On the geometric dilation of curves and point sets

Adrian Dumitrescu[*] *Ansgar Grüne*[†] *Günter Rote*[‡]

July 8, 2004

Abstract

Let G be an embedded planar graph whose edges are curves. The *detour* between two points u and v (on edges or vertices) of G is the ratio between the shortest path in G between u and v and their Euclidean distance. The maximum detour over all pairs of points is called the *geometric dilation* $\delta(G)$. Ebbers-Baumann, Grüne and Klein have recently shown that every finite point set is contained in a planar graph whose geometric dilation is at most 1.678, and some point sets require graphs with dilation $\delta \geq \pi/2 \approx 1.57$. We prove a stronger lower bound $\delta \geq (1 + 10^{-11})\pi/2$ by relating graphs with small dilation to a problem of packing and covering the plane by circular disks.

1 Introduction

Consider an embedded planar graph G, whose edges are arbitrary curves that do not intersect. Such graphs arise naturally in the study of transportation networks, like waterways, railroads or streets. For two points, u and v (on edges or vertices) of G, the *detour* between u and v in G is defined as

$$\delta_G(u, v) = \frac{d_G(u, v)}{|uv|}$$

where $|uv|$ is the Euclidean distance between u and v and $d_G(u, v)$ is the shortest path length in G between u and v, with the curve length taken as the length of an edge, see Figure 1a for an example. Edges of infinite length are useless in this context; thus, we will assume that all edges have finite length. To measure the quality of G as a transportation system, the *geometric dilation* $\delta(G)$ of G has been introduced [2]:

$$\delta(G) = \sup_{u,v \in G} \delta_G(u, v)$$

The older concept of *graph dilation* of G [6] is given by the restriction to pairs of vertices:

$$\delta'(G) = \max_{u,v \in V(G)} \delta_G(u, v)$$

That is, for δ', only the lengths of the edges but not their shapes are of interest. For a disconnected graph G, we have $\delta(G) = \delta'(G) = \infty$.

[*]Computer Science, University of Wisconsin–Milwaukee, 3200 N. Cramer Street, Milwaukee, WI 53211, USA; E-mail: `ad@cs.uwm.edu`

[†]Computer Science, Universität Bonn, Institut für Informatik I, D-53117 Bonn, GERMANY; E-mail: `gruene@informatik.uni-bonn.de`; partially supported by a DAAD PhD-grant

[‡]Freie Universität Berlin, Institut für Informatik, Takustraße 9, D-14195 Berlin, GERMANY; E-mail: `rote@inf.fu-berlin.de`

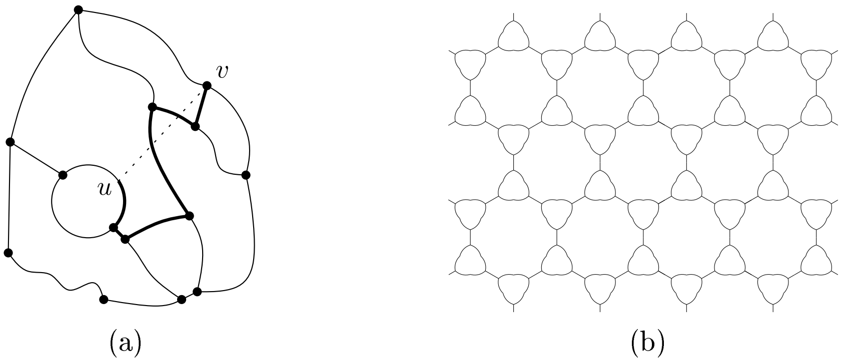

Figure 1: (a) the shortest path (fat) between u and v in the graph and the direct distance (dotted); (b) a section of a grid which has small dilation, less than 1.678.

The *geometric dilation* was only recently introduced. It is also the motivation of our paper. Several papers [1, 5, 10] have shown how to efficiently compute the geometric dilation of polygonal curves. Another question, investigated in [2] is that of constructing planar graphs with low dilation containing a given finite point set. For a finite point set P, the *geometric dilation of the point set* P is defined as

$$\Delta(P) = \inf_{P \subset G,\ G \text{ finite}} \delta(G).$$

Computing $\Delta(P)$ even for a set P with three points is a non-trivial task. The universal constant

$$\Delta = \sup_{P \text{ finite}} \Delta(P),$$

is defined as the maximal value of dilation over all finite point sets. Ebbers-Baumann, Grüne and Klein [2] established that $\Delta \leq 1.678$ by constructing a special grid-like structure which can be overlaid over any given finite set of rational points, see Figure 1b. They also proved that $\Delta \geq \pi/2$ and conjectured that the inequality is strict. We prove this conjecture by showing the new lower bound $\Delta \geq (1 + 10^{-11})\pi/2$.

Our proof relies on a stability result for the geometric dilation of closed curves in the plane: It has been shown that the only closed curves with dilation $\delta = \pi/2$ are circles [3, 4]. We show that curves whose dilation is *close* to $\pi/2$ are *close* to circles, in some well-defined sense (Lemma 1). This allows us to relate the dilation problem to a certain problem of packing and covering the plane by disks.

A key ingredient in our proof is a new inequality relating the length of a closed curve C, to the lengths of two other closed curves derived from it (Lemma 2): one curve is C^*, a centrally symmetric closed curve obtained from C via the *halving pair transformation* defined previously in [3] (under the name *partition pair transformation*); the other curve, M, is the *midpoint cycle* formed by the set of midpoints of chords which divide the length of C in half.

2 A bound on dilation from a packing/covering problem

The proof of the inequality $\Delta \geq \pi/2$ in [2] has two steps: (i) every graph G with $\delta(G) \leq \pi/2$ for embedding the vertices of a regular 10-gon must contain a cycle. (ii) any closed curve C has dilation at least $\pi/2$; therefore each graph containing a bounded face (i.e., closed curve) has dilation at least $\pi/2$. The proof of (ii) is based on the planar variant of Cauchy's surface area formula. Circles have dilation $\pi/2$. We prove in Section 2.1 that the converse also holds, in an approximate sense:

Lemma 1. *Let C be any simple closed curve whose dilation satisfies $\delta(C) \leq (1+\varepsilon)\pi/2$ for $\varepsilon \leq 0.0001$. Then C can be enclosed in the ring between two concentric circles of radii r and $r(1+3\sqrt{\varepsilon})$. This bound cannot be improved apart from the coefficient of $\sqrt{\varepsilon}$.*

The lemma can be extended to a larger, more practical range of ε, by increasing the coefficient of $\sqrt{\varepsilon}$. The case $\varepsilon = 0$ in Lemma 1 has already been established by Ebbers-Baumann et al. [3, 4]: The only closed curves with dilation $\pi/2$ are circles.

In this sense, the lemma can be seen as an instance of a *stability result* for a geometric inequality, see [7] for a survey. Such results complement geometric inequalities (like the isoperimetric inequality between the area and the perimeter of a planar region) with statements of the following kind: When the inequality is fulfilled "almost" as an equation, the object under investigation is "close" to the object or class of objects for which the inequality is tight.

We will combine Lemma 1 with a disk packing result from [9]. A (finite or infinite) set $\mathcal{C}$ of disks in the plane with disjoint interiors is called a *packing*.

Theorem 1. (Kuperberg, Kuperberg, Matoušek and Valtr [9]) *Let $\mathcal{C}$ be a packing in the plane with circular disks of radius at most* 1. *Consider the set of disks $\mathcal{C}'$ in which each disk $C \in \mathcal{C}$ is enlarged by a factor of* 1.00001 *from its center. Then $\mathcal{C}'$ covers no square with side length* 4.

From Lemma 1 and Theorem 1 we deduce our main result:

Theorem 2. *The minimum geometric dilation Δ necessary to embed any finite set of points in the plane satisfies $\Delta \geq (1+10^{-11})\pi/2$.*

Proof. Consider the set $P := \{\, (x,y) \mid x,y \in \{-9,-8,\ldots,9\} \,\}$ of grid points with integer coordinates in the square $Q_1 := [-9,9]^2 \subset \mathbb{R}^2$ (see Figure 2). We use a proof by contradiction and assume that there exists a planar connected graph G that contains P (as vertices or on its edges) and satisfies $\delta(G) \leq (1+10^{-11})\pi/2 < 2$.

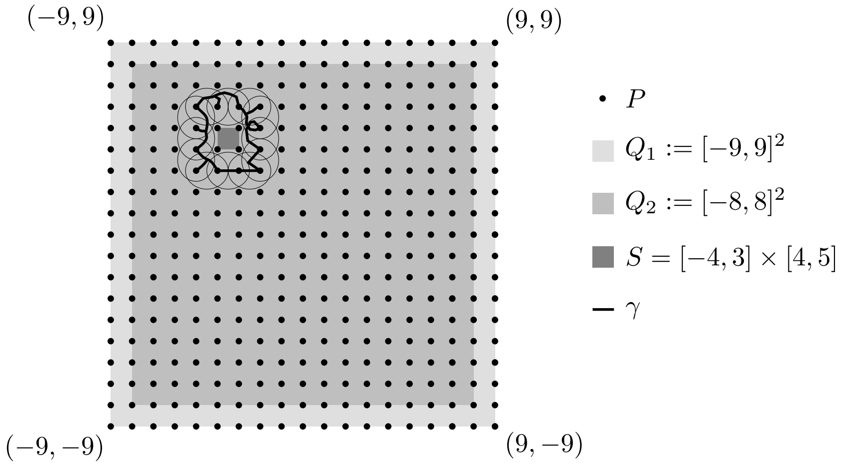

Figure 2: The point set P and some particular parts of the graph G.

We will show that if G attains such a low dilation, G contains a collection $\mathcal{M}$ of cycles with disjoint interiors, which are almost circles and which cover the smaller square $Q_2 := [-8, 8]^2$. The plan of the proof is as follows:

We would like to apply Lemma 1 to the collection of cycles (face boundaries) of G. However, this cannot be done directly because their dilation could be bigger than the dilation $\delta(G)$ of the graph: if C is such a cycle, G can offer shortcuts in the exterior of C, i.e., the shortest path between $a, b \in C$ does not necessarily use C, see Figure 3b. Therefore, we will find a different class of disjoint cycles covering Q_2 which do not allow shortcuts. The idea is to consider for every point in Q_2 the shortest cycle of G containing this point, and to keep only the maximal cycles from this family. We then apply Lemma 1 and conclude that every such cycle has to be contained in the ring between two concentric circles of radii r and $1.00001\,r$, where $r \leq 4$. The inner circles of cycles in $\mathcal{M}$ form a packing of disks, and the outer circles of these cycles form a covering of Q_2, in contradiction with Theorem 1 (situation scaled by a factor 4).

We now present the proof in detail.

Claim 1. *Every point $x \in Q_2$ is enclosed by a cycle C of G of length at most 8π.*

Proof. For every pair p, q of neighboring grid points of P, let $\pi_{p,q}$ be a shortest path in G connecting p and q. Every such path $\pi_{p,q}$ has length $\leq \delta_G(p, q) \cdot 1 < 2$ and stays therefore inside the ellipse $\{\, z \in \mathbb{R}^2 \mid |pz| + |zq| \leq 2 \,\}$. For a point x of Q_2, let $S := [i, i+1] \times [j, j+1]$ be the grid square which contains x, see Figure 2. Consider the closed path γ obtained by concatenating the 12 shortest paths between adjacent grid points on the boundary of the 3×3 square around S, see Figure 2. Due to the ellipses, γ encloses but does not enter S, thus it also encloses $x \in S$. The total length of γ is bounded by $|\gamma| \leq 12 \cdot \delta(G) \leq 12\frac{\pi}{2}(1 + 10^{-11}) = 6\pi + 6\pi \cdot 10^{-11} < 8\pi$. □

Since the length of each such cycle is bounded by $|C| \leq 8\pi$, its in-radius is at most $8\pi/2\pi = 4$. For any point $p \in Q_2 \setminus G$, let $C(p)$ denote a *shortest cycle* in G enclosing p. (Since G is finite, it is trivial that $C(p)$ exists.) If the shortest cycle is not unique, we pick one which encloses the smallest area. It follows from Claim 4 below that this defines the shortest cycle $C(p)$ uniquely. Obviously, $C(p)$ is a simple cycle (i.e., without self-intersections). Let $R(p)$ denote the (open) region enclosed by $C(p)$.

Claim 2. *(i) No shortest path of G can intersect $R(p)$.*

(ii) Between two points a, b on $C(p)$, there is always a shortest path on $C(p)$ itself.

Proof. (i) Since every subpath of a shortest path is a shortest path, it suffices to consider a path $\xi(a, b)$ between two points a, b on $C(p)$ whose interior is completely contained in $R(p)$ (Figure 3a). This path could replace one of the two arcs of $C(p)$ between a and b and yield a better cycle enclosing p, contradicting the definition of $C(p)$.

(ii) We have already excluded shortest paths which run through $R(p)$. We must exclude a path $\xi(a, b)$ between two points a, b on $C(p)$ which runs outside $C(p)$ and is strictly shorter than the two arcs of $C(p)$ between a and b (Figure 3b). Such a path would also lead to an immediate contradiction with the definition of $C(p)$. □

As an immediate consequence of statement (ii), we get:

Claim 3. *The dilation of every cycle $C(p)$ is at most the dilation $\delta(G)$ of the whole graph G.*

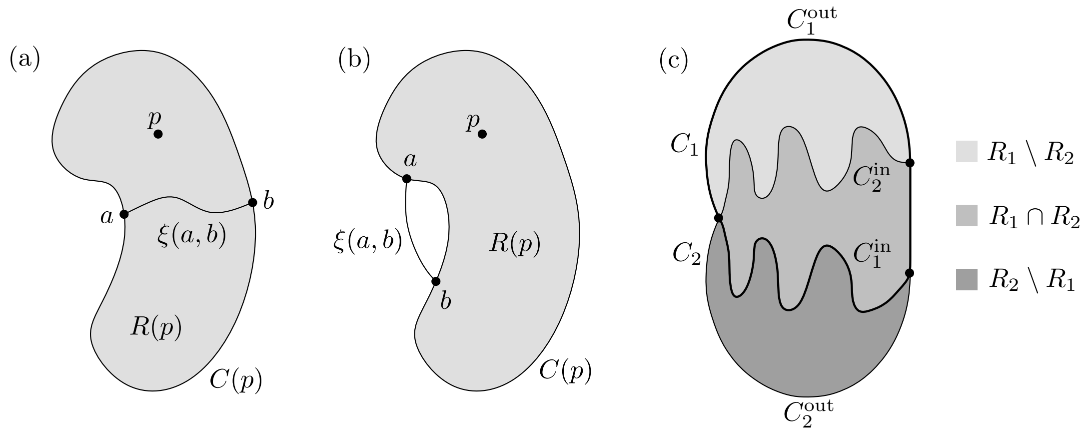

Figure 3: Impossible situations: (a) a shortest path $\xi(a,b)$ intersecting $R(p)$ or (b) being a shortcut; (c) two crossing shortest cycles.

This allows us to apply Lemma 1 to the cycles $C(p)$. However, to obtain a packing, we still have to select a cycle subset with disjoint regions. To this end we prove the following claim:

Claim 4. *For $p, q \in Q_2 \setminus G$, $C(p)$ and $C(q)$ are non-crossing, i.e., $R(p) \cap R(q) = \emptyset \;\vee\; R(p) \subseteq R(q) \;\vee\; R(p) \subseteq R(q)$.*

Proof. We use a proof by contradiction, see Figure 3c. Let us assume that the regions R_1 and R_2 of the shortest cycles $C_1 := C(p)$ and $C_2 := C(q)$ overlap, but none is fully contained inside the other. This implies that their union $R_1 \cup R_2$ is a bounded open connected set. Its boundary $\partial(R_1 \cup R_2)$ contains a simple cycle C enclosing $R_1 \cup R_2$.

By the assumptions we know that there is a part C_1^{in} of C_1 which connects two points $a, b \in C_2$ and is, apart from its endpoints, completely contained in R_2. Let C_1^{out} denote the other path on C_1 connecting a and b. By Claim 2(i), at least one of the paths C_1^{in} or C_1^{out} must be a shortest path. By Claim 2(ii), C_1^{in} cannot be a shortest path, since it intersects R_2. Hence, only C_1^{out} is a shortest path, implying $|C_1^{\text{out}}| < |C_1|/2$. Analogously, we can split C_2 into two paths C_2^{in} and C_2^{out} such that C_2^{in} is contained in R_1, apart from its endpoints, and $|C_2^{\text{out}}| < |C_2|/2$.

The boundary cycle C consists of parts of C_1 and parts of C_2. It cannot contain any part of C_1^{in} or C_2^{in} because it intersects neither with R_1 nor with R_2. Hence $|C| \leq |C_1^{\text{out}}| + |C_2^{\text{out}}| < (|C_1| + |C_2|)/2 \leq \max\{|C_1|, |C_2|\}$. Since C encloses $p \in R_1$ and $q \in R_2$, this contradicts the choice of $C_1 = C(p)$ or $C_2 = C(q)$. $\square$

Let $\mathcal{Y}$ be the set of shortest cycles $\mathcal{Y} = \{\, C(p) \mid p \in Q_2 \setminus G \,\}$, and let $\mathcal{M} \subset \mathcal{Y}$ be the set of *maximal shortest cycles* with respect to inclusion. Claim 4 implies that these cycles have disjoint interiors and cover Q_2. By Claim 3, the dilation of every cycle $C \in \mathcal{M}$ satisfies $\delta(C) \leq \delta(G) \leq \frac{\pi}{2}(1 + 10^{-11})$.

Via Theorem 1 we obtain a contradiction: since the geometric dilation of each cycle $C \in \mathcal{M}$ is small by Lemma 1, each such cycle lies in the ring between two concentric circles of radii r and $1.00001\, r$ (because $3\sqrt{10^{-11}} \leq 10^{-5}$), where $r \leq 4$. The inner circles of cycles in $\mathcal{M}$ form a packing of circles of radius at most 4 in Q_2. Since the cycles form a subdivision of Q_2, the outer circles of these cycles cover Q_2. This is a contradiction to Theorem 1 (situation scaled by a factor 4), because the side length of Q_2 is $16 \geq 4r$. The proof of the main result is now complete. $\square$

2.1 Proof of Lemma 1

Let C be a simple closed curve. We also refer to C as a *simple cycle*. We assume that C is given by an arc-length parameterization $c(t)$, $0 \leq t \leq |C|$, where $|C|$ denotes the length of C. Two points $p = c(t)$ and $\hat{p} = c(t \pm \frac{|C|}{2})$ on C that divide the length of C in two equal parts form a *halving pair* of C. The segment which connects them is a *halving chord*, and its length is the *halving distance*. We write $h = h(C)$ and $H = H(C)$ for the *minimum* and *maximum halving distance* of C.

The *midpoint cycle* M is the cycle formed by the midpoints of the halving chords of C, and is given by the parameterization

$$m(t) := \frac{1}{2}\left(c(t) + c(t + \tfrac{|C|}{2})\right). \tag{1}$$

The difference vector

$$c^*(t) := \frac{1}{2}\left(c(t) - c(t + \tfrac{|C|}{2})\right) \tag{2}$$

defines the *halving pair transformation* and the closed curve C^*, see Figure 4 for an illustration. Note that the difference vector is half the vector connecting the corresponding halving pair. By definition, $c^*(t) = -c^*(t + \frac{|C|}{2})$, hence C^* is centrally symmetric. On the other hand, we have $m(t) = m(t + \frac{|C|}{2})$, and thus, M traverses the same curve twice when C and C^* are traversed once. We define $|M|$ as the length of the curve $m(t)$ corresponding to one traversal, i.e., the parameter interval $0 \leq t \leq |C|/2$.

The curve C^* has the same set of halving distances as C; thus, $h(C^*) = h(C) = h$ and $H(C^*) = H(C) = H$.

The halving pair transformation decomposes the curve C into two components, from which C can be reconstructed:

$$c(t) = m(t) + c^*(t), \qquad c\left(t + \tfrac{|C|}{2}\right) = m(t) - c^*(t) \tag{3}$$

This is analogous the the decomposition of a function into an even and an odd function, or writing a matrix as a sum of a symmetric and a skew-symmetric matrix.

A key fact in our proof is the following lemma, which we think is of independent interest. We will use it as an upper bound on the length $|M|$ of the midpoint cycle M in terms of $|C|$ and $|C^*|$.

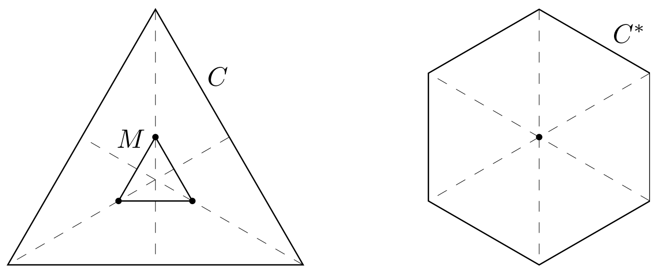

Figure 4: An equilateral triangle C, and the derived curves C^* and M

Lemma 2.

$$4|M|^2 + |C^*|^2 \le |C|^2.$$

Proof. If the curve C (and hence M and C^*) is piecewise differentiable, the proof is more transparent and less technical. This is the proof that we present here. The lemma holds for arbitrary rectifiable curves (i.e., curves with finite length), see Appendix B.

Using the linearity of the scalar product and $|\dot{c}(t)| = 1$, we obtain from (3)

$$\langle \dot{m}(t), \dot{c}^*(t)\rangle = \frac{1}{4}\left\langle \dot{c}(t) + \dot{c}(t+\tfrac{|C|}{2}), \dot{c}(t) - \dot{c}(t+\tfrac{|C|}{2})\right\rangle = \frac{1}{4}\left(|\dot{c}(t)|^2 - |\dot{c}(t+\tfrac{|C|}{2})|^2\right) = \frac{1}{4}(1-1) = 0$$

This means that the derivative of the difference vector $\dot{c}^*(t)$ and the derivative of the midpoint cycle are always orthogonal, thus

$$|\dot{m}(t)|^2 + |\dot{c}^*(t)|^2 = |\dot{c}(t)|^2 = 1$$

This implies

$$\begin{aligned} |C| &= \int_0^{|C|} |\dot{c}(t)|\ \mathrm{d}t = \int_0^{|C|} \sqrt{|\dot{m}(t)|^2 + |\dot{c}^*(t)|^2}\ \mathrm{d}t \\ &\ge \sqrt{\left(\int_0^{|C|} |\dot{m}(t)|\ \mathrm{d}t\right)^2 + \left(\int_0^{|C|} |\dot{c}^*(t)|\ \mathrm{d}t\right)^2} = \sqrt{4|M|^2 + |C^*|^2} \end{aligned} \tag{4}$$

The above inequality — from which the lemma follows — can be seen by a geometric argument: the left integral

$$\int_0^{|C|} \sqrt{|\dot{m}(t)|^2 + |\dot{c}^*(t)|^2}\ \mathrm{d}t$$

is the length of the curve

$$\gamma(s) := \left(\int_0^s |\dot{m}(t)|\ \mathrm{d}t, \int_0^s |\dot{c}^*(t)|\ \mathrm{d}t\right),$$

while the right expression

$$\sqrt{\left(\int_0^{|C|} |\dot{m}(t)|\ \mathrm{d}t\right)^2 + \left(\int_0^{|C|} |\dot{c}^*(t)|\ \mathrm{d}t\right)^2}$$

equals the distance of its end-points $\gamma(0) = (0,0)$ and $\gamma(|C|)$. $\square$

Corollary 1. $|C^*| \le |C|$.

To find an upper bound on the ratio H/h between the extreme halving distances we use the following lemma which extends an inequality of Ebbers-Baumann et al. [3, 4] to non-convex cycles.

Lemma 3. *The geometric dilation $\delta(C)$ of any closed curve C satisfies*

$$\delta(C) \ge \arcsin\left(\frac{h}{H}\right) + \sqrt{\left(\frac{H}{h}\right)^2 - 1}. \tag{5}$$

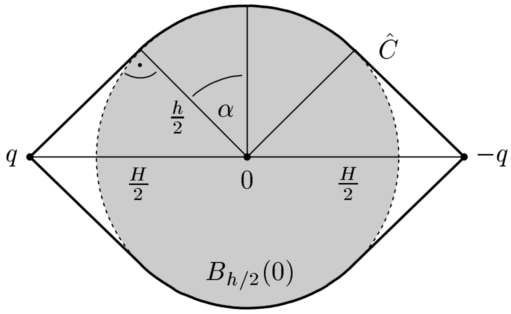

Figure 5: The cap-curve $\hat{C}$ has length $|\hat{C}| = 2h\Big(\alpha + \sqrt{(\frac{H}{h})^2 - 1}\,\Big)$, with $\sin\alpha = \frac{h}{H}$.

Note that the function $g(x) = \arcsin 1/x + \sqrt{x^2 - 1}$ appearing on the right side starts from $g(1) = \pi/2$ and is increasing on $[1, \infty)$. It is the length of the curve shown in Figure 5. Its Taylor expansion around $x = 1$ starts with $g(x) = \pi/2 + \sqrt{8}/3 \cdot (x-1)^{3/2} + O(x-1)^{5/2}$.

Proof. From the definition of dilation and halving pairs and Corollary 1, we get

$$\delta(C) \geq \frac{|C|}{2h} \geq \frac{|C^*|}{2h}.$$

We have seen above that $h(C^*) = h(C) = h$ and $H(C^*) = H(C) = H$. Then, by the symmetry of C^*, any two points p and $-p$ on C^* form a halving pair of distance $2|p| \geq h$. Hence, C^* contains the disk $B_{h/2}(0)$ of radius $h/2$ around the origin. On the other hand, C^* has to connect some halving pair $(q, -q)$ of maximum distance H. Therefore, C^* has at least the length of the cap-curve $\hat{C}$ depicted in Figure 5. Basic trigonometry shows that $|\hat{C}|$ equals the claimed bound on the right-hand side of (5). $\square$

In 1923, Kubota [8] used similar arguments to prove that the length of any convex closed curve C of diameter D and width w satisfies $|C| \geq 2w \arcsin(w/D) + 2\sqrt{D^2 - w^2}$. (The width w is the width of the narrowest strip between two parallel lines which encloses C.) This inequality implies the lower dilation bound $\delta(C) \geq \arcsin(w/D) + \sqrt{(D/w)^2 - 1}$ given in [3].

We continue with the proof of Lemma 1. The assumption in the lemma and the definition of dilation imply: $(1+\varepsilon)\pi/2 \geq \delta(C) \geq |C|/2h$, thus

$$|C| \leq (1+\varepsilon)\pi h. \tag{6}$$

Since $h(C^*) = h(C) = h$ and C^* is centrally symmetric, it encloses a circle with diameter h, thus

$$|C^*| \geq \pi h. \tag{7}$$

The next fact is well known (see e.g. [11]).

Lemma 4. *A closed curve C of length L can be enclosed in a circle of radius $L/4$.*

Proof. Fix a halving pair (a, b) of C. Then by definition, for any $p \in C$, we have $|pa| + |pb| \leq L/2$. It follows that C is contained in an ellipse with foci a and b and major axis $L/2$. This ellipse is included in a circle with radius $L/4$, and the lemma follows. $\square$

By Lemma 4, the midpoint cycle M can be enclosed in a circle of radius $|M|/4$ and some center z. By the triangle inequality, we immediately obtain:

$$|c(t) - z| \overset{(3)}{=} |m(t) + c^*(t) - z| \leq |c^*(t)| + |m(t) - z| \leq \frac{H}{2} + \frac{|M|}{4},$$

and

$$|c(t) - z| \overset{(3)}{=} |m(t) + c^*(t) - z| \geq |c^*(t)| - |m(t) - z| \geq \frac{h}{2} - \frac{|M|}{4}.$$

Thus, C can be enclosed in the ring between two concentric circles with radii $R = \frac{h}{2} + \frac{|M|}{4}$ and $r = \frac{h}{2} - \frac{|M|}{4}$. To prove Lemma 1, we have to bound the ratio R/r. For simplicity, we prove only the asymptotic bound $R/r \leq 1 + O(\sqrt{\varepsilon})$. The proof of the precise bound, which includes all numerical estimates, is given in Appendix A (for the interested reader).

Assume $H = h(1 + \beta)$. Lemma 3 and power series expansions yield the approximate lower bound

$$\delta(C) \geq \arcsin \frac{1}{1+\beta} + \sqrt{(1+\beta)^2 - 1} = \frac{\pi}{2} + \frac{2\sqrt{2}}{3}\beta^{3/2} + O(\beta^{5/2}).$$

With our initial assumption $\delta(C) \leq \frac{\pi}{2}(1 + \varepsilon)$ we get therefore $\beta = O(\varepsilon^{2/3})$. Lemma 2 and equations (6–7) imply

$$|M| \overset{\text{Lemma 2}}{\leq} \frac{1}{2}\sqrt{|C|^2 - |C^*|^2} \overset{(6),(7)}{\leq} \frac{1}{2}\sqrt{(1+\varepsilon)^2\pi^2h^2 - \pi^2h^2} = \frac{\pi h}{2}\sqrt{2\varepsilon + \varepsilon^2} = O(h\sqrt{\varepsilon}),$$

which yields

$$\frac{R}{r} = \frac{H/2 + |M|/4}{h/2 - |M|/4} = \frac{h(1+\beta) + |M|/2}{h - |M|/2} \leq \frac{1 + O(\varepsilon^{2/3}) + O(\varepsilon^{1/2})}{1 - O(\varepsilon^{1/2})} = 1 + O(\varepsilon^{1/2}),$$

completing the proof of the asymptotic bound in Lemma 1. □

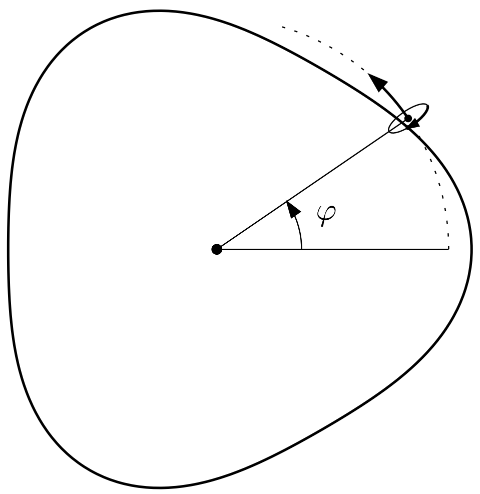

Figure 6: A moon's orbit; the figure shows the curve C for $s = 0.1$.

Tightness of the bound in Lemma 1. The curve C defined below and illustrated in Figure 6 shows that the order of magnitude in the bound cannot be improved.

$$C(\varphi) := \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} (1 + s\cos 3\varphi) + \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} (-\frac{s}{3}\sin 3\varphi).$$

This curve is the path of a moon orbiting around the earth on a small elliptic orbit with major axis s (collinear to the line earth–sun), with a frequency three times that of the earth's own circular orbit around the sun. A ring with outer radius R and inner radius r containing the curve satisfies $R/r \geq (1+s)/(1-s) = 1 + \Theta(s)$. One can show that the length is bounded by $|C| \leq 2\pi + O(s^2)$, and the halving distances are bounded by $2 - O(s^3) \leq h < H \leq 2 + O(s^3)$ (see Appendix C). If s is not too large, C is convex, and this implies [3, 4] that the dilation is given by

$$\delta(C) = \frac{|C|/2}{h} \leq \frac{\pi + O(s^2)}{2 - O(s^3)} = (1 + O(s^2))\pi/2.$$

Thus, we have dilation $\delta = (1+\varepsilon)\pi/2$ with $\varepsilon = O(s^2)$, but the ratio of the radii of the enclosing ring is $1 + \Theta(s) = 1 + \Omega(\sqrt{\varepsilon})$. A more careful estimate shows that the ratio of the enclosing ring is $1 + \frac{3}{4}\sqrt{\varepsilon} + O(\varepsilon)$ (see Appendix C). Thus, the even coefficient 3 in Lemma 1 cannot be improved very much. $\square$

3 Conclusion

Our result looks like a very minor improvement over the easier bound $\Delta \geq \pi/2$, but it is a first step and has required the introduction of new techniques. Our approximations are not very far from optimal, and we believe that new ideas are required to improve the lower bound to, say, $\pi/2+0.01$. An improvement of the constant 1.00001 in the disk packing result of [9] (Theorem 1) would of course immediately imply a better bound for the dilation. The authors of [9] did not attempt to optimize the parameters of their proof in order to get the strongest possible bound, but rather tried to choose parameters which would permit realistic pictures of the situations in the proof. However, the choice of parameters in their proof is very delicate, and it is not straightforward to improve it.

We do not know whether the link between disk packing (Theorem 1) and dilation that we have established works in the opposite direction as well: Can one construct a graph of small dilation from a "good" circle packing (whose enlargement by a "small" factor covers a large area)? If this were true (in some meaningful sense which would have to be made precise) it would mean that a substantial improvement of the lower bound on dilation cannot be obtained without proving, at the same time, a strengthening of Theorem 1 with a larger constant than 1.00001.

References

[1] P. Agarwal, R. Klein, C. Knauer and M. Sharir, Computing the detour of polygonal curves, Technical Report B 02-03, FU Berlin, January 2002.

[2] A. Ebbers-Baumann, A. Grüne and R. Klein, On the geometric dilation of planar point sets, *Proceedings of the the 14th Annual International Symposium on Algorithms and Computation* (ISAAC '03), Lecture Notes in Computer Science, Vol. 2906, 2003, pp. 250–259.

[3] A. Ebbers-Baumann, A. Grüne and R. Klein, Geometric dilation of closed planar curves: a new lower bound, *Abstracts of the 20th European Workshop on Computational Geometry*, Seville, March 2004, pp. 123–126.

[4] A. Ebbers-Baumann, A. Grüne and R. Klein, Geometric dilation of closed planar curves: new lower bounds, submitted for publication, June 2004, *Computational Geometry, Theory and Applications*, special issue for the 20th European Workshop on Computational Geometry.

[5] A. Ebbers-Baumann, R. Klein, E. Langetepe and A. Lingas, A fast algorithm for approximating the detour of a polygonal chain, *Proceedings of the the 9th European Symposium on Algorithms* (ESA '01), Lecture Notes in Computer Science, Vol. 2161, 2001, pp. 321–332.

[6] D. Eppstein, Spanning trees and spanners, in J.-R. Sack and J. Urrutia, editors, *Handbook of Computational Geometry*, Elsevier, 1999, pp. 425–461.

[7] H. Groemer, Stability of geometric inequalities, in P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, Vol. A, North-Holland, Amsterdam 1993, pp. 125–150.

[8] T. Kubota, Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen, *Sci. Rep. Tôhoku Univ.* **12** (1923), 45–65.

[9] K. Kuperberg, W. Kuperberg, J. Matoušek and P. Valtr, Almost-tiling the plane by ellipses, *Discrete & Computational Geometry*, **22** (1999), 367–375.

[10] S. Langerman, P. Morin and M. Soss, Computing the maximum detour and spanning ratio of planar chains, trees and cycles, *Proceedings of the the 19-th International Symposium on Theoretical Aspects of Computer Science* (STACS '02), Lecture Notes in Computer Science, Vol. 2285, 2002, pp. 250–261.

[11] Paul R. Scott and Poh Wah Awyong, Inequalities for Convex Sets, *Journal of Inequalities in Pure and Applied Mathematics* **1** (1) Art. 6 (2000), 6 pp. http://jipam.vu.edu.au/

[12] I. M. Yaglom and V. G. Boltyanskiĭ, *Convex Figures*, Holt, Rinehart and Winston, New York 1961.

A Proof of the Precise Bound in Lemma 1

The main assumption of the lemma is $\delta(C) \leq (1+\varepsilon)\pi/2$ for $\varepsilon \leq 0.0001$. Assume $H = h(1+\beta)$. Lemma 3 yields the lower bound:

$$\delta(C) \geq \arcsin\frac{1}{1+\beta} + \sqrt{(1+\beta)^2 - 1} = \arcsin\frac{1}{1+\beta} + \sqrt{2\beta+\beta^2}$$

We have $\beta \leq 0.01$, otherwise this implies $\delta(C) > 1.0001\,\pi/2$, which contradicts the assumption of the lemma.

It is well known that for $x \in [0, \pi/2]$,

$$\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

By setting $x = \sqrt{2\beta}$, we obtain the following inequality, for the given β-range:

$$\sin\left(\frac{\pi}{2} - \sqrt{2\beta}\right) = \cos\sqrt{2\beta} \leq 1 - \beta + \frac{\beta^2}{6} \overset{\beta \leq 0.01}{\leq} 1 - \frac{\beta}{\beta+1} = \frac{1}{\beta+1}.$$

Thus

$$\arcsin\frac{1}{1+\beta} \geq \frac{\pi}{2} - \sqrt{2\beta},$$

and therefore

$$\delta(C) \geq \frac{\pi}{2} - \sqrt{2\beta} + \sqrt{2\beta+\beta^2} = \frac{\pi}{2} + \frac{\beta^2}{\sqrt{2\beta}+\sqrt{2\beta+\beta^2}} \overset{\beta \leq 0.01}{\geq} \frac{\pi}{2} + \frac{\beta^2}{(\sqrt{2}+\sqrt{2.01})\sqrt{\beta}} \geq \frac{\pi}{2} + \frac{\beta^{3/2}}{3}.$$

With our initial assumption, we get

$$\frac{\pi}{2} + \frac{\beta^{3/2}}{3} \leq \delta(C) \leq \frac{\pi}{2}(1+\varepsilon),$$

which yields

$$\beta \leq \left(\frac{3\pi}{2}\right)^{2/3} \varepsilon^{2/3} \leq 2.9\varepsilon^{2/3} \overset{\varepsilon \leq 10^{-4}}{\leq} 0.7\varepsilon^{1/2}. \tag{8}$$

Lemma 2, (6), and (7) imply that

$$|M| \overset{\text{Lemma 2}}{\leq} \frac{1}{2}\sqrt{|C|^2 - |C^*|^2} \overset{(6),(7)}{\leq} \frac{1}{2}\sqrt{(1+\varepsilon)^2\pi^2h^2 - \pi^2h^2} = \frac{\pi h}{2}\sqrt{2\varepsilon+\varepsilon^2} \overset{\varepsilon \leq 10^{-4}}{\leq} 2.24\, h\sqrt{\varepsilon}.$$

We have to bound the ratio R/r between the two concentric circles containing C:

$$\frac{R}{r} = \frac{H/2 + |M|/4}{h/2 - |M|/4} = \frac{h(1+\beta) + |M|/2}{h - |M|/2} \leq \frac{1+\beta+1.12\sqrt{\varepsilon}}{1-1.12\sqrt{\varepsilon}} \overset{(8)}{\leq} \frac{1+1.82\sqrt{\varepsilon}}{1-1.12\sqrt{\varepsilon}} \overset{\varepsilon \leq 10^{-4}}{\leq} 1+3\sqrt{\varepsilon}$$

This completes the proof of the bound in Lemma 1. □

B Proof of Lemma 2 for Arbitrary Rectifiable Curves

Let $0 = t_0, t_1, \ldots, t_i, t_{i+1}, \ldots, t_n = |C|$ be a (common) subdivision for the parameter interval of C, C^*, and M, where an arc-length parameterization $c(t)$, $0 \leq t \leq |C|$, is assumed for C. We use the Δ operator as an abbreviation for the difference between two successive values of an expression depending on t_i, for example, $\Delta c(t_i) := c(t_{i+1}) - c(t_i)$. (In this section, Δ does not denote the dilation constant.) We have $|\Delta c(t_i)| \leq \Delta t_i$.

The relation $c(t) = m(t) + c^*(t)$ gives

$$|\Delta m(t_i)|^2 + |\Delta c^*(t_i)|^2 = |\Delta c(t_i)|^2 - 2\langle \Delta m(t_i), \Delta c^*(t_i)\rangle. \tag{9}$$

A straightforward substitution of the definitions (1–2) of m and c^* yields

$$\langle \Delta m(t_i), \Delta c^*(t_i)\rangle = \frac{1}{4}\left(|\Delta c(t_i)|^2 - |\Delta c(t_i + \tfrac{|C|}{2})|^2\right).$$

Substituting this into (9) gives

$$|\Delta m(t_i)|^2 + |\Delta c^*(t_i)|^2 = \frac{1}{2}|\Delta c(t_i)|^2 + \frac{1}{2}|\Delta c(t_i + \tfrac{|C|}{2})|^2 \leq \frac{1}{2}(\Delta t_i)^2 + \frac{1}{2}(\Delta t_i)^2 = (\Delta t_i)^2$$

This yields

$$|C| = \sum_{i=0}^{n-1} \Delta t_i \geq \sum_i \sqrt{|\Delta m(t_i)|^2 + |\Delta c^*(t_i)|^2} \geq \sqrt{\left(\sum_i |\Delta m(t_i)|\right)^2 + \left(\sum_i |\Delta c^*(t_i)|\right)^2}.$$

The last inequality is analogous to the inequality in (4) which was used in the continuous case. The sums on the right-hand side of the last expression converge to $2|M|$ and $|C^*|$, respectively, if the limit is taken for successively refined subdivisions of the parameter interval. □

C The Example of Figure 6 Showing Asymptotic Tightness of the Bound in Lemma 1

The norm of the derivative of C in the given parameterization can be calculated exactly:

$$|C'(\varphi)| = \sqrt{1 + (64/9)s^2(1 - \cos^2(3\varphi))} = 1 + O(s^2)$$

This means that the length of the curve piece $C[\varphi_1, \varphi_2]$ between two parameter values $\varphi_1 < \varphi_2$ is closely approximated by the difference of parameter values.

$$|C[\varphi_1, \varphi_2]| = \int_{\varphi_1}^{\varphi_2} |C'(\varphi)| \, \mathrm{d}\varphi = (\varphi_2 - \varphi_1)(1 + O(s^2)) \tag{10}$$

In particular, the total length is $2\pi + O(s^2)$. It follows from (10) that halving pairs are defined by parameter values φ and $\hat{\varphi} = \varphi \pm \pi \pm O(s^2)$. The motion of C can be decomposed into a circular orbit of the earth and a local elliptic orbit of the moon:

$$C(\varphi) = E(\varphi) + M(\varphi),$$

with

$$E(\varphi) := \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

and

$$M(\varphi) := s \cdot \left(\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \cos 3\varphi + \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} (-\frac{1}{3}\sin 3\varphi) \right)$$

Points of "opposite" parameter values φ and $\bar{\varphi} := \varphi + \pi$ have exactly distance 2, since the terms in M cancel: $M(\varphi + \pi) = M(\varphi)$, and hence

$$C(\varphi) - C(\bar{\varphi}) = 2\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}.$$

Halving distances can be estimated as follows:

$$\begin{aligned} |C(\varphi) - C(\hat{\varphi})|^2 &= \left|\big(C(\varphi) - C(\bar{\varphi})\big) + \big(C(\bar{\varphi}) - C(\hat{\varphi})\big)\right|^2 \\ &= |C(\varphi) - C(\bar{\varphi})|^2 + 2\,\langle C(\varphi) - C(\bar{\varphi}), C(\bar{\varphi}) - C(\hat{\varphi})\rangle + |C(\bar{\varphi}) - C(\hat{\varphi})|^2 \\ &= 4 + 2\left\langle \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, E(\bar{\varphi}) - E(\hat{\varphi}) + M(\bar{\varphi}) - M(\hat{\varphi}) \right\rangle + [O(s^2)(1 + O(s^2))]^2 \end{aligned}$$

The estimate for the last expression follows from $|\bar{\varphi} - \hat{\varphi}| = O(s^2)$ and (10). The scalar product can be decomposed into two terms. The first term can be evaluated directly:

$$\left\langle \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, E(\bar{\varphi}) - E(\hat{\varphi}) \right\rangle = -(1 - \cos(\bar{\varphi} - \hat{\varphi})) = O(\bar{\varphi} - \hat{\varphi})^2 = O(s^4)$$

The second term can be bounded by noting that moon's speed is bounded: $|M'| = O(s)$.

$$\left\langle \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, M(\bar{\varphi}) - M(\hat{\varphi}) \right\rangle \leq 1 \cdot |M(\bar{\varphi}) - M(\hat{\varphi})| = O(s) \cdot |\bar{\varphi} - \hat{\varphi}| = O(s^3)$$

Putting everything together, every halving distance is bounded as follows:

$$|C(\varphi) - C(\hat{\varphi})| = \sqrt{4 - O(s^4) \pm O(s^3) + O(s^4)} = 2 \pm O(s^3).$$

H and h are bounded by the same estimate.

A more precise estimate for the length is $|C| = 2\pi(1 + \frac{16}{9}s^2 + O(s^4))$. Substituting this into the derivation at the end of Section 2.1 gives a dilation of $\delta = (1+\varepsilon)\pi/2$ with $\varepsilon = \frac{16}{9}s^2 + O(s^3)$. The ratio of the radii of the enclosing ring is $(1+s)/(1-s) = 1 + 2s + O(s^2) = 1 + \frac{3}{4}\sqrt{\varepsilon} + O(\varepsilon)$. This means that the coefficient 3 of $\sqrt{\varepsilon}$ in Lemma 1 cannot be reduced below 3/4. $\square$