Expansive Motions on the Line and the Associahedron

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Abstract. We investigate the polytope that describes the motions of a set of points on a line, subject to certain conditions on the increase of their distances. It turns out that this polytope has the combinatorial structure of the associahedron. In other words, it gives a geometric representation of the set of triangulations of an n-gon, or of the set of binary trees on n vertices, or of many other combinatorial objects that are counted by the Catalan numbers. The neighborhood in the combinatorial sense is reflected by the adjacency in this representation. Our geometric representation of the associahedron has a large number of free parameters, allowing representations distinct from the other known representations of the associahedron.

1 Introduction

The associahedron. One of the purposes of graph drawing is to have geometric realizations or pictures that reveal something about the underlying structure of some object or some set of objects. The *associahedron* is a particularly nice example where the structure of a set of combinatorial objects, the Catalan structures, are realized by a geometric object, a polytope. The Catalan structures refer to any of a great number of combinatorial objects which are counted by the Catalan numbers (see the extensive list in Stanley [12]), some of the most notable being the triangulations of a convex polygon, binary trees, the ways of evaluate a product of n factors when multiplication is not associative (hence the name associahedron), and monotone lattice paths that go from one corner of a square to the opposite corner without crossing the diagonal. For the sake of illustration, let us focus the attention on the triangulations of a convex n-gon. The associahedron is a polytope which has a vertex for every triangulation, and in which two vertices are connected by an edge of the polytope if the two triangulations are connected by an edge flip. Fig. 1 shows an example of an associahedron.

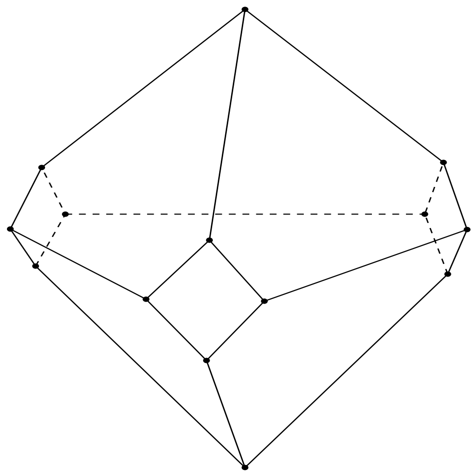

Fig. 1. The three-dimensional associahedron. The vertices represent all triangulations of a convex hexagon or all possible ways to insert parentheses into the product $a*b*c*d$.

There is an easy geometric realization of this polytope as a special case of a secondary polytope (Gel′fand, Zelevinskiĭ, and Kapranov [4], see also Ziegler [14]). Every triangulation is represented by a vector $(a_1, \ldots, a_n)$ of n components. The entry a_i is simply the sum of the areas of all triangles of the triangulation that are incident to the i-th vertex. We will refer to this realization as the *classical realization* of the associahedron. It depends on the location of the vertices of the convex n-gon, but all polytopes that one gets in this way are combinatorially equivalent. Dantzig, Hoffman, and Hu [2, Section 2], and independently de Loera et al. [7] in a more general setting, have given other representations of the triangulations as the vertices a 0-1-polytope in $\binom{n}{3}$ variables corresponding to the possible triangles of a triangulation (the *universal* polytope), or in $\binom{n}{2}$ variables corresponding to the possible edges of a triangulation. These realizations are in a sense most natural, but they have higher dimensions and have more adjacencies between vertices than the associahedron. Every classical associahedron, however, arises as a projection of the universal polytope.

The first published realization of an associahedron is due to Lee [6], but it is not fully explicit. A few earlier and more complicated ad-hoc realizations that were never published are mentioned in Ziegler [14, Section 0.10].

In this paper we will give another, different family of geometric realizations.

Expansive motions. We are given a set of n points $x_1 < \cdots < x_n$ on the real line that are to move with (unknown) velocities v_i, $i = 1, \ldots, n$. An *expansive* motion is a motion in which no inter-point distance decreases. This can easily be written as follows:

$$v_j - v_i \geq 0, \text{ for } 1 \leq i < j \leq n \tag{1}$$

These constraints in the variables v_i define a polyhedral cone. Since a translation of the whole point set (addition of a constant to all variables v_i) does not change these constraints, we may normalize one variable:

$$v_1 = 0 \tag{2}$$

This yields a pointed polyhedral cone with the origin as a vertex. This cone is not very interesting. Its $n-1$ extreme rays correspond to the motions where $x_1, \ldots, x_i$ remain stationary and the points $x_{i+1}, \ldots, x_n$ move away from them at uniform speed:

$$0 = v_1 = v_2 = \cdots = v_i < v_{i+1} = \cdots = v_n$$

We get a richer structure by perturbing the constraints (1):

$$v_j - v_i \geq f_{ij}, \text{ for } 1 \leq i < j \leq n, \tag{3}$$

for some numbers f_{ij}. (Note that the values x_i play actually no role in these constraints.) For an appropriate choice of these numbers, the vertices of the resulting polytope will correspond to *non-crossing alternating trees*, which are Catalan structures.

Related Work. Expansive motions were instrumental in showing that every polygon in the plane can be unfolded into convex position, see Connelly, Demaine and Rote [1]. More recently, the expansion cone for a planar set of points was studied as an object in its own right (Rote, Santos, and Streinu [11]), and certain perturbations of this cone lead to polyhedra whose vertices correspond to so-called *minimums pseudo-triangulations.* Pseudo-triangulations were introduced by Pocchiola and Vegter [8] for computing visibility graphs and have been useful in other areas [5, 13]. It turns out that the perturbations chosen in [11] do not work for degenerate point sets. In particular, for points on a line, one gets a polyhedron equivalent to the one given by (1). For point sets in convex position, however, pseudo-triangulations coincide with triangulations, and one gets yet another representation of the associahedron. This representation is however affinely equivalent to the classical representation of the associahedron [11].

One can also look at the whole *arrangement* of hyperplanes of the form

$$v_j - v_i = f_{ij}. \tag{4}$$

Such arrangements for various special values of f, like $f \equiv 0$ or $f \equiv 1$, have been the object of extensive combinatorial studies, see for example Postnikov and Stanley [10]. In this paper, we study only one *cell* of this arrangement, and moreover, we are trying to avoid degeneracies, in contrast to the above-mentioned choices of f which lead to highly degenerate arrangements.

2 The Expansion Polytope

It is easy to see that the polytope P defined by (2–3) is full-dimensional, after eliminating the constant variable $v_1 = 0$, i. e., it has dimension $n-1$. P contains no line, so it must have vertices. For any vertex v, or for any feasible point $v \in P$, we may look at the set $E(v)$ of tight inequalities at v:

$$E(v) := \{\, ij \mid 1 \le i < j \le n, v_j - v_i = f_{ij} \,\}$$

We regard $E(v)$ as the set of edges of a graph on the vertices $\{1, \ldots, n\}$.

One may get various polyhedra by choosing different numbers f_{ij} in (3). We choose them with the following properties.

$$f_{il} + f_{jk} > f_{ik} + f_{jl}, \text{ for } 1 \le i < j \le k < l \le n. \tag{5}$$

For $j = k$ we use this with the interpretation $f_{jj} = 0$, so we require

$$f_{il} > f_{ik} + f_{kl}, \text{ for } 1 \le i < k < l \le n. \tag{6}$$

One way to satisfy these conditions is to select

$$f_{ij} := h(x_j - x_i), \text{ for } i < j \tag{7}$$

for an arbitrary strictly convex function h with $h(0) = 0$. The simplest choice is $h(x) = x^2$ and $x_i = i$, yielding $f_{ij} = (i-j)^2$.

Two edges ij and jk with $i < j < k$ are called *transitive edges*, and edges ik and jl with $i < j < k < l$ are called *crossing edges.*

Lemma 1. *If f satisfies (5–6) and $v \in P$, then $E(v)$ cannot contain transitive or crossing edges.*

Proof. If we have two transitive edges $ij, jk \in E(v)$ this means that $v_j - v_i = f_{ij}$ and $v_k - v_j = f_{jk}$. This gives $v_k - v_i = f_{ij} + f_{jk} < f_{ik}$, by (6), and thus v cannot be in P because it violates (3). The other statement follows similarly. □

3 Non-crossing Alternating Trees

A graph without transitive edges is called an *alternating* or *intransitive* graph: every path in an alternating path changes continually between up and down.

Lemma 2. *A graph on the vertex set* $\{1, \ldots, n\}$ *without transitive or crossing edges cannot contain a cycle.*

Proof. Assume that C is a cycle without transitive edges. Let i and m be the lowest and the highest-numbered vertex of a cycle C, and let ik be an edge of C incident to i, but different from im. The next vertex on the cycle after k must be between i and k; continuing the cycle, we must eventually reach m, so there must be an edge jl which jumps over k, and we have a pair ik, jl of crossing edges. □

Since the polyhedron is $(n-1)$-dimensional, the set $E(v)$ of a vertex v must contain at least $n-1$ edges. We have just seen that it is acyclic, and hence it must be a tree and contain exactly $n-1$ edges. So we get

Proposition 1. *P is a simple polyhedron. The tight inequalities for each vertex correspond to non-crossing alternating trees.* □

We will see below that P contains in fact *all* non-crossing alternating trees as vertices.

First, we will study a few combinatorial properties of these trees. Alternating trees have been studied in combinatorics in several papers, see for example [9, 10] or [12, Exercise 5.41, pp. 90–92] and the references given there.

Non-crossing alternating trees were only studied by Gelfand, Graev, and Postnikov, under the name of "standard trees". They proved the following fact [3, Theorem 6.4].

Proposition 2. *The non-crossing alternating trees non $n+1$ points are in one-to-one correspondence with the binary trees on n vertices, and hence their number is the n-th Catalan number $\binom{2n}{n}/(n+1)$.* □

The bijection given in [3] to prove this fact is very straightforward. The vertices of the binary tree correspond to the edges of the alternating tree. It is easy to see that every non-crossing alternating tree must contain the edge $1n$. Removing this edge splits the tree into two parts; this corresponds to the two subtrees of the root in the binary tree. The two parts are handled recursively. Fig. 2 gives an example of this correspondence.

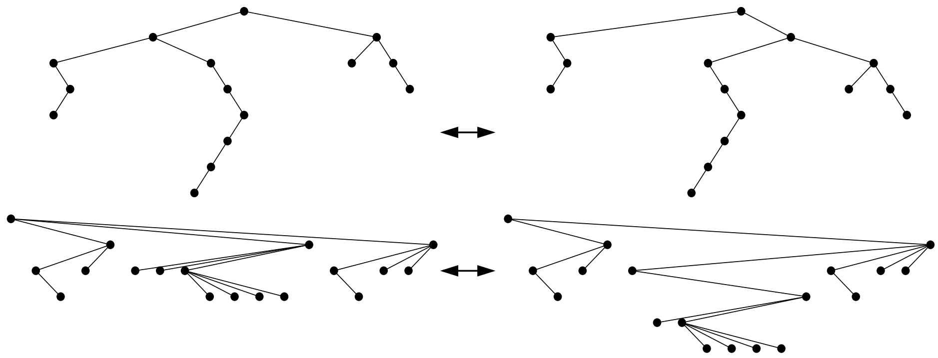

Fig. 2. The bijection with binary trees, and a rotation of binary trees (upper part) together with the corresponding edge exchange (lower part).

We extend this correspondence to the adjacency structure between trees:

Lemma 3. *If we remove any edge $e \neq 1n$ from a non-crossing alternating tree T, there is precisely one other non-crossing alternating tree T' which shares the edges $T - \{e\}$ with T. This exchange operation between non-crossing alternating trees corresponds to a rotation of the binary tree under the above bijection.* □

4 A New Realization of the Associahedron

Proposition 1 implies that the edge $1n$ belongs to $E(v)$ for all vertices v. This means that all vertices lie on the hyperplane

$$v_n - v_1 = f_{1n}. \tag{8}$$

Therefore, if we intersect P with this hyperplane, we get another polytope P_0 which is a facet of P and which has the same set of vertices of P. It is clear that P_0 is bounded: v_1 and v_n are fixed and the other values are locked in between v_1 and v_n.

Lemma 4. *Let v be a vertex of P_0. Then v has $n-2$ adjacent vertices, and they correspond to the $n-2$ non-crossing alternating trees that can be obtained from $E(v)$ by exchanging an edge different from the edge $1n$.*

Proof. Since v is a vertex of an $(n-1)$-dimensional simple polytope, it has $n-2$ outgoing edges and $n-2$ neighbors on P_0. Each edge is obtained by relaxing one of the defining equations of v, and hence an adjacent vertex v' shares all but one of the tight inequalities with v. It follows that $E(v)$ and $E(v')$ have $n-2$ common edges. Hence $E(v')$ must be one of the "exchange neighbors" according to Lemma 3. There are $n-2$ of these neighbors. Therefore, all of them must appear as neighboring vertices of v on the polytope. □

Theorem 1. *P_0 is a polytope whose vertices are in one-to-one correspondence with the non-crossing alternating trees on n vertices, or with the binary trees on $n-1$ vertices. Two vertices are adjacent if and only if the two non-crossing alternating trees differ in a single edge (or if the two binary trees differ by a rotation). Hence it is an associahedron.*

P is an unbounded polyhedron with the same vertex set as P_0. The extreme rays correspond to the non-crossing alternating trees with the edge $1n$ removed.

Proof. For lack of effort, we prove only the statements regarding P_0. We know that it has at least one vertex. By Proposition 1, that vertex must correspond to a non-crossing alternating tree. By Lemma 4, every exchange neighbor of a non-crossing alternating tree that is represented in the polytope must also appear as a vertex. Since the set of all non-crossing alternating trees is connected under the edge exchange operation (like the set of binary trees under tree rotations), we conclude that all trees appear as vertices. □

We remark that we have obtained this result in a somewhat indirect way, by combining combinatorial properties with general structural knowledge about simple polytopes. We have not explicitly proved that any single tree $E(v)$ is in fact feasible, i. e., satisfies the constraints (3).

A result which is related to Theorem 1 was proved by Gelfand, Graev, and Postnikov [3, Theorem 6.3], in a setting dual to ours: Here a triangulation of a certain polytope was constructed. The non-crossing alternating trees correspond to the *simplices* of the triangulation. It is shown explicitly that the simplices form a partition of the polytope. Certain numbers f_{ij} are then associated to the *vertices* of the polytope to show that the triangulation is a projection of the boundary of a higher-dimensional polytope. Incidentally, the numbers that were suggested for this purpose are $(i-j)^2$, which coincides with the simple proposal given in Section 2, but the calculations are not given in the paper.

One easily sees that the conditions (5–6) on f are also necessary for the theorem to hold: If any of these conditions would hold as an equality or as an inequality in the opposite direction, the argument of Lemma 1 would work in the opposite direction, and certain non-crossing alternating trees would be excluded. Thus, (5–6) gives complete characterization of the possible parameter values f_{ij}.

5 Conclusion

The conditions (5–6) leave a lot of freedom for the choice of the variables f_{ij}. We have an $\binom{n}{2}$-dimensional parameter space. This is in contrast to the classical representation mentioned in the introduction, which has $2n$ parameters (the coordinates of n points in the plane). A few of these dimensions only lead to scalings or other trivial transformations of the polytope, but most of them lead to genuinely different polytopes. We haven't checked how the appearance of the associahedron changes under different choices of the parameters. A systematic way of trying different choices would be to select a convex function h in (7), and to play around with the values x_i. It might also be interesting to observe in what way the polytope degenerates as h varies from a "strongly" convex function to a more and more linear shape.

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