Flip Graphs of Degree-Bounded (Pseudo-)Triangulations*

Oswin Aichholzer[1], Thomas Hackl[1], David Orden[2], Pedro Ramos[2], Günter Rote[3], André Schulz[3], and Bettina Speckmann[4]

[1] Institute for Softwaretechnology, Graz University of Technology
[oaich|thackl]@ist.tugraz.at
[2] Departamento de Matemáticas, Universidad de Alcalá
[david.orden|pedro.ramos]@uah.es
[3] Institute of Computer Science, FU Berlin
[rote|schulza]@inf.fu-berlin.de
[4] Department of Mathematics and Computer Science, TU Eindhoven
speckman@win.tue.nl

Abstract. We study flip graphs of (pseudo-)triangulations whose maximum vertex degree is bounded by a constant k. In particular, we consider (pseudo-)triangulations of sets of n points in convex position in the plane and prove that their flip graph is connected if and only if $k > 6$; the diameter of the flip graph is $O(n^2)$. We also show that for general point sets flip graphs of minimum pseudo-triangulations can be disconnected for $k \leq 9$, and flip graphs of triangulations can be disconnected for any k.

1 Introduction

An edge flip is a common local and constant size operation that transforms one triangulation into another. It exchanges a diagonal of a convex quadrilateral, formed by two triangles, with its counterpart. The *flip graph* $\mathcal{F}_T(S)$ of triangulations of a planar point set S has a vertex for every triangulation of S, and two vertices are connected by an edge if there is a flip that transforms the corresponding triangulations into each other. One of the first and most fundamental results concerning edge flips in triangulations is the fact that flips can be used repeatedly to convert any triangulation into the Delaunay triangulation [7, 9]. This implies immediately that $\mathcal{F}_T(S)$ is connected for any planar point set S.

The *flip distance* between two triangulations is the minimum number of flips needed to convert one triangulation into the other. The diameter of $\mathcal{F}_T(S)$ is an upper bound on the flip distance. For a set S of n points in the plane it is known that the diameter of $\mathcal{F}_T(S)$ is $\Theta(n)$ if S is in convex position, and $\Theta(n^2)$ if S is in general position. However, the computational complexity of computing the flip distance between two particular triangulations is not known, even for convex sets [5]. In higher dimensions the flip graph does not even have to be connected [8].

Of growing interest are also subgraphs of flip graphs which correspond to particular classes of triangulations. Houle et al. [4] consider triangulations which contain a perfect matching of the underlying point set. They show that this class of triangulations is connected via flips, that is, the corresponding subgraph of the flip graph is connected. Related results exist for order-k Delaunay graphs, which consist of a subset of k-edges, where a k-edge is an edge for which a covering disk exists which covers at most k other points of the set. For general point sets the graph of order-k Delaunay graphs is connected via edge flips for $k \leq 1$, but there exist examples for $k \geq 2$ that can not be converted into each other without leaving this class [1]. If the underlying point set is in convex position, then [1] also shows that the resulting flip graph is connected

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for any $k \geq 0$. The flip operation has been extended to other planar graphs, see [3] for a very recent and extensive survey.

Pseudo-triangulations are a generalization of triangulations. A *pseudo-triangle* is a planar polygon with exactly three internal angles less than π. A pseudo-triangulation of a point set S is a partition of the convex hull of S into pseudo-triangles whose vertex set is S. Of particular interest are the so-called *minimum* pseudo-triangulations which have exactly $2n-3$ edges. Contrary to triangulations each internal edge of a pseudo-triangulation can be flipped. A flip in a pseudo-triangulation exchanges the diagonal of a pseudo-quadrilateral with its unique counterpart. The flip graph $\mathcal{F}_{PT}(S)$ of minimum pseudo-triangulations of a point set S is connected and Bereg [2] showed that its diameter is $O(n \log n)$.

There are point sets for which every triangulation has a vertex of degree $n-1$. But for pseudo-triangulations it is known [6] that any point set in general position has a pseudo-triangulation of maximum vertex degree 5. For point sets in convex position every pseudo-triangulation is in fact a triangulation and indeed, convex point sets always have triangulations of maximum vertex degree 4. Hence the question arises if the flip graphs of (pseudo-)triangulations whose maximum vertex degree is bounded by a constant k are connected for certain values of k.

Results. The majority of our results concern (pseudo-)triangulations of point sets in convex position. So let S be a set of n points in convex position in the plane. In Section 2 we show that the flip graphs of triangulations of S of maximum vertex degree $k = 4$, $k = 5$, and $k = 6$ are not connected. Then we prove that the flip graphs are connected for any $k > 6$ and argue that they have diameter $O(n^2)$. In Section 3 we briefly consider point sets in general position and show that flip graphs of minimum pseudo-triangulations can be disconnected for $k \leq 9$, and flip graphs of triangulations can be disconnected for any constant k.

2 Point sets in convex position

In this section we study the flip graphs of degree-bounded triangulations of a set S of n points in convex position in the plane. As mentioned above, arbitrary point sets do not necessarily have a triangulation of bounded vertex degree, but point sets in convex position always have a zigzag triangulation of maximum vertex degree 4. Let k denote the maximum vertex degree of a triangulation T on S. If S has $n \geq 5$ points, then k must be at least 4. Every point set S in convex position has in fact $\Theta(n)$ different zigzag triangulations. It is easy to see, though, that one can not flip even a single edge in such a zigzag triangulation without exceeding a vertex degree of 4 (see Fig. 1(a)).

For $k = 5$ consider the triangulation depicted in Fig. 1(b). Only the dashed edges can be flipped, but there are $\Theta(n)$ rotationally symmetric versions of these triangulations, none of which can be reached from any other without exceeding a vertex degree of 5. For $k = 6$ consider the triangulation depicted in Fig. 1(c). No edge of this triangulation can be flipped but again there are $\Theta(n)$ rotationally symmetric versions of this triangulation, none of which can be reached from any other without exceeding a vertex degree of 6.

Definitions and notation. Let S be a set of n points in convex position in the plane, let T be a triangulation of S, and let D be the dual graph of T. Clearly, D is a tree. We distinguish three different types of triangles in T: *ears*, which have two edges on the convex hull of S, *path triangles*, which have one edge on the convex hull of S, and *inner triangles*, which have no edge on the convex hull of S. The *tip* of an ear is the vertex that

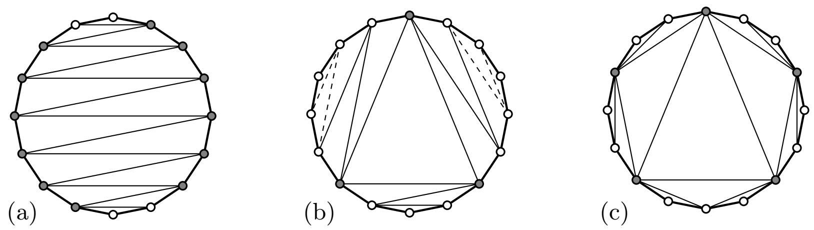

Fig. 1. Triangulations with vertex degree $k = 4$ (a), $k = 5$ (b), and $k = 6$ (c).

is adjacent only to two convex hull vertices. The ears of T are dual to the *leaves* of D and inner triangles of T are dual to *branching vertices* of degree three. A *path* in D is any connected sub-graph of D that consists only of vertices of degree two, so in particular, any vertex of a path is dual to a path triangle. Note that vertices of degree one (leaves) can not be part of a path with this definition. The *length* of a path is its number of vertices. A path that is adjacent to at least one leaf is called a *leaf path*. All other paths are called *inner paths*. An inner triangle that is adjacent to at least two leaf paths is referred to as *merge triangle*.

Consider a path in D of length at least two. If the convex hull edges of its dual path triangles are adjacent on the convex hull of S, then we say that they form a *fan*. The triangles of a fan all share one common vertex, the *fan handle*. The degree in T of a fan handle is always at least five. The size of a fan is the degree of the fan handle minus two, that is, the number of diagonals of the triangles that make up the fan. Path triangles such that the convex hull edges of every second path triangle are adjacent on the convex hull are said to form a *zigzag*. Flipping every second edge of a zigzag is called an *inversion* of the zigzag.

A triangulation of S is called a *zigzag triangulation* if it has precisely two ears which are connected by a zigzag. Clearly, the maximum vertex degree of a zigzag triangulation is four and its dual graph is a path. A zigzag triangulation is uniquely defined (up to inversion) by the location of one of its ears. We call the zigzag triangulation that has the tip of one of its ears on the left-most vertex of S the *left-most zigzag triangulation* of S. Finally, a *fringe triangulation* is a triangulation which has no fans of size greater than four, where each fan is adjacent to an inner triangle, and where every leaf path is dual to a zigzag. In particular, every zigzag triangulation is a fringe triangulation.

Algorithmic outline. Let T be a triangulation of S with maximum vertex degree $k > 6$. In the following we show how to flip from T to the left-most zigzag triangulation of S without ever exceeding vertex degree k. In particular, we show in Subsection 2.1 how to first convert T into a fringe triangulation with $O(n)$ flips. In Subsection 2.2 we prove that each fringe triangulation always has a *light* merge triangle, that is, a merge triangle with two vertices of degree $< k$. We then show how to remove this light merge triangle by merging its adjacent zigzags with $O(n)$ flips, resulting in a fringe triangulation that has one less inner triangle. After repeating this step $O(n)$ times we have converted T into a zigzag triangulation. Finally, in Subsection 2.3 we demonstrate how to "rotate" T into any other zigzag triangulation of S, again with $O(n)$ flips.

Theorem 1. *Let S be a set of n points in convex position and let T be a triangulation of S with maximum vertex degree $k > 6$. Then T can be flipped into the left-most zigzag triangulation of S in $O(n^2)$ flips while at no time exceeding a vertex degree of k.*

Corollary 1. *Let S be a set of n points in convex position. Then for any $k > 6$ the set of (pseudo-)triangulations of S with maximum vertex degree k is connected by flips. The diameter of the corresponding flip graph is $O(n^2)$.*

2.1 Creating a fringe triangulation

Recall that a fringe triangulation has no fans of size greater than four, each fan is adjacent to an inner triangle, and every leaf path is dual to a zigzag. If T is not a fringe triangulation, then it has at least one fan F. Let $3 \leq f \leq n-2$ be the size of F and let $v_1 \dots v_f$ denote the "non fan handle" vertices of the edges of F ordered cyclically around the fan handle v_0 (see Fig. 2(a)).

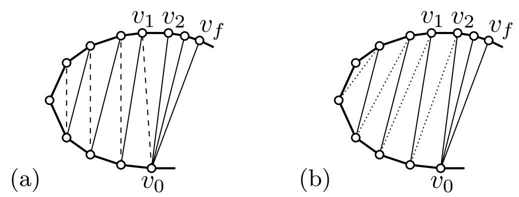

Fig. 2. A fan adjacent to a zigzag. Inverting the zigzag (flipping the dashed edges to the dotted edges) reduces the degree of the fan.

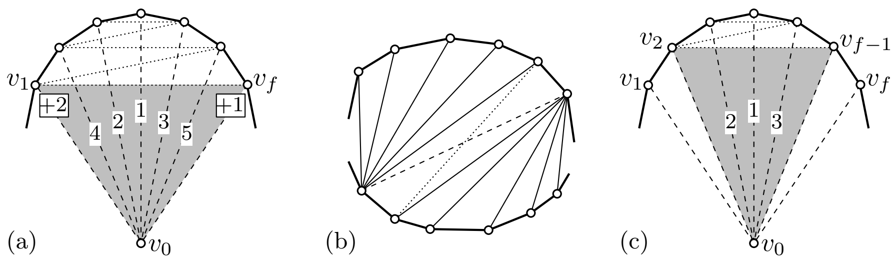

Fig. 3. Handling fans: The boxed numbers show the degree change, the other numbers indicate the order in which flips are performed.

If a zigzag Z, which is dual to a leaf path, is adjacent to F, then one inversion of Z can decrease f by one (see Fig. 2(b)). After the inversion the reduced fan is again adjacent to a zigzag which is dual to a leaf path. Repeated inversion of the (stepwise expanded) zigzag will eventually remove the fan at a cost of $\Theta(f \cdot (\tilde{n} + f))$ flips where $\tilde{n}$ is the length of the dual path of Z. In the following we present a more efficient method to remove or reduce fans.

Consider the fan depicted in Fig. 3(a). If one of v_1 and v_f has degree less than k and the other one less than $k-1$, then we can convert the fan to an inner triangle and a zigzag (which might be empty) ending in an ear, see Fig. 3(a). The degree requirement for v_1 and v_f is always satisfied, unless v_1 or v_f is the fan handle of another fan or is adjacent to an inner triangle.

In the first case the common diagonal of the two fans can be flipped to create two fans separated by a zigzag while decreasing the fan degrees by one each, see Fig. 3(b) (the dashed edge shows the situation before, the dotted edge after the flip). In the second case we distinguish two sub cases.

If $f \geq 5$, then we convert the fan without the diagonals incident to points of too high degree (v_1 and/or v_f) to an inner triangle and a zigzag, as before. If, for example, both, v_1 and v_f, have degree k, then the new inner triangle will be spanned by v_0, v_2, and v_{f-1}, see Fig. 3(c). If $f < 5$, then we can only remove this (constant sized) fan if it is adjacent to a zigzag Z dual to a leaf path. As described above, we can invert Z at most three times to remove the fan.

Lemma 1. *Let S be a set of n points in convex position and let T be a triangulation of S with maximum vertex degree $k > 6$. Then T can be transformed into a fringe triangulation of S with maximum vertex degree k in $O(n)$ flips, while at no time exceeding a vertex degree of k.*

Proof. We first separate any fans whose handles are connected by an edge with one flip per pair. Then we turn any fan that satisfies the degree condition into an inner triangle and a zigzag. This requires a number of flips linear in the sizes of the converted fans. The remaining fans are necessarily adjacent to an inner triangle. We either remove them or reduce them to a fan of size at most four, again with a number of flips linear in their sizes. If one of the remaining, constant size, fans is adjacent to a zigzag that is dual to a leaf path, then we remove this fan by inverting the zigzag at most three times. Since each zigzag dual to a leaf path can be adjacent to at most one constant size fan, the bound follows. ◻

2.2 Merging zigzags

Recall that a merge triangle is an inner triangle that is adjacent to at least two leaf paths. In a fringe triangulation, all leaf paths are dual to a zigzag. In the following we first prove that each fringe triangulation has a light merge triangle, that is, a merge triangle with two vertices of degree $< k$. Then we show how to remove a light merge triangle by merging two of the adjacent zigzags.

Let T be a fringe triangulation with maximum vertex degree $k > 6$. If T is not a zigzag triangulation, then T has at least one merge triangle $\triangle$. $\triangle$ is adjacent to two zigzags Z_1 and Z_2 which are dual to leaf paths. Either or both of these zigzags can be empty, that is, $\triangle$ can be adjacent to one or two ears. W.l.o.g. we assume that both zigzags are non-empty – empty zigzags contribute less edges and hence are never involved in worst case situations for our proofs. We call the vertex v_{tip} of $\triangle$ that is also a vertex of Z_1 and Z_2 the *tip* of $\triangle$. The edge e of $\triangle$ that is not adjacent to v_{tip} is the *base edge* of $\triangle$. Observe that the degree of v_{tip} is at most 6: Two edges from $\triangle$, two convex hull edges, and at most one edge each from Z_1 and Z_2.

Lemma 2. *Let S be a set of n points in convex position and let T be a fringe triangulation of S with maximum vertex degree $k > 6$. Then T has a light merge triangle.*

Proof. Since T is not a zigzag triangulation it has at least one merge triangle $\triangle$ which is adjacent to two zigzags Z_1 and Z_2 which are dual to leaf paths. Since the tip v_{tip} of each merge triangle has degree at most 6 we have to prove that there exists a merge triangle whose base edge e has a vertex with degree $< k$.

If there is a merge triangle where v_{tip} has degree < 6, then we can invert either Z_1 or Z_2 and so decrease the degree of one endpoint of e. Since all vertices of a zigzag, except v_{tip} and the endpoints of e, have degree at most 4, this inversion maintains the degree bound. Hence we can assume that the tip of each merge triangle has degree 6. Let D be the dual graph of T. We create the graph D' by removing all leafs and leaf paths from D. Observe that each triangle $\triangle'$ which is dual to a leaf vertex of D' is a merge triangle in T. We now create a second graph D'' by removing each leaf of D' (see Fig. 4).

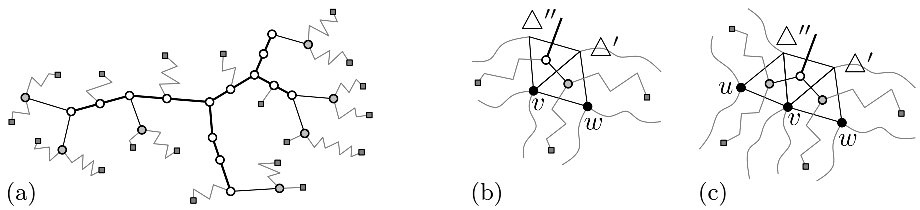

Fig. 4. The graphs D, D', and D'' (a); a leaf of D'' with one/two children in D' (b)/(c).

Since D is a tree, both D' and D'' are trees as well. Hence D'' has at least two leafs. Let $\triangle''$ be dual to a leaf of D'' and let $\triangle'$ be a child of $\triangle''$ in D'. Consider the vertex v which is a vertex of both $\triangle''$ and $\triangle'$, but not a vertex of the parent of $\triangle''$ in D'' (see Fig. 4(b) and (c)). By assumption the tip w of $\triangle'$ has degree 6, which implies that the zigzag adjacent to the edge (v, w) has no edge with endpoint v. Hence, if $\triangle''$ has only one child in D' (see Fig. 4(b)) then v has at most degree 6. If $\triangle''$ has two children in D' (see Fig. 4(c)) then also consider the tip u of $\triangle'$'s sibling. Since u also has degree 6, we again know that the zigzag adjacent to the edge (u, v) has no edge with endpoint v. So also in this case we can conclude that v has at most degree 6. $\square$

Lemma 2 states that each fringe triangulation has a light merge triangle $\triangle$. We now show how to remove $\triangle$ by merging its adjacent zigzags Z_1 and Z_2. Let us denote the edge of $\triangle$ that is adjacent to Z_1 with e_1 and the one adjacent to Z_2 with e_2. Further, let us denote the vertex of $\triangle$ that is shared by e and e_1 by v_1 and the one shared by e and e_2 by v_2 (see Fig. 5(a)). We can assume w.l.o.g. that v_{tip} has degree 6, that v_1 has degree k, and that v_2 has degree $k - 1$.

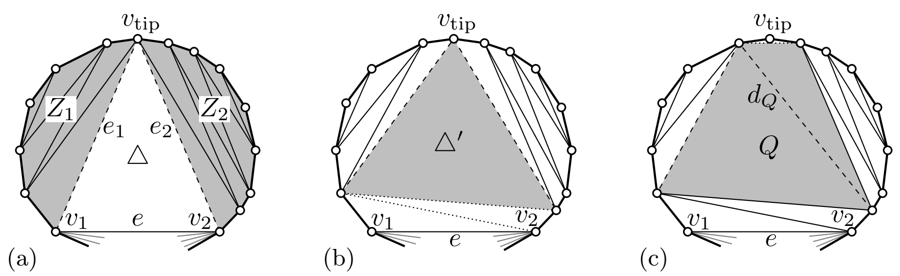

Fig. 5. Merging two zigzags: Start configuration (a) and first steps (b), (c).

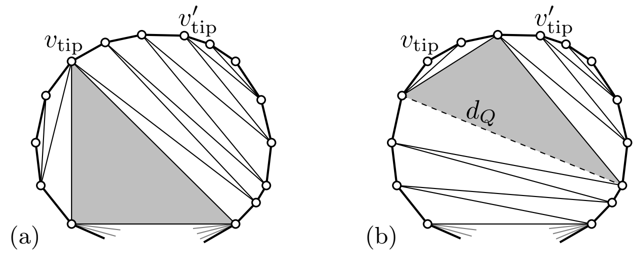

Fig. 6. The repeating case: Start configuration (a), Z' and Z_t meet (b).

We want to merge Z_1 and Z_2 together with $\triangle$ into a new zigzag Z' that starts at e. We start by flipping first e_1 and then e_2. These flips create the first two edges of Z' and do not violate the degree bound (see Fig. 5(b)). Now there is a new triangle $\triangle'$ that lies between (the remains of) Z_1 and Z_2, similar to $\triangle$ before. We continue by flipping the two edges of $\triangle'$ that are part of Z_1 and Z_2, respectively (see Fig. 5(c)). Now we created a quadrilateral Q with a diagonal d_Q that separates four zigzags: The shrinking zigzags Z_1 and Z_2, the zigzag Z' growing from e, and a temporary zigzag Z_t growing from v_{tip} (see Fig. 5(c)).

We continue to take turns flipping edges common to Q and Z_1 and Z_2. This alternatingly adds two edges to Z' and to Z_t and requires at most one additional flip of d_Q per flip. The maximal vertex degree reached during all flips occurs at the vertices of Q, which can have maximal degree at most $7 \leq k$: two convex hull vertices, at most two edges each from both adjacent zigzags, and d_Q.

If both Z_1 and Z_2 have equal size – the *symmetric* case – then Z' and Z_t ultimately grow into each other, forming one zigzag. In this case at least every second flip adds a new edge to Z' which is never flipped again. Let m be the total number of vertices involved in the merge. Then the merge in the symmetric case can by executed with at most $2m$ flips.

If Z_1 and Z_2 do not have equal size, consider the location of Z' at the end of the merge, which is uniquely determined by e, independent of how the merge is executed. Z' will end at an ear that has a tip $v'_{\rm tip}$, which might lie to the right or to the left of $v_{\rm tip}$. ($v_{\rm tip}$ and $v'_{\rm tip}$ are identical iff Z_1 and Z_2 have equal length.) We distinguish two sub cases, depending on the interaction of Z_t with $v'_{\rm tip}$. The first is the *repeating* case, which occurs if Z' and Z_t meet before Z_t reaches $v'_{\rm tip}$ (see Fig. 6), and the second is the *recursive* case, which occurs if Z_t reaches $v'_{\rm tip}$ before it meets Z' (see Fig. 7).

In the repeating case we are in the same situation as the one we started from, just with smaller zigzags. We have to merge two zigzags – the remainder of either Z_1 or Z_2 and Z_t – and we have a light merge triangle to merge them at: The last edge of Z' (d_Q in Fig. 6(b)) becomes our new base edge e' and both endpoints of e' have degree at most $6 < k$ (two convex hull edges and at most two edges each from both adjacent zigzags). We can now continue flipping in such a way that Z' is continued. Unlike in the symmetric case, the edges of Z_t are flipped again. However, recall that our flips alternatingly add two edges to Z_t and to Z'. Hence we can charge each edge of Z' – which is never flipped again – with an additional two flips to account for flipping the edges of Z_t again, which implies that the total number of flips in the repeating case is also linear in m.

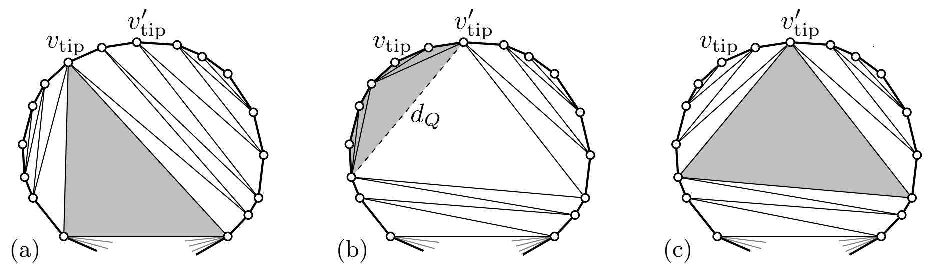

Fig. 7. The recursive case: Start configuration (a), Z_t reaches $v'_{\rm tip}$ (b), after merging recursively (c).

In the recursive case d_Q becomes the base edge of a new light merge triangle, at which two zigzags – the remainder of either Z_1 or Z_2 and Z_t – have to be merged. Possibly after inverting one or both zigzags the new v_{tip} has again degree 6 and both end-points of the new base edge also have at most degree $6 < k$ (see Fig. 7(b)). After solving this instance recursively, the remaining problem is in the symmetric case (see Fig. 7(c)). Let m again be the total number of vertices involved in the merge and let m' be the number of vertices in Z' when we recurse. The total number of flips satisfies the following recursion:

$$T(m) = \Theta(m') + T\left(\frac{m}{2} - \frac{m'}{2}\right) + \Theta(m - m') \leq O(m) + T\left(\frac{m}{2}\right) = O(m) \ .$$

Thus also in the recursive case, the number of flips is linear in the size of the two zigzags which are merged.

Lemma 3. *Let S be a set of n points in convex position, let T be a fringe triangulation of S with maximum vertex degree $k > 6$, let $\triangle$ be a light merge triangle of T, and let m be the total number of vertices of $\triangle$ and its two adjacent zigzags that end in ears. $\triangle$ and both zigzags can be merged into one zigzag ending in an ear in $O(m)$ time, while at no time exceeding a vertex degree of k.*

Since the light merge triangle $\triangle$ is an inner triangle of a fringe triangulation, it might be adjacent to a fan of size at most four, which in turn is adjacent to a zigzag Z'', followed by another fan of size at most four, which is adjacent to the next inner triangle. After removing $\triangle$ by merging it with its two adjacent zigzags into one new zigzag Z', we might have to invert Z' and $Z' + Z''$ a constant number of times to remove the fans and ensure that the new triangulation is again a fringe triangulation. The inversions of Z' can be charged to the merge operation and the inversions of Z'' can be charged to Z'' – since any inner path becomes a leaf path only once, the total costs for all these inversions is $O(n)$.

2.3 Rotating a zigzag triangulation

Recall that a zigzag triangulation is uniquely defined (up to inversion) by the location of one if its ears. In this section we show how to flip or "rotate" any zigzag triangulation of S into any other zigzag triangulation of S. We distinguish the rotations by the "angle" between the source and target ears. Specifically, consider the line ℓ_s through the ears of the source zigzag and the line ℓ_t through the ears of the target zigzag. If these lines partition the vertices of S into four equal sets (plus or minus one) then we say that they form a 90° angle. Note that this angle definition is not geometric but relies solely on the distribution of the points of S. If S forms a regular convex n-gon then the regular geometric and our combinatoric angle definition are the same.

First, as a basic step, we describe a rotation by 90°. In this special case the tips of the ears u and v of the target zigzag are connected by an edge of the source zigzag (see Fig. 8(a)). Flipping the edge between u and v creates a diagonal d_Q, flipping the remaining edges adjacent to u and v creates a splitting quadrilateral Q between the shrinking source and the growing target zigzags (see Fig. 8(b)-(c)). We can now flip to alternatingly add edges to the zigzags growing from u and v, just as described before in the symmetric case of the merge. Hence a rotation by 90° can be executed with a linear number of flips. The maximal vertex degree reached during all flips occurs again at the vertices of Q, which can have maximal degree at most $7 \leq k$: two convex hull vertices, at most two edges each from both adjacent zigzags, and d_Q.

Next, we consider rotations by 45°, that is, rotations where $1/8$ of the vertices lie between the source and target ears in either clockwise or counter-clockwise direction (see Fig. 9(a)). The two target ears u and

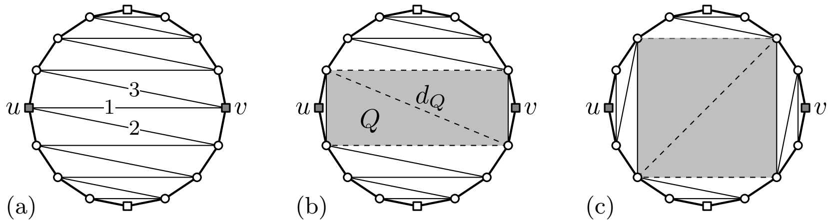

Fig. 8. Zigzag rotation by 90°: Start configuration (a), first steps (b), (c).

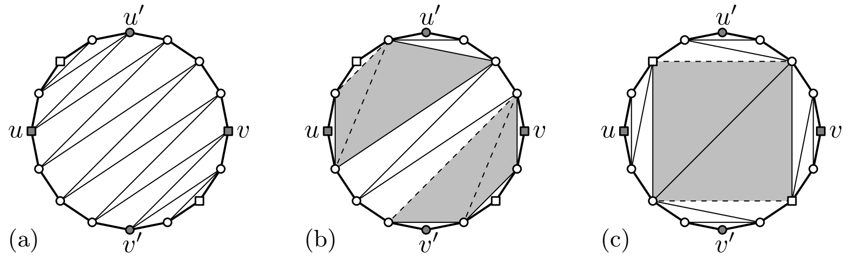

Fig. 9. Zigzag rotation by 45°: Start configuration (a), first steps (b), the zigzags growing from u, u', v, and v' meet (c).

v are no longer connected by an edge of the source zigzag. We select two temporary tips u' and v' which are connected to u and v, respectively, and which have the same distance (in vertices) to the ears of the source zigzag as u and v (see Fig. 9(b)). We now grow zigzags from u, u', v, and v' as described before for the 90° rotation. When these four zigzags meet (see Fig. 9(c)) then we can finish the rotation as in the 90° setting. Using the same arguments as before (twice) we can conclude that also the rotation by 45° can be executed with a linear number of flips, while at no time exceeding a vertex degree of $7 \leq k$.

The final case which we consider are rotations between 45° and 90°. All other rotations can be composed of a rotation by 90° followed by a rotation between 45° and 90°. We select two temporary tips u' and v' as before and start growing four zigzags, also as before (see Fig. 10(a) and (b)). But in this case, when two of the growing zigzags meet, they do not form a 90° setting (see Fig. 10(b)). But we can execute two merges at two light merge triangles $\triangle_u$ and $\triangle_v$, each time involving one of the zigzags growing from a temporary tip and a tip from the source zigzag. After the two merges, we are again in the 90° setting and can finish the rotation (see Fig. 10(c)). With the same arguments as before we can conclude that also rotations by any angle between 45° and 90° can be executed with a linear number of flips, while at no time exceeding a vertex degree of $7 \leq k$.

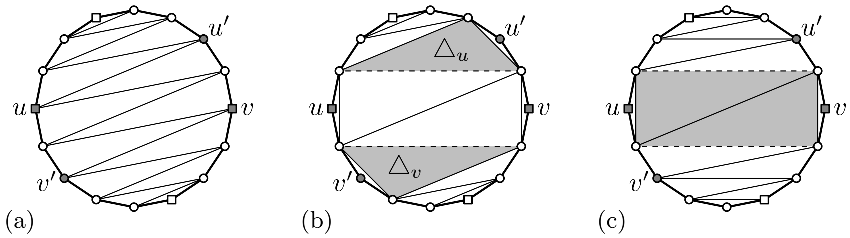

Fig. 10. Zigzag rotation between 45° and 90°: Start configuration (a), the zigzags growing from u, u', v, and v' meet (b), after the merge (c).

Lemma 4. *Let S be a set of n points in convex position and let T be any zigzag triangulation of S. T can be rotated into any other zigzag triangulation of S with $O(n)$ flips, while at no time exceeding a vertex degree of k.*

3 Point sets in general position

In this section we study flip graphs of bounded degree triangulations and pseudo-triangulations of a set S of n points in general position in the plane. There are point sets for which every triangulation must have a vertex of degree $n-1$. Nevertheless we can ask the following question: If there are two triangulations T_1 and T_2 of a point set S, both of which have maximum vertex degree k, is it possible to flip from T_1 to T_2 while at no time exceeding a vertex degree of k? For pseudo-triangulations it is know [6] that any point set S in general position has a pseudo-triangulation of maximum vertex degree 5. Hence the question arises if there is a $k \geq 5$ such that the flip graph of pseudo-triangulations with maximum vertex degree k is connected.

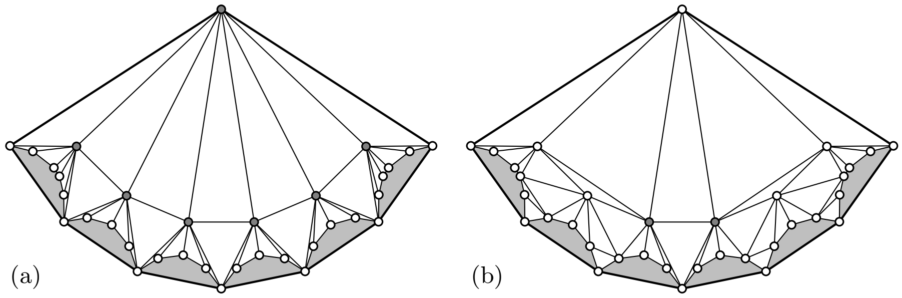

Fig. 11. Two triangulations which cannot be flipped into each other.

For any k there is a point set which has two triangulations T_1 and T_2 of maximum vertex degree k which cannot be flipped into each other without exceeding a vertex degree of k. Consider the example for $k = 8$ depicted in Fig. 11. The shaded parts indicate zigzag triangulations and the dark vertices have degree 8. In the left triangulation only edges of the zigzags can be flipped without exceeding vertex degree 8, hence it is impossible to reach the triangulation on the right. This example can be easily modified for any constant $k > 3$.

We do not know if there is a k such that the flip graph of pseudo-triangulations with maximum vertex degree k is connected, but we do know that k, if it exists, needs to be larger than 9. Consider the pseudo-triangulation P depicted in Fig. 12. P has maximal vertex degree 9, but no edge of P can be flipped. However, P is clearly not the only pseudo-triangulation of this point set with maximum vertex degree 9.

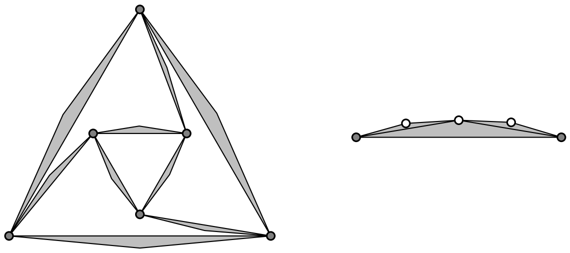

Fig. 12. The "triangular edges" in the left drawing consist of the structure shown on the right, with the indicated orientation. Dark vertices have degree 9.

References

1. M. Abellanas, J. Bose, J. García, F. Hurtado, M. Nicolás, and P. Ramos. On properties of higher order delaunay graphs with applications. In *Abstr. 20th European Workshop on Computational Geometry*, pages 119–122, 2005.
2. S. Bereg. Transforming pseudo-triangulations. *Information Processing Letters*, 90:141–145, 2004.
3. P. Bose and F. Hurtado. Flips in planar graphs. *Computational Geometry*, 42:60–80, 2009.
4. M. Houle, F. Hurtado, M. Noy, and E. Rivera-Campo. Graphs of triangulations and perfect matchings. *Graphs and Combinatorics*, 21(3):325–331, 2005.
5. F. Hurtado, M. Noy, and J. Urrutia. Flipping edges in triangulations. *Discrete and Computational Geometry*, 22(3):333–346, 1999.
6. L. Kettner, D. Kirkpatrick, A. Mantler, J. Snoeyink, B. Speckmann, and F. Takeuchi. Tight degree bounds for pseudo-triangulations of points. *Computational Geometry: Theory and Applications*, 25:1–12, 2003.
7. C. Lawson. Generation of a triangular grid with applications to contour plotting. Memo 299, Jet Propulsion Laboratory, 1972.
8. F. Santos. A point set whose space of triangulations is disconnected. *Journal of the American Mathematical Society*, 13(3):611–637, 2000.
9. R. Sipson. Locally equiangular triangulations. *The Computer Journal*, 21(3):243–245, 1973.