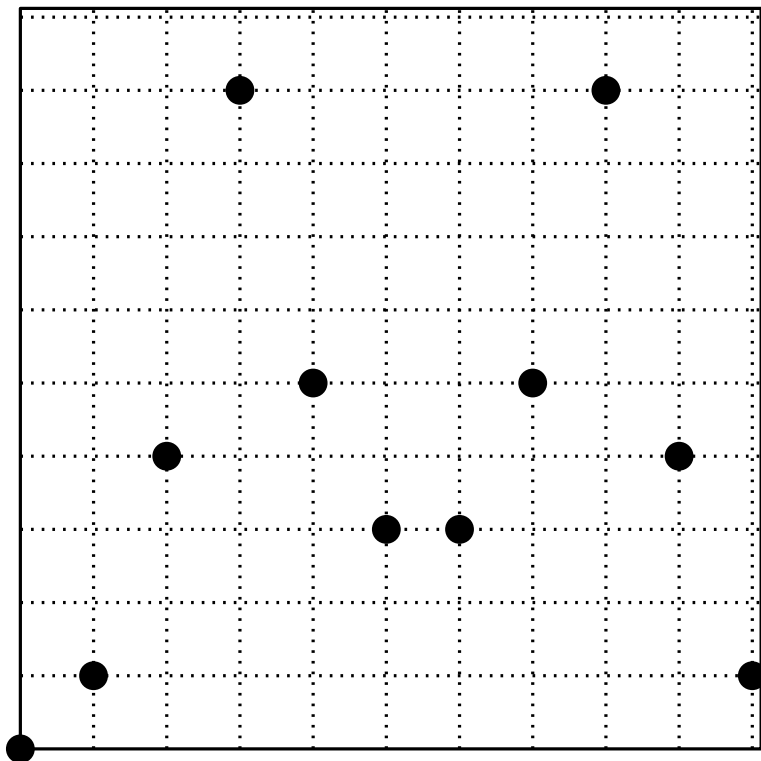
Maximize $\varepsilon =$

- the shortest edge
- the smallest feature distance
- the smallest angle
- the smallest face area
- ...

<http://userpage.fu-berlin.de/~uibyo/geometrica/>

Related Problem: The Moment Curve

Draw n points in general position, with small coordinates



Example: $p = 11$

$$P_i := \begin{pmatrix} 1 & i & i^2 \pmod p \end{pmatrix}, \quad i = 0, \dots, p-1$$

$(p \text{ prime})$

$$\begin{vmatrix} 1 & i & i^2 \pmod p \\ 1 & j & j^2 \pmod p \\ 1 & k & k^2 \pmod p \end{vmatrix} \equiv (j-i)(k-i)(k-j) \pmod p \neq 0 \pmod p$$

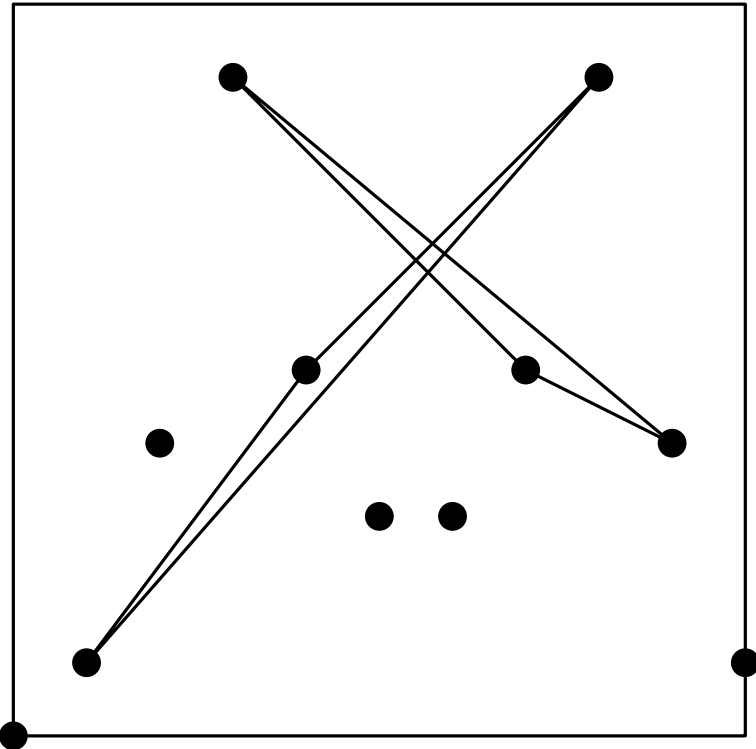
Application:

Perturbation techniques for
geometric algorithms

[Emiris & Canny 1995, Seidel 1998]

Related Problem: Heilbronn's Problem

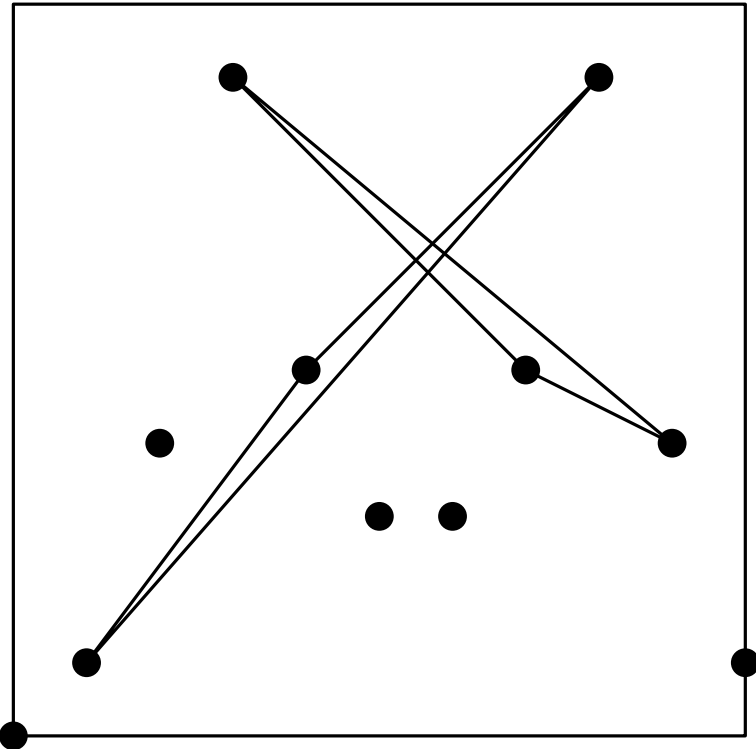
Place n points in the unit square and maximize the
smallest triangle area Δ_{\min}



$$\Delta_{\min} \geq \Omega(1/n^2)$$

Related Problem: Heilbronn's Problem

Place n points in the unit square and maximize the
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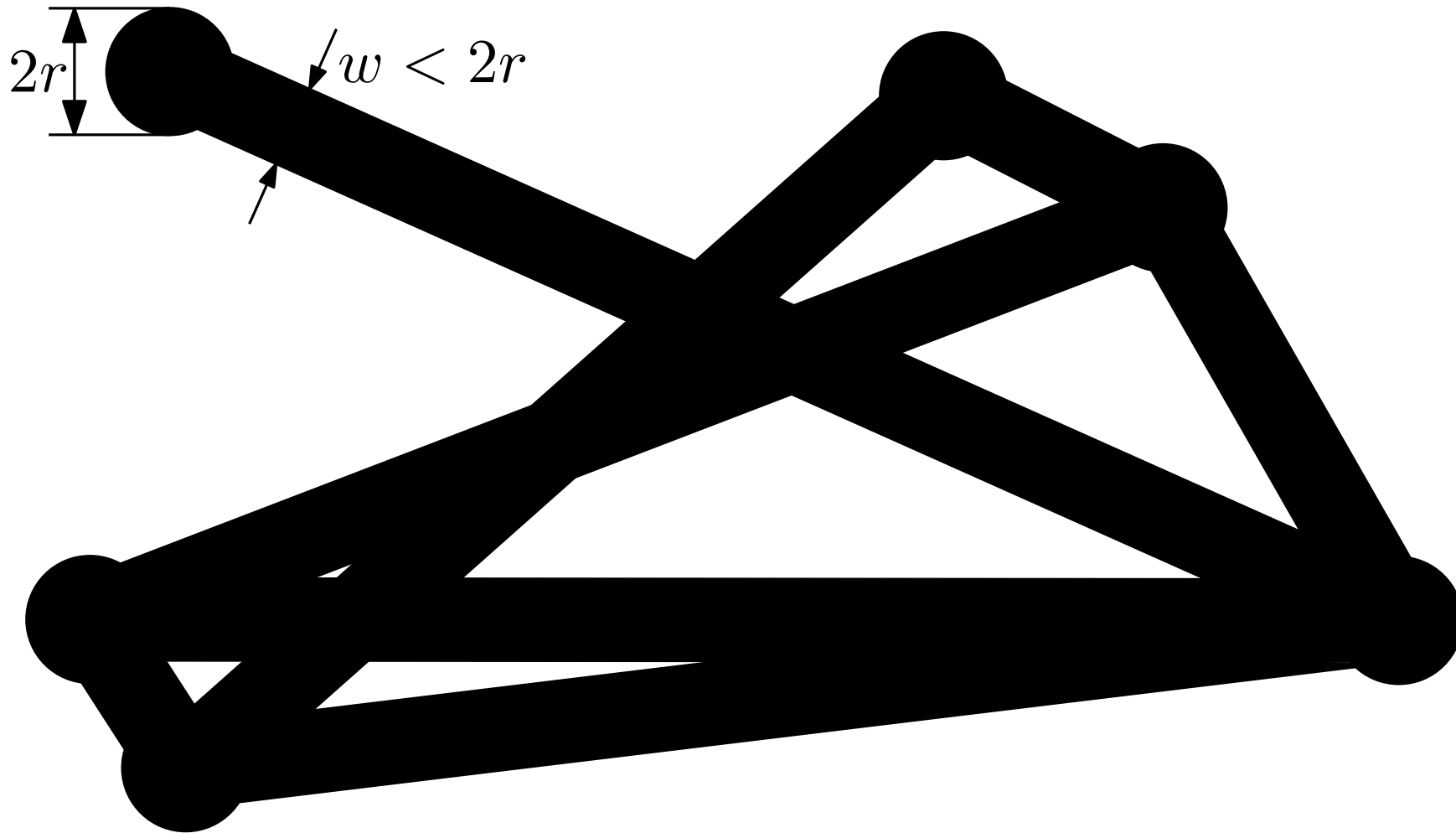
$$\Delta_{\min} \geq \Omega(1/n^2)$$

$$\frac{\ln n}{n^2} \leq \Delta_{\min} \leq \frac{1}{n^{8/7-\varepsilon}}$$

[Komlós, Pintz & Szemerédi 1981,1982]

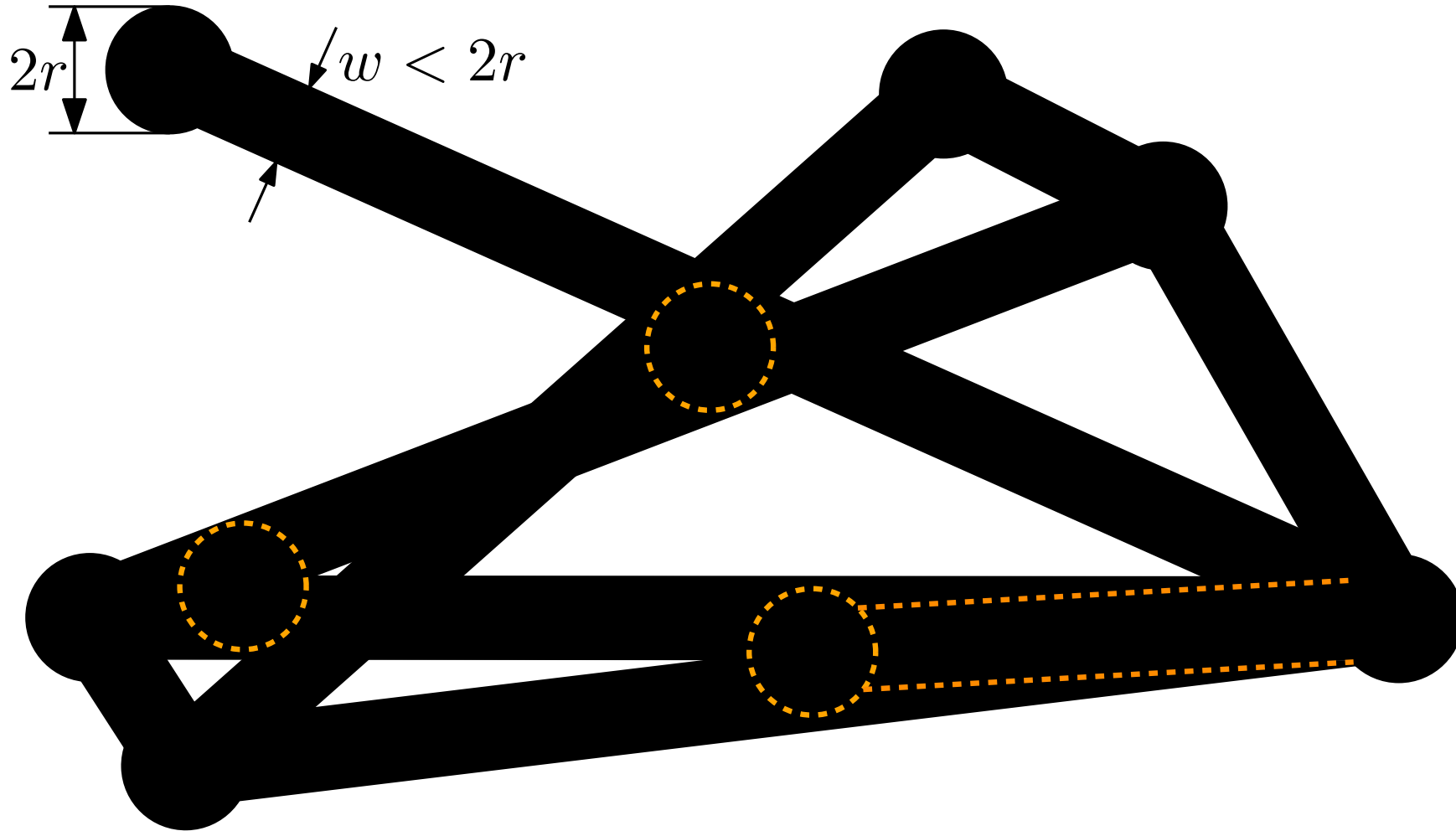
Related Problem: Bold Drawings

[van Kreveld 2009]



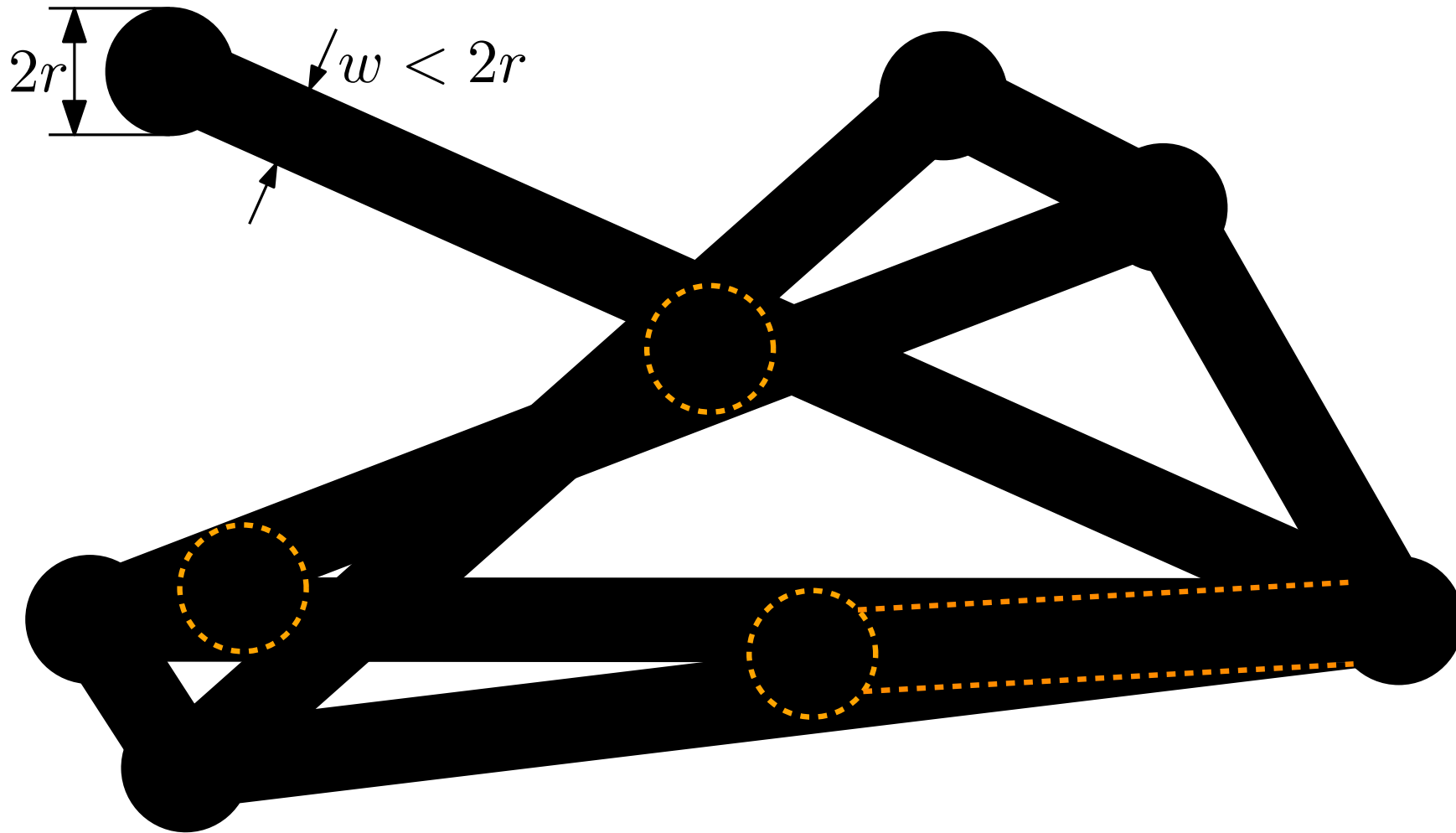
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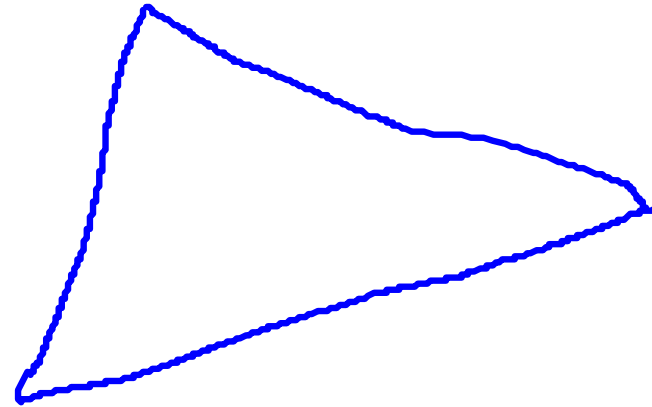
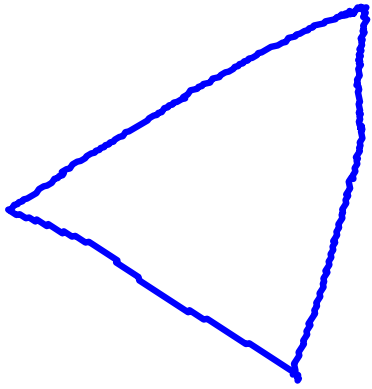
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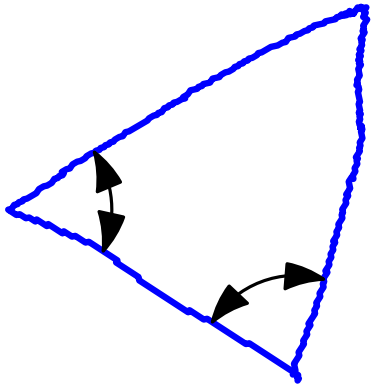


[Pach 2011] If $w < r$ then bold drawings of every K_n exist.

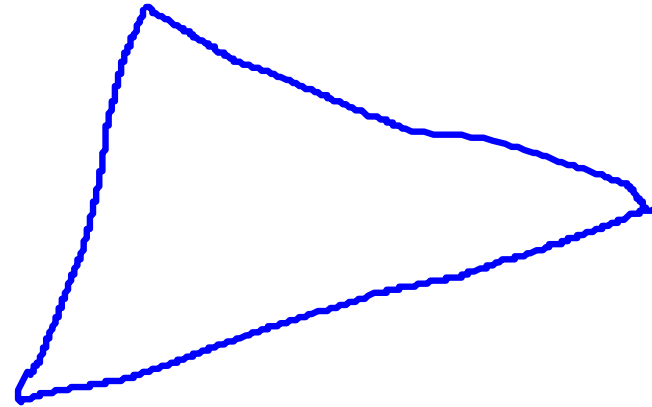
Related Problem: The General Triangle



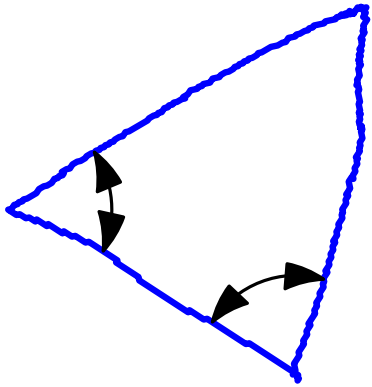
Related Problem: The General Triangle



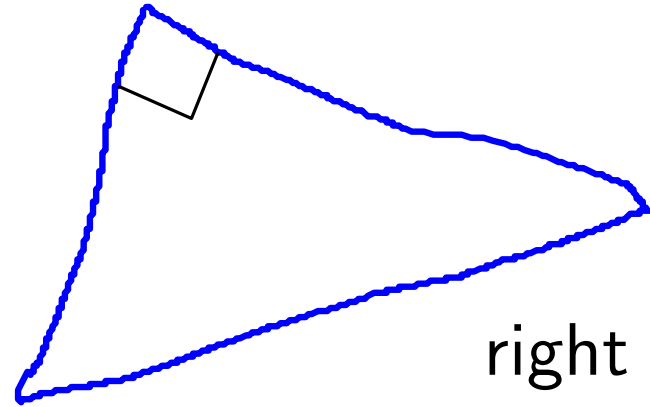
isosceles



Related Problem: The General Triangle

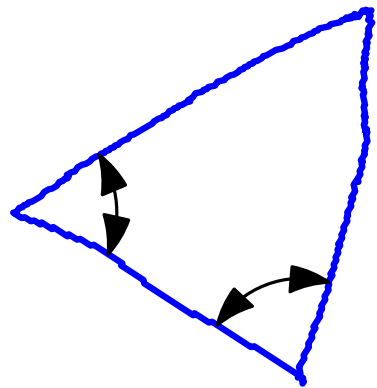


isosceles

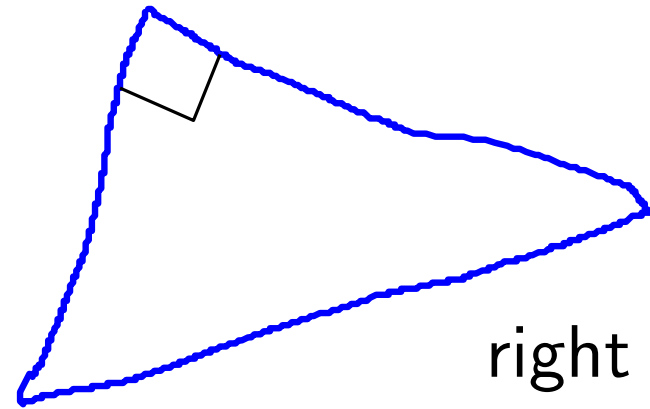


right

Related Problem: The General Triangle



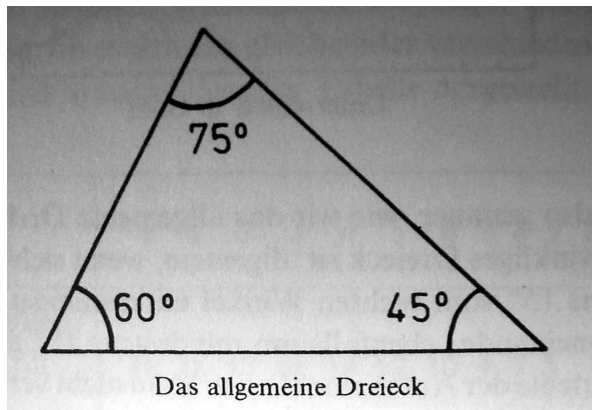
isosceles



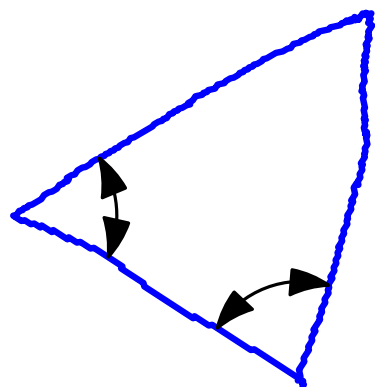
right

Theorem. There is a unique (up to similarity) general triangle. Its angles are 45° , 60° , and 75° .

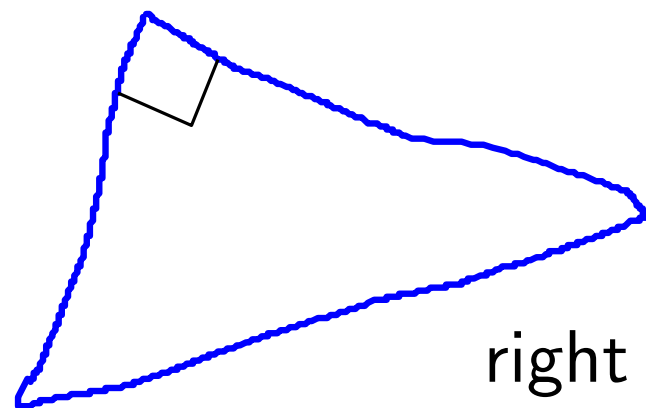
[B. Tergan, Das allgemeine Dreieck, *J. Math. Did.* **1**, 1980]



Related Problem: The General Triangle



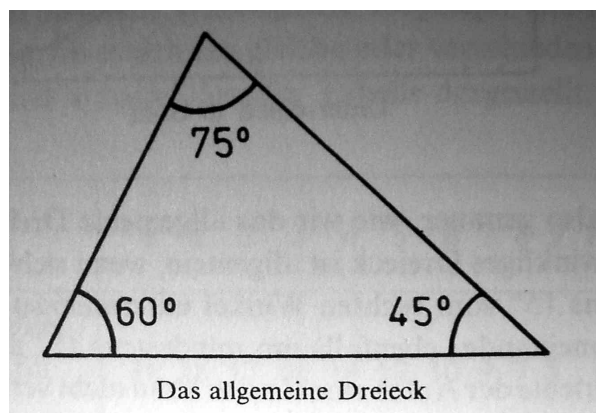
isosceles



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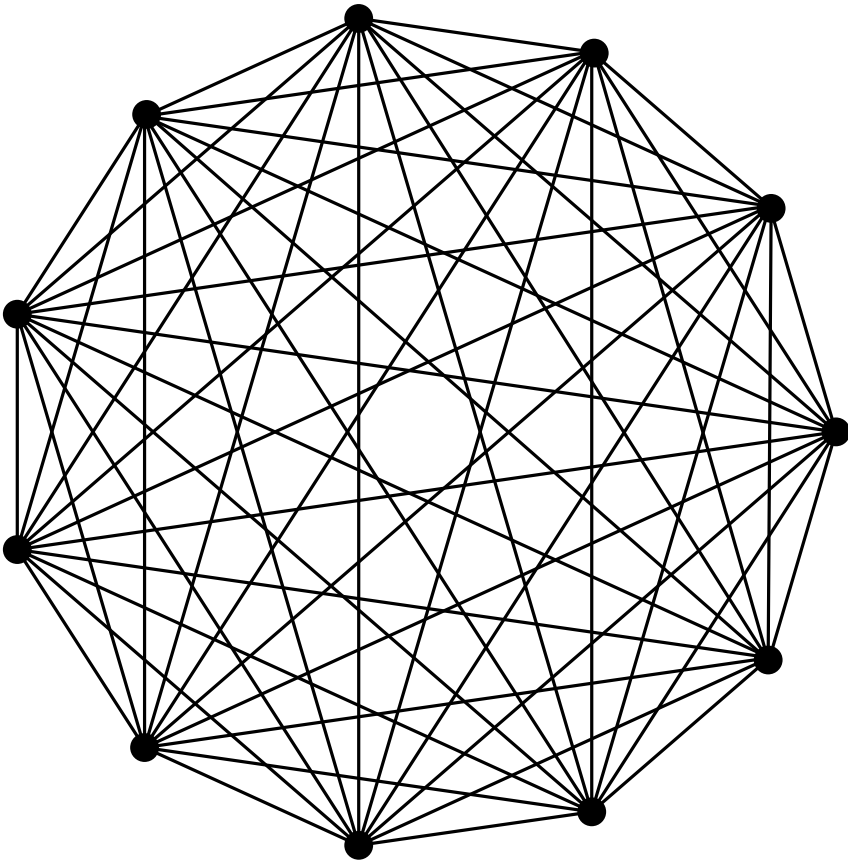
[B. Tergan, Das allgemeine Dreieck, *J. Math. Did.* **1**, 1980]



G. A. Jürgen, Das allgemeinste Dreieck, *Praxis d. Math.* **23**, 1981.

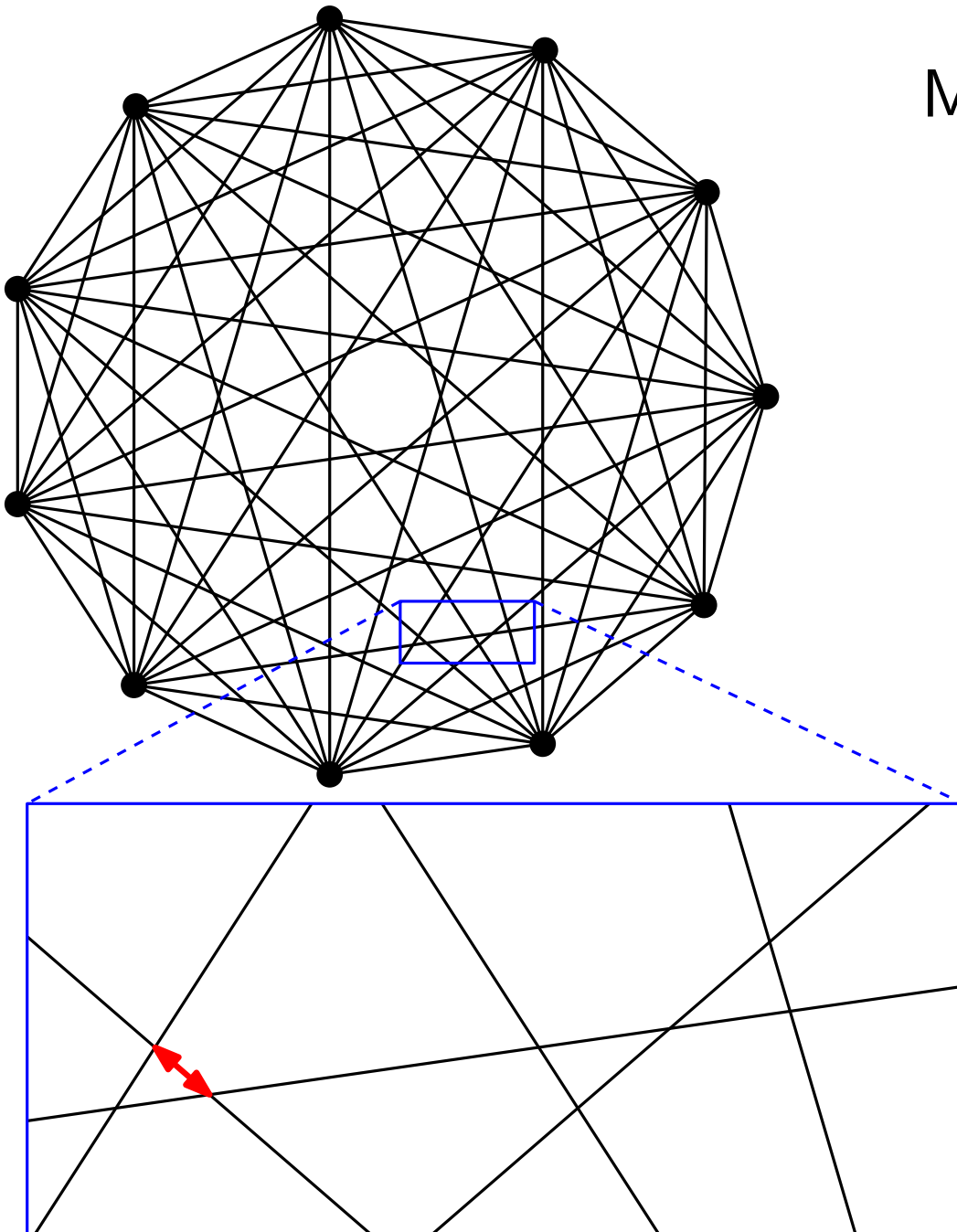
F. U. M. Weg, Das allgemeine Viereck, 1984.

Drawing K_n



Drawing K_n

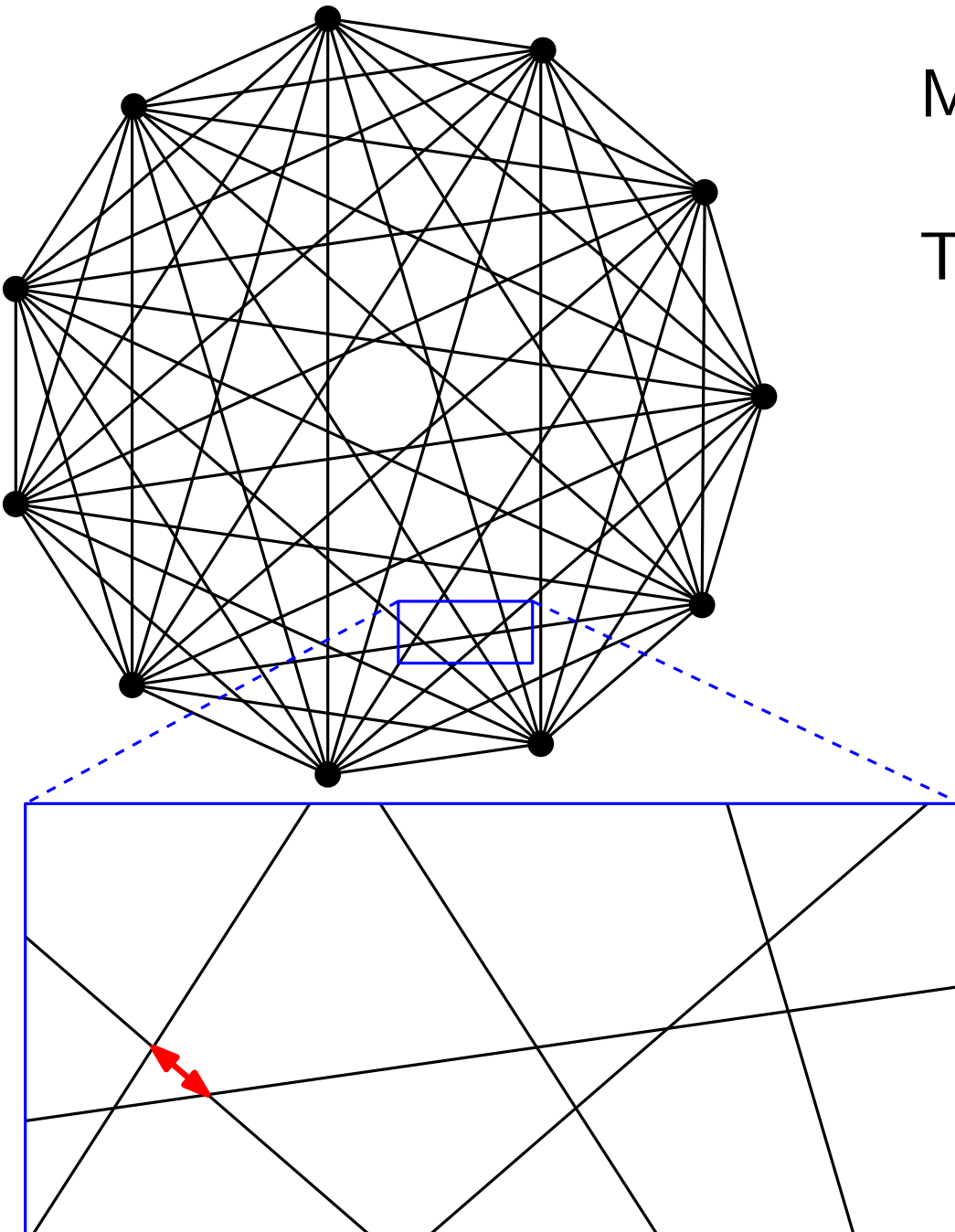
Maximize the shortest edge!



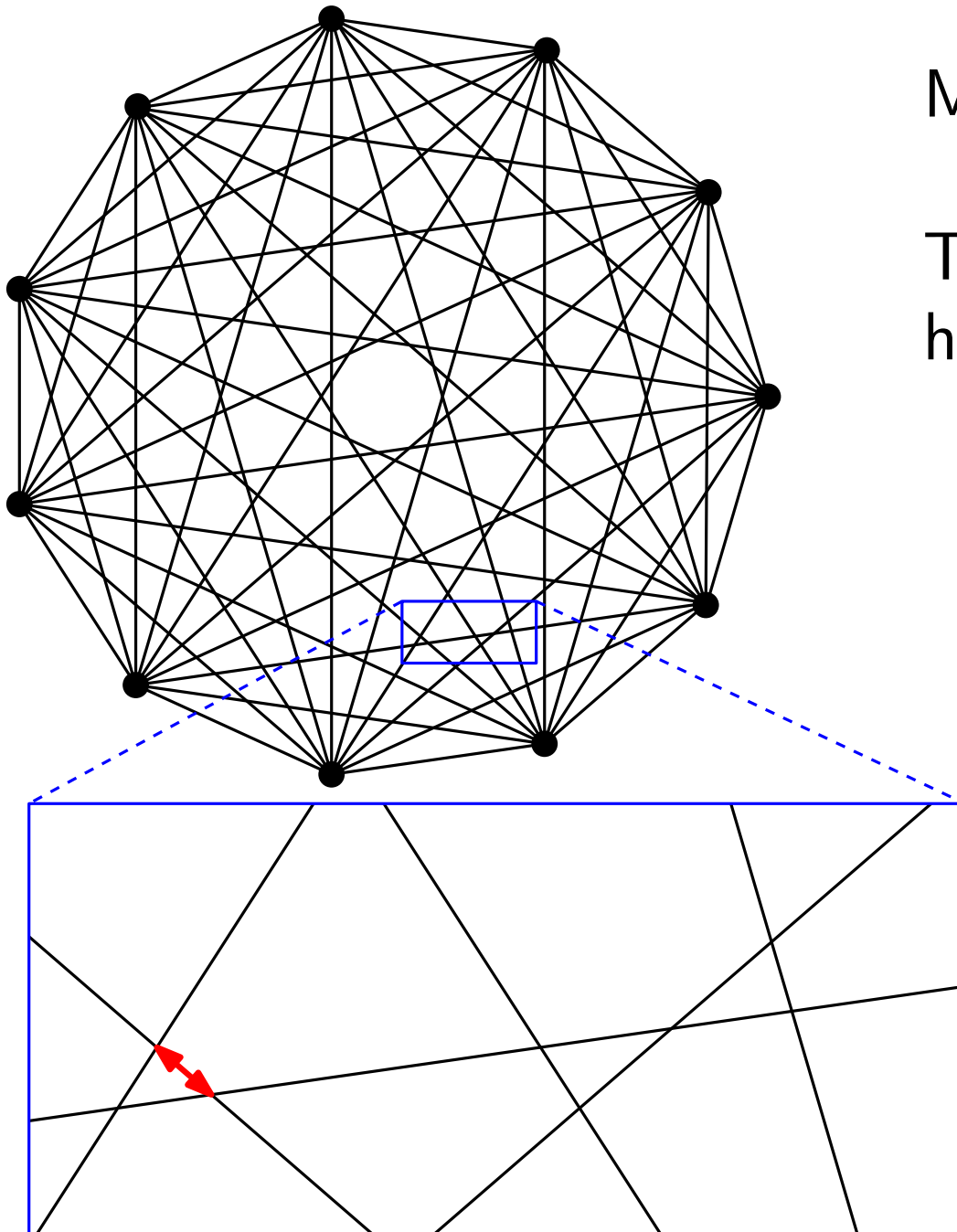
Drawing K_n

Maximize the shortest edge!

The regular n -gon (n odd)



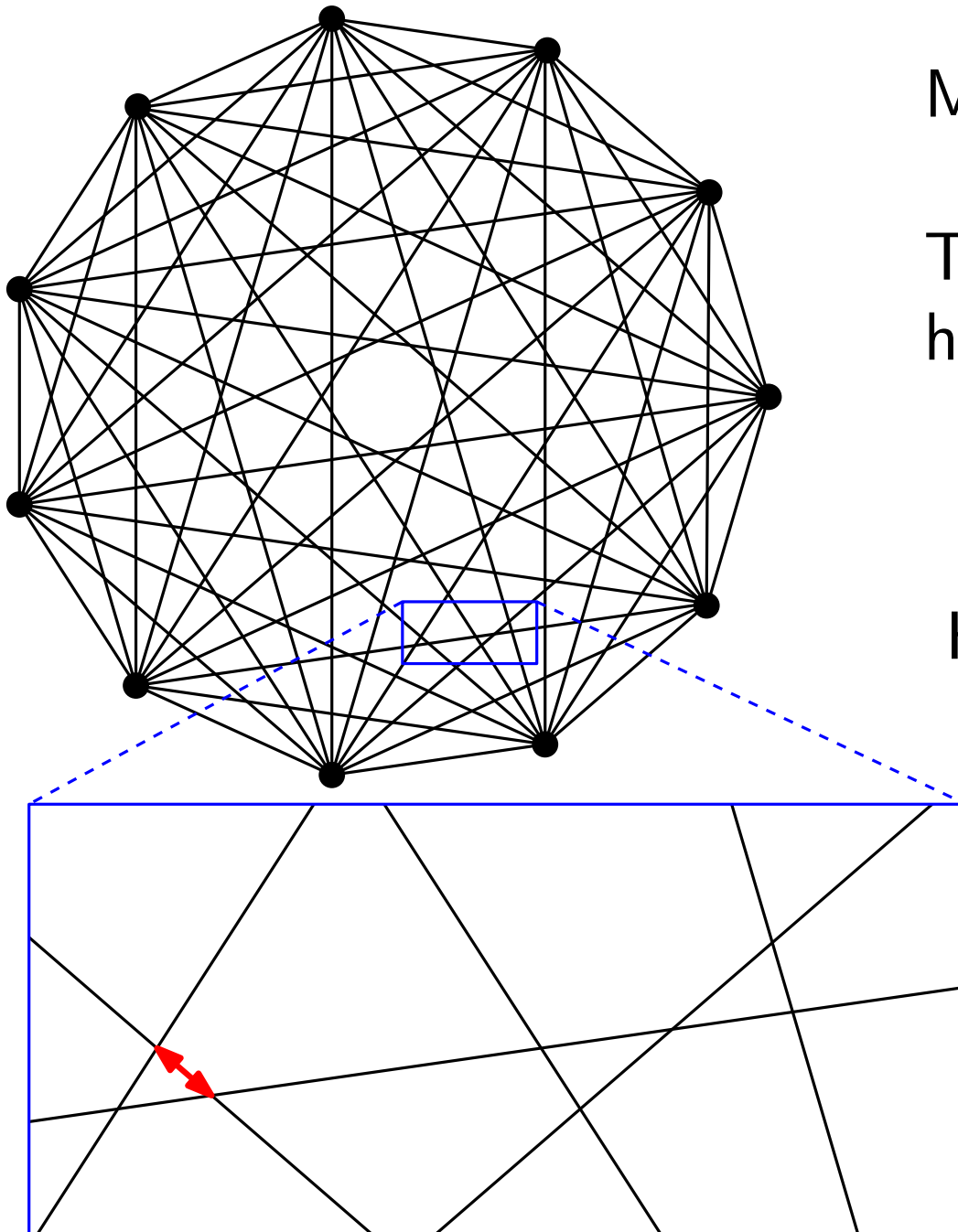
Drawing K_n



Maximize the shortest edge!

The regular n -gon (n odd)
has no triple diagonal crossings.
[Heineken 1962, Bol 1936]

Drawing K_n

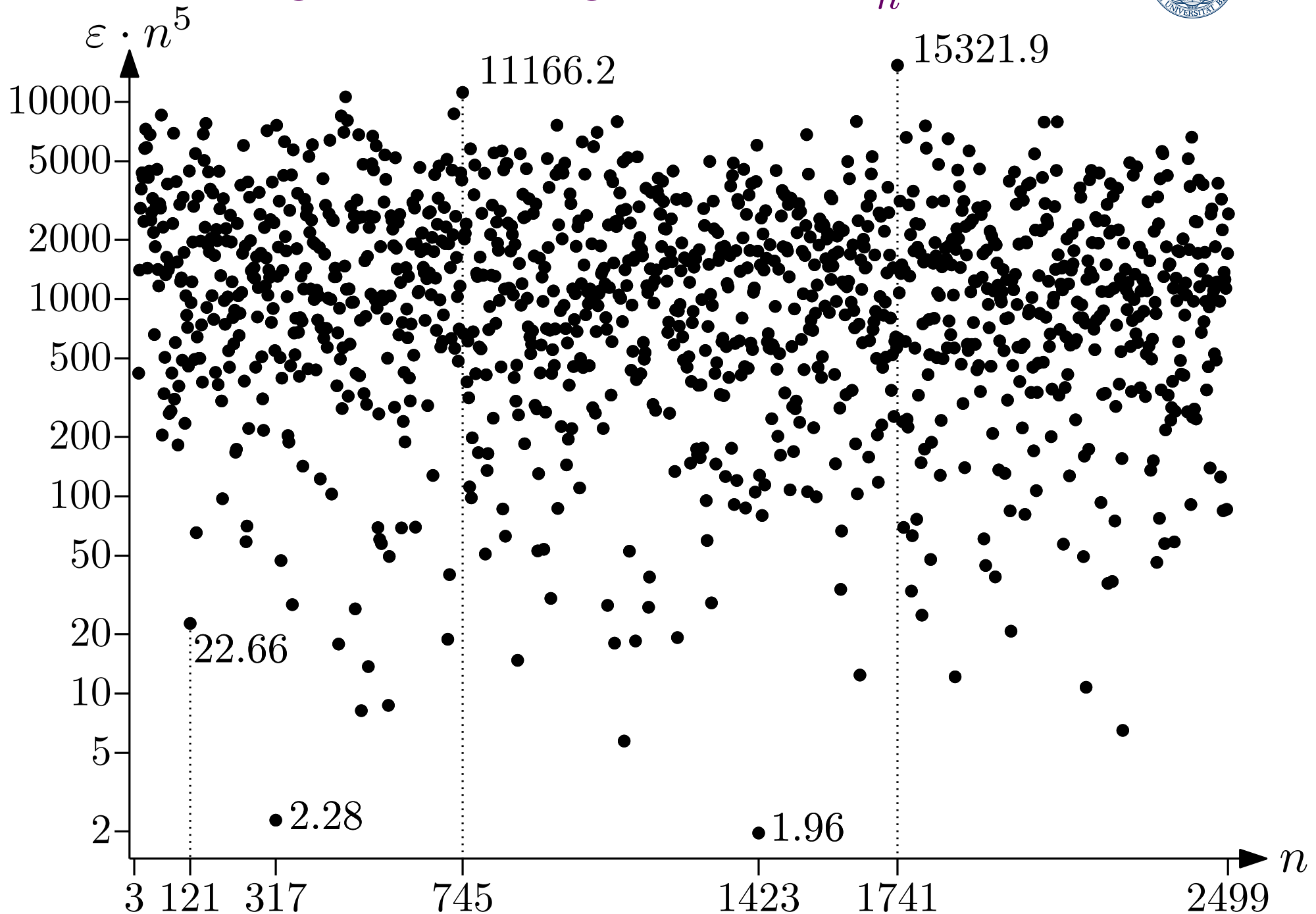


Maximize the shortest edge!

The regular n -gon (n odd)
has no triple diagonal crossings.
[Heineken 1962, Bol 1936]

How long is the shortest edge?

Shortest Edge in the Regular $K_n \approx \frac{1}{n^5}$



No Triple Diagonal Intersections

The regular n -gon (n odd) has no triple diagonal crossings.

[Heineken 1962, Bol 1936]

$z = e^{2\pi/n}$. Triple intersection of $\overline{z^0}, \overline{z^r}, \overline{z^s}, \overline{z^t}, \overline{z^u}, \overline{z^v} \implies$

$$\begin{vmatrix} 1 & z^0 z^r & z^0 + z^r \\ 1 & z^s z^t & z^s + z^t \\ 1 & z^u z^v & z^u + z^v \end{vmatrix} = 0$$

$\dots \implies F(z) = 0$ with

$$F(z) = (1 - z^t)(1 - z^s)(1 - z^{u+v-w}) - (1 - z^u)(1 - z^v)(1 - z^{s+t-r})$$

z is primitive n -th root of unity $\implies F(z^2) = 0 \implies \dots$

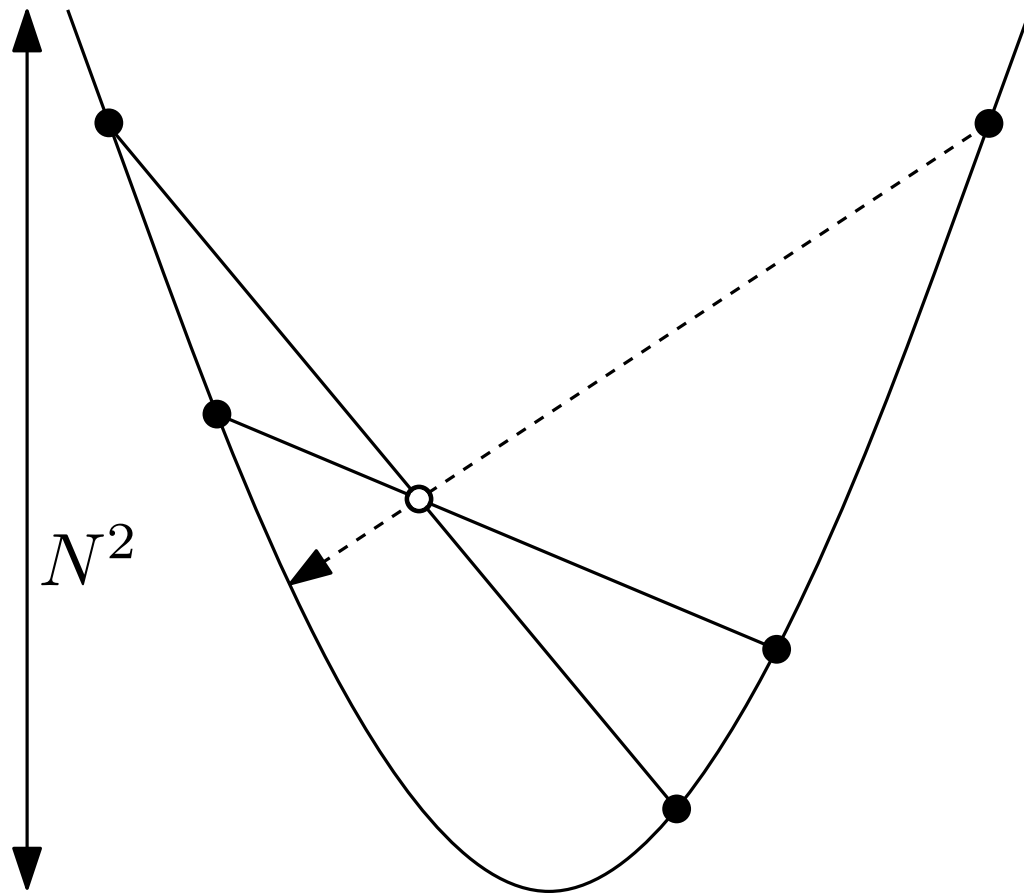
$\dots \implies z^{u+v} = z^{s+t}$

□

Separation bounds give something like $\varepsilon \geq 1/2^{O(n \log n)}$.

Incremental Construction

Add points one by one on the parabola $\binom{i}{i^2}$, $i = -N, \dots, N$

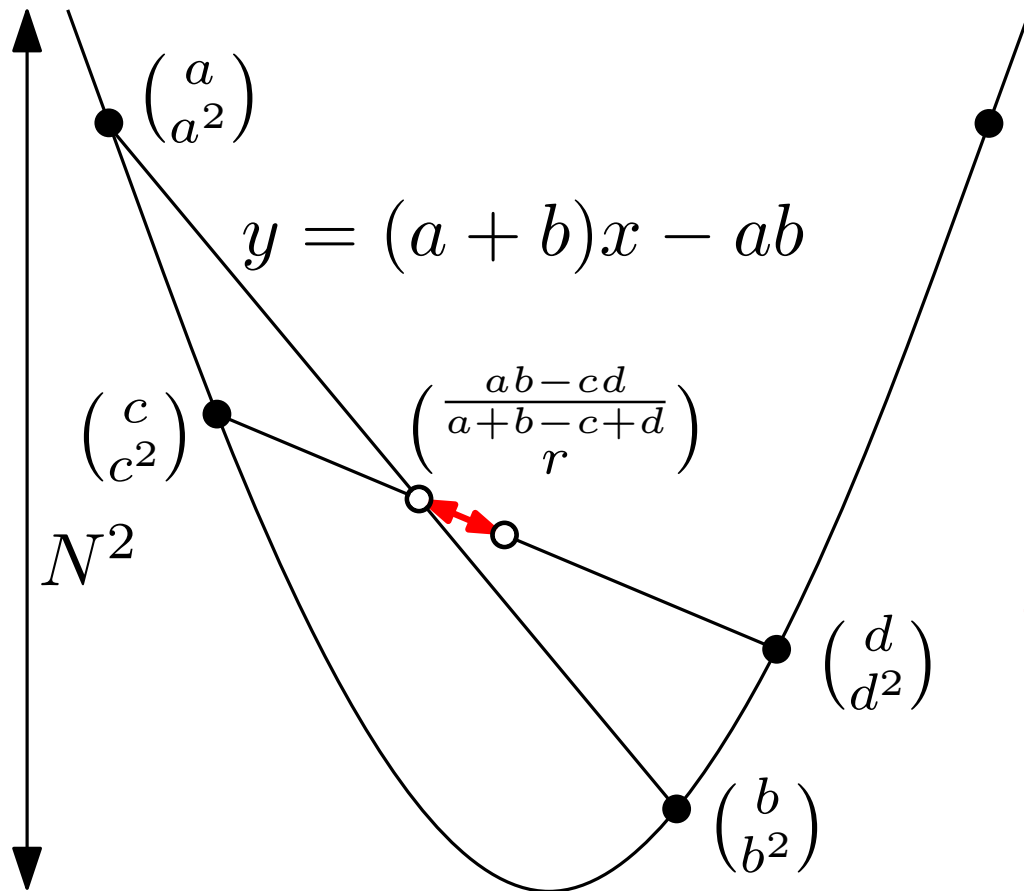


Every 5-tuple induces a “forbidden” placement for future points.

$N = n^5 \implies$ can place n points without triple crossings

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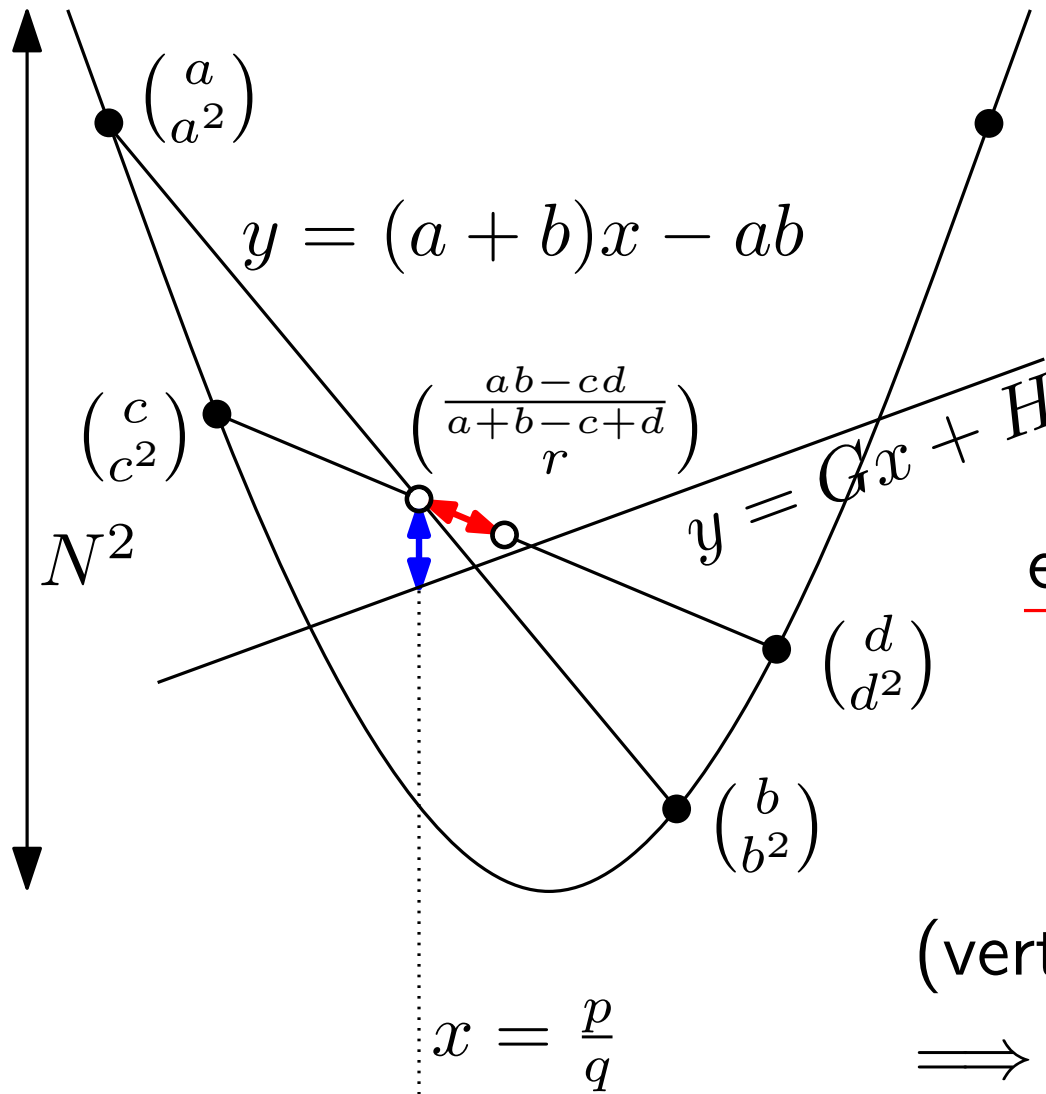
edge length \geq horizontal distance

$$\left| \frac{p}{q} - \frac{p'}{q'} \right| \geq \frac{1}{|qq'|} \geq \frac{1}{4N^2}$$

$\implies \geq \Omega(1/N^3)$ after scaling

Incremental Construction

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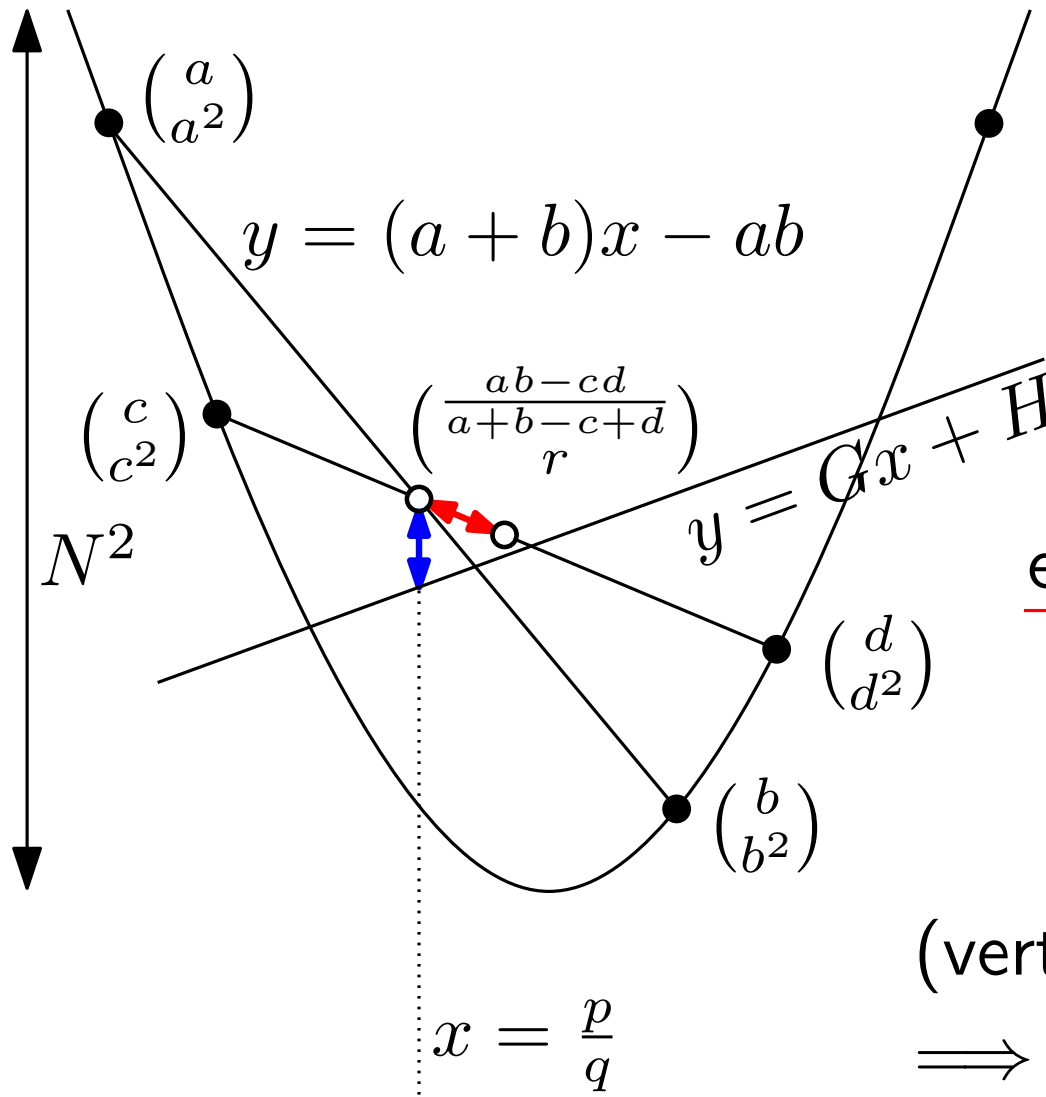
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(vertical) feature distance $\geq \frac{1}{q} \geq \frac{1}{2N}$

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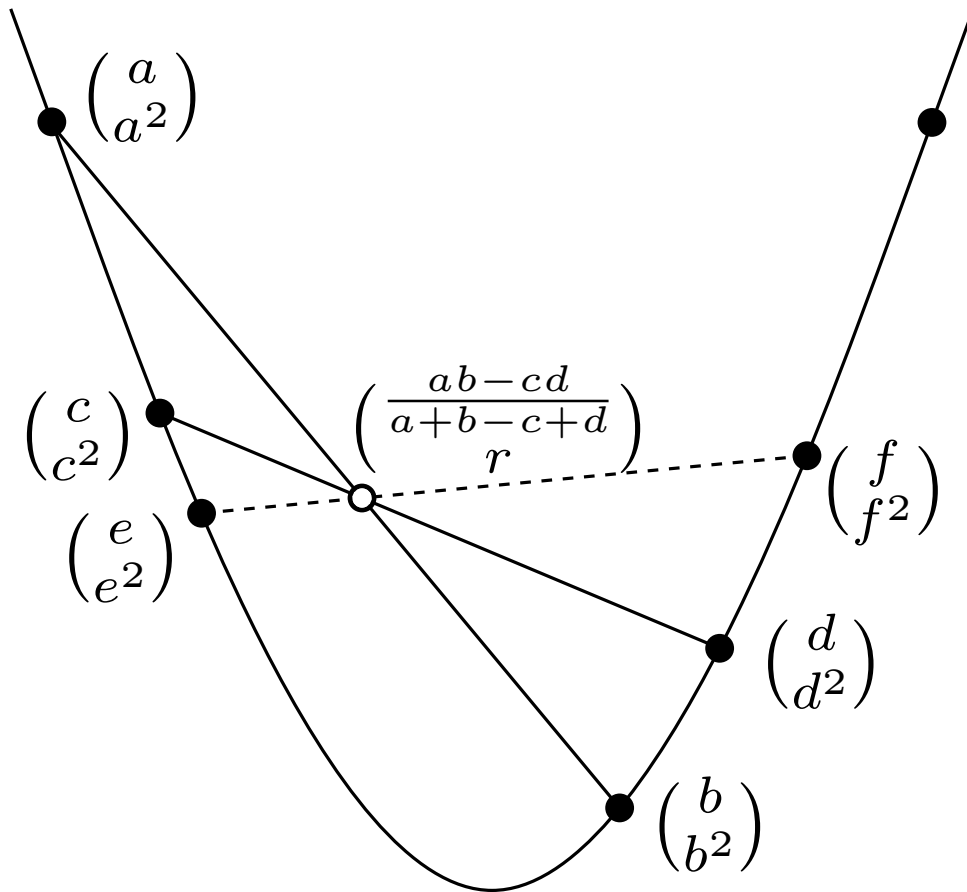
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Incremental Construction

Add points one by one on the parabola $\binom{i}{i^2}$, $i = -N, \dots, N$



How many diagonals go through the point $\binom{p/q}{r}$?

$$(e - \frac{p}{q})(f - \frac{p}{q}) = (a - \frac{p}{q})(b - \frac{p}{q})$$

$$(qe - p)(qf - p) = R \quad (\text{const.})$$

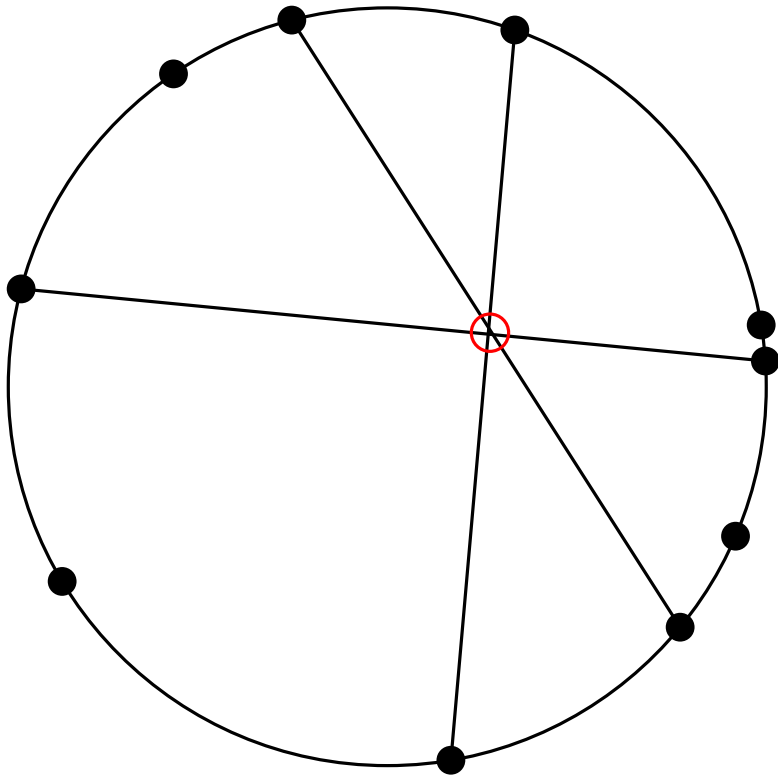
$$\begin{aligned} \# \text{ diagonals } \overline{\binom{e}{e^2}, \binom{f}{f^2}} \\ \leq \#(\text{divisors of } R) \\ \leq N^{O(1/\log \log N)} \quad (R = O(N^2)) \end{aligned}$$

Every 4-tuple eliminates $\leq N^{O(1/\log \log N)}$ possibilities.

$\implies N \geq n^4 \cdot N^{O(1/\log \log N)}$ is sufficient.

$\implies N := n^{4+O(1/\log \log n)} \implies \varepsilon = 1/n^{12+O(1/\log \log n)}$

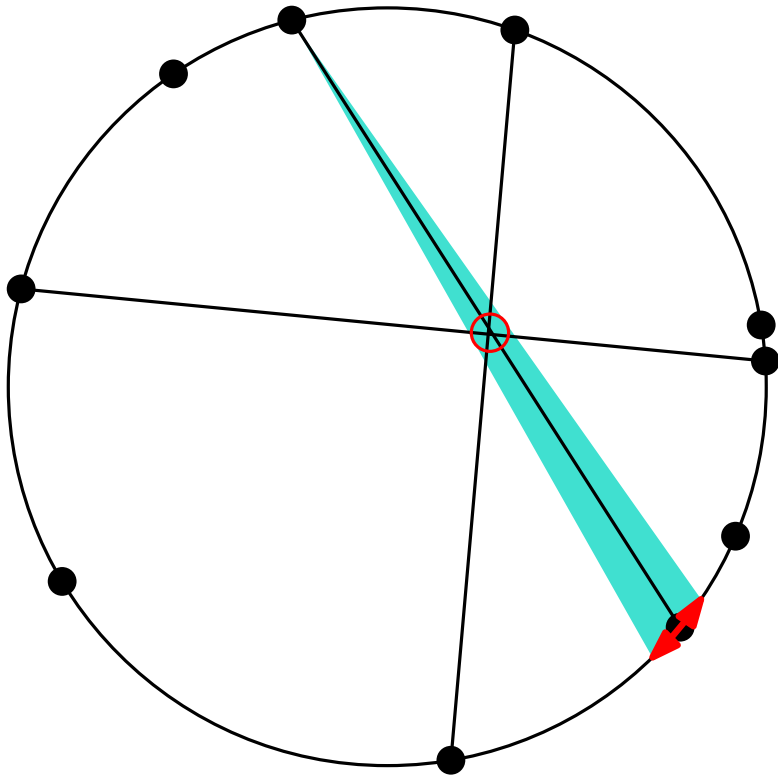
Randomized Construction



Place $2n$ random points on the circle
 $E[\# \text{ triplets of diagonals}$

closer than $\varepsilon] \leq n$

Remove $\leq n$ “bad” points



Place $2n$ random points on the circle
 $E[\# \text{ triplets of diagonals}$
 $\text{closer than } \varepsilon] \leq n$

Remove $\leq n$ “bad” points

$P[\text{a random triplet of diagonals}$
 $\text{is closer than } \varepsilon] \leq O(\varepsilon)$

Want:

$$E[\# \text{bad triplets}] = O(n^6) \cdot O(\varepsilon) \leq n$$

$$\implies \varepsilon = \text{const} \cdot \frac{1}{n^5}.$$

Direct constructions?

Points $\binom{i}{i^k}$, $i = 1, \dots, n$? $k = 5$ works for $n \leq 350$.

Greedy algorithm on the parabola $\binom{i}{i^2}$, $i = 1, 2, \dots$

Select $i = 1, 2, 3, 4, 5, 6, 7, 8, 14, 16, 24, 27, 30, 33, 50, 55, 61, 75, 77,$
102, 103, 106, 140, 144, 150, 189, 201, 229, 257, 259, 268, 309, 317,
338, 353, 357, 368, 412, 456, 471, 514, 560, 563, 566, 568, 580, 594,
646, 655, 739, 778, 831, 865, 895, 921, 937, 1019, 1038, 1079, 1147,
1185, 1195, 1235, 1352, 1412, 1439, 1440, 1488, 1511, 1529, 1536, 1551,
1562, 1586, 1619, 1788, 1791, 1851, 1930, 1990, 2052, 2057, 2169, 2187,
2198, 2234, 2237, 2342, 2444, 2450, 2518, 2603, 2695, 2723, 2726,
2771, 2841, 2959, 3080, 3113, 3262, 3277, 3285, 3286, 3413, 3444,
3652, 3666, 3731, 3772, 3817, 3991, 4041, 4051, 4216, 4217, 4225,
4236, 4478, 4572, 4582, 4872, 4909, 4945, 5013, 5254, 5274, 5303,
5415, 5498, 5543, 5572, 5608, 5613, 5722, 5786, 5794, 5820, 6273,
6321, 6387, 6506, 6669, 6687, 6879, 7061, 7170, 7199, 7473, 7520,
7566, 7596, 7608, 7609, 7692, 7753, 7832, 8016, 8221, 8248, 8694,
8811, 8956, 8967, 8975, 9003, 9148, 9180, 9198, 9354, 9387, 9418,

9572, 9638, 9754, 10092, 10109, 10112, 10126, 10168, 10196, 10333.