Flip Graphs of
Bounded-Degree Triangulations

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Edge Flips in Triangulations
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EDGE FLIPS IN TRIANGULATIONS

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The Flip Graph
The Flip Graph
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- connected.
- diameter $O(n^2)$
  $O(n)$ in convex position
The Flip Graph

- connected.
- diameter
  \[ O(n^2) \]
  \[ O(n) \] in convex position

Subgraphs of the flip graph:
maximum degree \( \leq k \) (small)

- connected?
- diameter?

General question: Local transformations between combinatorial structures
Results

For points in convex position, the flip graph of triangulations with degree \( \leq k \) is

- disconnected, for \( k = 4, 5, 6 \)
- connected, for \( k \geq 7 \). The diameter is \( O(n^2) \).

For points in general position, the flip graph of triangulations with degree \( \leq k \) can be disconnected, for any \( k \).
Points in Convex Position

maximum degree $k \leq 4$: zigzag triangulations

No edge can be flipped without creating a degree-5 vertex.
Points in Convex Position

maximum degree $k \leq 5$:  

maximum degree $k \leq 6$:  

![Graphs with maximum degree 5 and 6](image-url)
Theorem. Let $k \geq 7$. Any two triangulations of a set of $n$ points with maximum degree $\leq k$ can be transformed into each other by a sequence of $O(n^2)$ flips, without exceeding vertex degree $k$.

Proof strategy:

any triangulation $T \rightsquigarrow$ “canonical” triangulation $C$
Convex Position, $k \geq 7$

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Proof strategy:

any triangulation $T$ $\leadsto$ “canonical” triangulation $C$

“canonical” = zigzag triangulation
Convex Position, $k \geq 7$

**Theorem.** Let $k \geq 7$.
Any two triangulations of a set of $n$ points with maximum degree $\leq k$ can be transformed into each other by a sequence of $O(n^2)$ flips, without exceeding vertex degree $k$.

Proof strategy:

any triangulation $T$ \sim “canonical” triangulation $C$

another triangulation $T'$ \sim triangulation $C'$
Convex Position, \( k \geq 7 \)

**Theorem.** Let \( k \geq 7 \).
Any two triangulations of a set of \( n \) points with maximum degree \( \leq k \) can be transformed into each other by a sequence of \( O(n^2) \) flips, without exceeding vertex degree \( k \).

**Proof strategy:**

any triangulation \( T \) \[ \rightarrow \] “canonical” triangulation \( C \) \[ \leftarrow \] another triangulation \( T' \) \[ \rightarrow \] triangulation \( C' \)

“canonical” = zigzag triangulation
The Dual Graph
The Dual Graph
The Dual Graph

schematic drawing
The Dual Graph

ears (degree 1, leaves)

schematic drawing
The Dual Graph

ears (degree 1, leaves)
path triangles (degree 2)
schematic drawing
The Dual Graph

ears (degree 1, leaves)
path triangles (degree 2)
inner triangles (degree 3, branching vertices)
Paths in the Dual Tree

inner path

leaf path
Paths in the Dual Tree

inner path

leaf path

goal: zigzag
Overall Strategy

- Turn leaf paths into zigzags \( O(n^2) \) flips
- Eliminate inner triangles \( O(n^2) \) flips
- Rotate final zigzag \( O(n^2) \) flips
Turn Leaf Path into Zigzag

\[ O(n) \text{ flips for extending the zigzag by 1} \rightarrow O(n^2) \text{ total} \]
Zigzag Rotation

STRATEGY: Always turn leaf paths into zigzags.

can also rotate final zigzag:

\[ O(n^2) \text{ flips in total.} \quad \text{(can even be done in } O(n) \text{ flips)} \]
Create a Zigzag Triangulation

Goal: dual tree $D \rightarrow$ a zigzag path

Strategy:
Process the tree from the leaves towards the center.
Find a *good merge triangle* and eliminate it:

leaf path $\leq k$ $\leq k$ $\leq k$

leaf path $\leq k$ $\leq k$ $\leq k$
Create a Zigzag Triangulation

Goal: dual tree $D \rightarrow$ a zigzag path

Strategy:
Process the tree from the leaves towards the center.
Find a *good merge triangle* and eliminate it:

leaf path

$\leq k \quad \quad \quad \quad \leq k$

$\Delta'$

leaf path

$\leq k \quad \quad \quad \quad \leq k$

$v$
Find a Good Merge Triangle $\Delta'$

start with the dual tree $D$
Find a Good Merge Triangle $\Delta'$

start with the dual tree $D$

remove all leaf paths: $\rightarrow D'$
(leaves of $D'$ = merge triangles)

merge triangles
Find a Good Merge Triangle $\Delta'$

start with the dual tree $D$

remove all leaf paths: $\rightarrow D'$
(leaves of $D'$ = merge triangles)

remove all leaves of $D'$: $\rightarrow D''$
merge triangles

take a merge triangle $\Delta'$ adjacent to a leaf $\Delta''$ of $D''$
Find a Good Merge Triangle $\Delta'$

start with the dual tree $D$

remove all leaf paths: $\rightarrow D'$
(leaves of $D'$ = merge triangles)

remove all leaves of $D'$: $\rightarrow D''$

take a merge triangle $\Delta'$ adjacent to a leaf $\Delta''$ of $D''$
Find a Good Merge Triangle $\Delta'$
Find a Good Merge Triangle $\Delta'$

leaf path

(leaf path)

leaf path

leaf path

leaf path

leaf path
Find a Good Merge Triangle $\Delta'$

\[ \Delta' \leq k \]

(leaf path)

\[ \Delta'' \]

\[ \Delta'' \leq 6 < k \]

leaf path

\[ = 6 \]

leaf path

leaf path

\[ = 6 \]

leaf path

\[ \leq k \]

(leaf path)

\[ = 6 \leq k \]
Eliminate a Merge Triangle

\[ \text{leaf path} \]

\[ \Delta' \]

\[ \deg \leq k \]

\[ \deg < k \]
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

leaf path

\[ \Delta' \]

\[ \text{deg} \leq k \]

\[ \text{deg} < k \]
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

leaf path

\[ \Delta' \]

\[ \deg \leq k \]

\[ \deg < k \]
Eliminate a Merge Triangle

$$v_{\text{tip}}$$

$$\deg \leq k$$

$$\deg \leq k$$

$$+1$$
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

\[ \deg \leq k \]

\[ \deg \leq k \]

\[ +1 \]
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

\[ \text{deg} \leq k \]

\[ \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ \deg \leq k \]

\[ v_{\text{tip}} \]

\[ \deg \leq k \]
Eliminate a Merge Triangle

\[ \nu_{\text{tip}} \]

\[ \text{deg} \leq k \]

\[ \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

\[ \text{deg} \leq k \]

\[ \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

\[
\text{deg} \leq k
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\[
\text{deg} \leq k
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Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

\[ \text{deg} \leq k \]

\[ \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

\[ \deg \leq k \]

\[ \deg \leq k \]
Eliminate a Merge Triangle

$v_{tip}$

$\deg \leq k$

$\deg \leq k$
Eliminate a Merge Triangle

\[ \text{deg} \leq k \quad \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

\[ \deg \leq k \]

\[ \deg \leq k \]
Eliminate a Merge Triangle

\[ \text{deg} \leq k \]

\[ v_{\text{tip}} \]

\[ \text{deg} \leq k \]
Eliminate a Merge Triangle

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\[ v_{\text{tip}} \]

\[ \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ v_{\text{tip}} \]

\[ \text{deg} \leq k \quad \text{deg} \leq k \]
Eliminate a Merge Triangle
Eliminate a Merge Triangle

$v_{\text{tip}}$

$\deg \leq k$

$\deg \leq k$

$\deg \leq k$

$\deg \leq k$
Eliminate a Merge Triangle

\[ \text{deg} \leq k \]

\[ v_{\text{tip}} \]

\[ \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ \nu_{\text{tip}} \]

\[ \deg \leq k \]

\[ \deg \leq k \]
Eliminate a Merge Triangle

\[ \text{deg} \leq k \quad \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ \text{deg} < k \]

starting situation

\[ \text{deg} \leq k \]
Eliminate a Merge Triangle

\[ \deg < k \]

\[ \deg \leq k \]

starting situation
Eliminate a Merge Triangle

\[ \text{deg} < k \]

\[ \text{deg} \leq k \]

starting situation

starting situation
Eliminate a Merge Triangle
Eliminate a Merge Triangle
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Eliminate a Merge Triangle
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Eliminate a Merge Triangle

\[ O(n) \text{ flips.} \]
Schematic view

leaf path

\( v_{\text{tip}} \)

leaf path

\( v_{\text{tip}} \)

OK
Schematic view

leaf path

$v_{\text{tip}}$

leaf path

OK

OK
Schematic view

ANALYSIS:
n operations:
- extending a zigzag by 1
- merging to leaf paths
Each operation takes $O(n)$ flips.  \[ \rightarrow O(n^2) \text{ in total.} \]
+ final rotation: $O(n^2)$ flips.
General Point Sets

Bounded degree is not always possible:

These edges are part of every triangulation.
maximum degree $= n - 1$
General Point Sets

**Theorem.** For any $k \geq 4$, there are arbitrarily large point sets with two triangulations $T, T'$ with maximum degree $k$ that cannot be transformed into each other by a sequence of flips without exceeding vertex degree $k$. 
General Point Sets

Example: $k = 8$

no flips possible

some freedom inside the shaded regions

(for example: appropriate zigzag triangulations)
General Point Sets

\[ T \quad T' \]
OPEN QUESTION:

1. Can two triangulations with degree $\leq k$ be transformed into each other without exceeding degree $k + 1$? (Or $2k$? Or $f(k)$?)
Open Questions

1. Can two triangulations with degree \( \leq k \) be transformed into each other without exceeding degree \( k + 1 \)? (Or \( 2k \)? Or \( f(k) \)?)

2. How about pseudotriangulations? (\( k \geq 10 \) !)

3. Flip diameter of bounded-degree triangulations of convex point sets is \( O(n) \)? \( O(n \log n) \)?