Piecewise Linear Morse Theory

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Classical Morse Theory [8] considers the topological changes of the level sets $M_h = \{ x \in M \mid f(x) = h \}$ of a smooth function f defined on a manifold M as the height h varies. At *critical points*, where the gradient of f vanishes, the topology changes. These changes can be classified locally, and they can be related to global topological properties of M. Between critical values, the level sets vary smoothly.

This talk concerns Morse Theory of *piecewise linear* functions, and in particular, the "uninteresting" part of Morse theory, the level sets *between* the critical values, where "nothing happens". Spatial data coming from data acquisition processes (like medical imaging) or numerical simulations (like fluid dynamics) need to be represented for the purpose of storage on a computer, visualization, or further processing. Commonly they are represented as piecewise linear functions. My interest in Morse theory arose out of a fast and simple algorithm [4] for constructing the *contour tree* (or *Reeb graph*) of a piecewise linear function, a tree that represents how the connected components of the level sets, the *contours*, split and merge, are created and destroyed. While writing up this algorithm, I felt that I should say something about the obvious absence of topological changes when passing over "non-critical" vertices, but I could not find any results in the literature that I could readily apply. The results below are a contribution towards the foundations of Morse theory for piecewise linear functions of up to three variables.

1. Results and Open Questions

We assume that the domain M is a triangulation of a convex region in $\mathbb{R}^3$. The function f is given at the vertices and extended to M by linear interpolation. For simplicity, we restrict our attention to vertices in the interior of M. For our purposes, the *link* of a vertex v is the graph consisting of the neighbors of v, with an edge between two neighbors u and w if the triangle uvw is in the triangulation. The upper (lower) link is generated by the vertices whose value is bigger (smaller) than $f(v)$. We assume that no two vertices have the same value.

Theorem 1. *Let v be a vertex in the interior of M. The topology of the level sets M_h is the same for all values h in a sufficiently small interval $f(v) - \varepsilon \le h \le f(v) + \varepsilon$ if and only if the upper and the lower link are both non-empty and connected.*

The criterion can be adapted for boundary vertices, and also for two-dimensional domains. If the condition if the theorem is fulfilled, we call v a regular (or ordinary) point, otherwise it is a *critical point*, and $f(v)$ is a *critical value*.

Theorem 2. *If the interval $[a, b]$ contains no critical value, then there is an isotopy between all level sets in this range, i. e., a continuous bijection*

$$g\colon M_b \times [a, b] \to \{ x \in M \mid a \le f(x) \le b \}$$

that is level-preserving: $f(g(x, h)) = h$.

For a fixed height h, the homeomorphism $g(\cdot, h)$ between M_b and M_h is piecewise linear. However, the isotopy is not piecewise linear when regarded on its domain $M_b \times [a, b]$. (It should not be too difficult to strengthen the proof to achieve this.)

The proof is given in the appendix of [4] by an explicit construction: very roughly, the upper link of v is embedded as a planar straight-line graph inside the convex polygon whose sides correspond to the tetrahedra incident to v that are intersected by the level set through v. This graph has the same face structure as the level set above v. As the level set proceeds downwards towards v, the graph of the upper link shrinks towards the center, and at v, the result is a wheel.

The part of the proof that relies on drawing a graph with straight lines does not carry over to higher dimensions. An alternative approach that has not been tried might be to use a sequence of elementary subdivision operations (by inserting a new vertex into a cell) and their inverse "welding" operations [5, Theorem II.11].

There is a natural conjecture for the extension of the characterization of critical points to 4 dimensions: the link of a vertex is a 3-sphere, and for a regular vertex, it should be necessary and sufficient that the level set through this vertex cuts this 3-sphere into two 3-balls that are glued together along a 2-sphere forming their common boundary. This condition is straightforward to test. In five and higher dimensions, the problem of recognizing a critical point becomes more difficult, and it is probably even undecidable, for some high enough dimension.

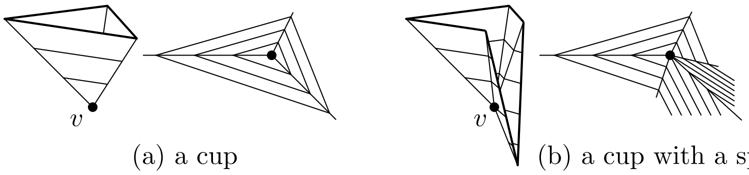
2. Related Literature

Interestingly, in Morse Theory for *continuous* functions [9], the criterion for the *definition* of a regular point v is just a local version of the *conclusion* of our Theorem 2: the existence of an isotopy between level sets in the neighborhood of v, i. e., some height-preserving homeomorphism between some neighborhood of v and the Cartesian product of a manifold with an interval of height values.

Tom Banchoff introduced Morse theory for piecewise linear functions in a widely known and often cited paper [2] about critical points, which even contains a *Critical Point Theorem*, without ever defining critical points, however. The results concern the Euler characteristic of the manifold and its relation to an appropriately defined index of a critical point. They remain at the level of counting, and no connection to the topology of level sets is made.

Morse Theory for piecewise linear functions has also been treated by Brehm and Kühnel [3, Section 2]; see also Kühnel [7, Chapter 7] for a more detailed account. Critical points are defined and the topology changes at these points are analyzed at the level of homology. For our case of two and three dimensions, this implies that regular points, where the homology is trivial, do not incur a topology change when the level set passes them, and there is a piecewise linear homeomorphism between different level sets [5]. However, the existence of this homeomorphism does not lead to the isotopy of Theorem 2.

In a related paper, Agrachev, Pallaschke and Scholtes [1] get a conclusion like in Theorem 2 under a stronger condition. To classify a vertex v in a piecewise linear function as a regular vertex, they require the existence of a direction that has a positive scalar product with the gradients of f on all cells incident to v. The example of a bivariate function in the figure shows that this is stronger than necessary. The graph of the function has three faces forming a "cup". The figure shows a side view and a top view with level lines. The vertex v at the bottom is a minimum and therefore certainly not regular. If we add a "spout" that makes if possible for the water to flow out, we only add more faces incident to v, and this cannot make the vertex regular, according to this definition. However, one sees that the level lines are isotopic as they pass through v.



(a) a cup (b) a cup with a spout

The *Stratified Morse Theory* of Goresky and MacPherson [6] seems like a natural candidate to apply to our setting: It treats smooth functions on non-smooth manifolds. If we look at the height function on the graph of the function f, which is a polyhedral hyper-surface in four dimensions, we are precisely in the situation that we need. One problem is that all vertices v of the domain are regarded as critical points by definition. The theory makes statements about the nature of the topology changes in this case, but only about the homeomorphism type of the level sets above ($M_{f(v)+\varepsilon}$) and below ($M_{f(v)-\varepsilon}$) the value $f(v)$, and thus they cannot be used straight from the book in order to get our results about the isotopy *across* the value $f(v)$. Moreover, this theory depends on heavy tools like René Thom's isotopy lemma, which is a powerful and deep statement with a very long proof (by John Mather) that has never been formally published. It might be rewarding to try to follow the proofs of [6] for the special case considered here, possibly replacing the application of Thom's Lemma by something that can be proved directly.

References

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