Counting triangulations and pseudo-triangulations of wheels

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Abstract

Motivated by several open questions on triangulations and pseudotriangulations, we give closed form expressions for the number of triangulations and the number of minimum pseudo-triangulations of n points in wheel configurations, that is, with $n-1$ in convex position. Although the numbers of triangulations and pseudotriangulations vary depending on the placement of the interior point, their difference is always the $(n-2)$nd Catalan number.

We also prove an inequality $\#PT \leq 3^i \#T$ for the numbers of minimum pseudo-triangulations and triangulations of any point configuration with i interior points.

1 Introduction

A *triangulation* of a set P of n points in the plane is a partition of the convex hull of P into triangles using as corners all the points of P and no other point. Equivalently, a triangulation is a straight-line embedding of a planar graph whose vertex set is P, and whose faces are triangles, except for the outer face which is the exterior of the convex hull of the points. (We assume that all point sets in this paper are in general position: that no three points lie on a common line.)

Although there have been many interesting results on counting and enumerating triangulations in the plane, there remain elementary open questions, such as what point sets have the most and the fewest triangulations. A series of upper bounds are known on the number of triangulations of a given point set, with a recent count of $59^{n+o(n)}$ by Santos and Seidel [14] replacing the previous best of $2^{8.12n+O(\log n)}$ by Denny and Sohler [6]. Bespamyatnikh [3] has the fastest enumeration algorithm for triangulations, and an algorithm of Pocchiola can be generalized to enumerate pseudotriangulations [4]. Aichholzer [2] maintains a list of the leading examples that are known for up to 20 points.

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A *pseudo-triangle* is a planar polygon that has exactly three convex vertices, called *corners,* with internal angles less than π. A *pseudo-triangulation* for a set P of n points in the plane is a partition of the convex hull of P into pseudo-triangles whose vertex set is exactly P. Pseudo-triangulations, which have also been called geodesic triangulations, have been studied because of their applications to visibility [11, 12], ray shooting [5, 7], kinetic collision detection [1, 8, 9], and rigidity [15] in the plane.

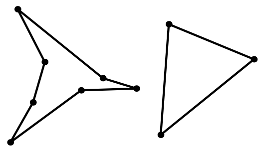

Figure 1: Pseudo-triangles

A pseudo-triangulation is *minimum* if it uses $n-2$ pseudo-triangles [15]. In a minimum pseudo-triangulation, every vertex is *pointed*—all of the edges emanating from it lie in a cone of angle less than π. In a triangulation, only the vertices of the convex hull are pointed. Minimum pseudo-triangulations have nice properties: Every edge except the convex hull edges can be flipped, and the flip incidence graph is connected, regular and polytopal [13].

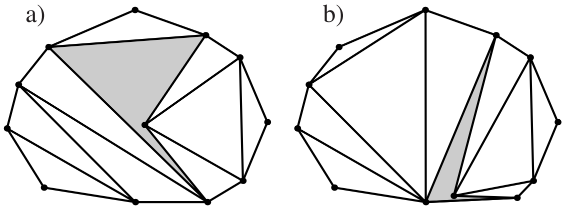

Figure 2: Pseudotriangulations of wheels

For n points in convex position, the only pseudo-triangulations are the triangulations. Their number is well-known to be the $(n-2)$nd Catalan number, $\frac{1}{n-1}\binom{2(n-2)}{n-2}$. For convenience, we denote this number C_n, instead of the standard notation C_{n-2}.

We are interested in counting the number of triangulations, $\#T$, and minimum pseudo-triangulations, $\#PT$, for given point configurations. We would like to know the relationships between these numbers, and the n-point sets that give rise to their minimum and maximum values. For example, from the data in [2] it seems plausible that a double circle has the smallest number of triangulations among all planar configurations with n points. (A double circle consists of $n/2$ vertices of a convex polygon and an additional interior point very close to each edge of the polygon.) We conjecture that $\#T \leq \#PT$, with equality only for points in convex position. In this note we merely scratch the surface of these questions, and show that for *wheel configurations*, with $n-1$ points in convex position and one somewhere inside the convex hull, the difference $\#PT - \#T$ equals the $(n-2)$nd Catalan number, although the numbers $\#T$ and $\#PT$ can vary (Lemmas 1 to 4 and Corollary 6). We also prove that $\#PT \leq 3^i \#T$ for any configuration (Theorem 8), where i is the number of interior points (those which are not vertices of the convex hull).

Note that we consider triangulations and pseudotriangulations of a fixed point set. Topological triangulations, which fix the topology of the embedding but not the geometry, are different. In 1962, Tutte [16] gave an elegant expression for the number of rooted topological triangulations on n vertices and an asymptotic count of about $2^{cn+o(n)}$ for $3.245 < c < 3.246$. In the upcoming ICALP, Li and Nakano [10] give a nice enumeration algorithm.

2 Triangulations of the wheel

We will consider point configurations that we call *wheels*: n points with $n-1$ in convex position and one point inside. Two examples are given in Figure 2a and b. We determine $\#T$ and $\#PT$ for wheels by counting in a special case in which the inner point is close to one side of the convex hull, as in Figure 2b.

Lemma 1 *An n-point wheel, with inner point near one side, has $\#T = C_n - C_{n-1}$.*

Proof: Any triangulation of a wheel with inner point q close to edge pr includes the triangle $\triangle pqr$. By removing edge pr and moving q into convex position we have a bijection between triangulations of this wheel and the $C_n - C_{n-1}$ triangulations of a convex n-gon that do not use the edge pr. ∎

Lemma 2 *An n-point wheel, with inner point near one side, has $\#PT = 2C_n - C_{n-1}$.*

Proof: With only one inside point q close to edge pr, a minimum pseudo-triangulation will be a triangulation with exception of one quadrilateral that involves p, q, and r. We can form such pseudo-triangulations in two ways: First, from any of the $C_n - C_{n-1}$ triangulations of the wheel, we may remove either edge qp or qr, producing $2(C_n - C_{n-1})$ distinct pseudo-triangulations. Second, we may add edges qp and qr to any of the C_{n-1} triangulations of the $n-1$ points in convex position. ∎

We are now interested in moving the inner point and observe how $\#T$ and $\#PT$ change. Change occurs when three points become colinear; in the wheel, when the inner point crosses a chord of the convex $(n-1)$-gon. We use the term *i-chord* to refer to a directed chord whose introduction splits the convex $(n-1)$-gon into a convex $(i+1)$-gon on the left and an $(n-i)$-gon on the right, as in Figure 3. Crossing an i-chord will always be meant from the left side.

Lemma 3 *As the inner point in an n-point wheel crosses an i-chord, #T increases by $C_{i+1}C_{n-i+1} - C_{i+2}C_{n-i}$.*

Proof: When q crosses i-chord pr from left to right, triangulations that do not use edge pq are unchanged. Those that change consist of a triangulation of a convex polygon and a wheel of the type counted by Lemma 1. Thus, there are $C_{i+1}(C_{n-i+1} - C_{n-i})$ triangulations gained and $C_{n-i}(C_{i+2} - C_{i+1})$ lost. ∎

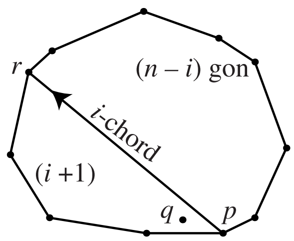

Figure 3: i-chord

Lemma 4 *As the inner point in an n-point wheel crosses an i-chord, #PT increases by $C_{i+1}C_{n-i+1} - C_{i+2}C_{n-i}$.*

Proof: When q crosses i-chord pr from left to right, two types of pseudo-triangulations change. The first type are those that use edge pr: as in the proof of Lemma 3, we gain $C_{i+1}(2C_{n-i+1} - C_{n-i})$ pseudo-triangulations and loose $C_{n-i}(2C_{i+2} - C_{i+1})$. The second type are those using both pq and pr. Of these we gain $C_{n-i}C_{i+2}$ and loose $C_{i+1}C_{n-i+1}$. Thus, the net gain is $C_{i+1}C_{n-i+1} - C_{i+2}C_{n-i}$. ∎

Corollary 5 *For any wheel with n points in general position, $\#PT = \#T + C_n$.*

Proof: By Lemmas 1 and 2, when the inner point is close to a side, the difference $\#PT - \#T = C_n$. By Lemmas 3 and 4, this difference does not change as the inner point moves across chords. ∎

Lemmas 1 to 4 also imply that the minimum of $\#PT$ and $\#T$ are obtained when the interior point is close to an edge and the maximum when the interior point is "at the center" (except the center is only well-defined for a regular wheel). The next result gives the value of this maximum:

Corollary 6 *Among all odd wheels, with $2m-1$ points in convex position and one point inside, the maximum numbers of triangulations and pseudo-triangulations occur when the vertices are evenly spaced along a circle and the interior point is in the center. The numbers are:*

$$\begin{aligned}
\#T &= \frac{1}{2}\left(C_{2m+1} - (2m-1)(C_{m+1})^2\right) \\
\#PT &= \frac{1}{2}\left(C_{2m+1} + 2C_{2m} - (2m-1)(C_{m+1})^2\right).
\end{aligned}$$

Proof: Sketch: Starting with a point near an edge, the numbers of triangulations and pseudo-triangulations increase as we cross i-chords with $1 < i < m$. The maximum number of such chords which we can cross for a given i is $(i+1)$, and in the regular wheel we have crossed exactly that number when the point is in the center. By Lemmas 1 and 3 the number of triangulations is

$$\begin{aligned}
\#T &= C_{2m} - C_{2m-1} \\
&\quad + \sum_{1 \le i < m} (i+1)[C_{i+1}C_{2m-i+1} - C_{i+2}C_{2m-i}].
\end{aligned}$$

This can be reduced to the closed form of the corollary. ∎

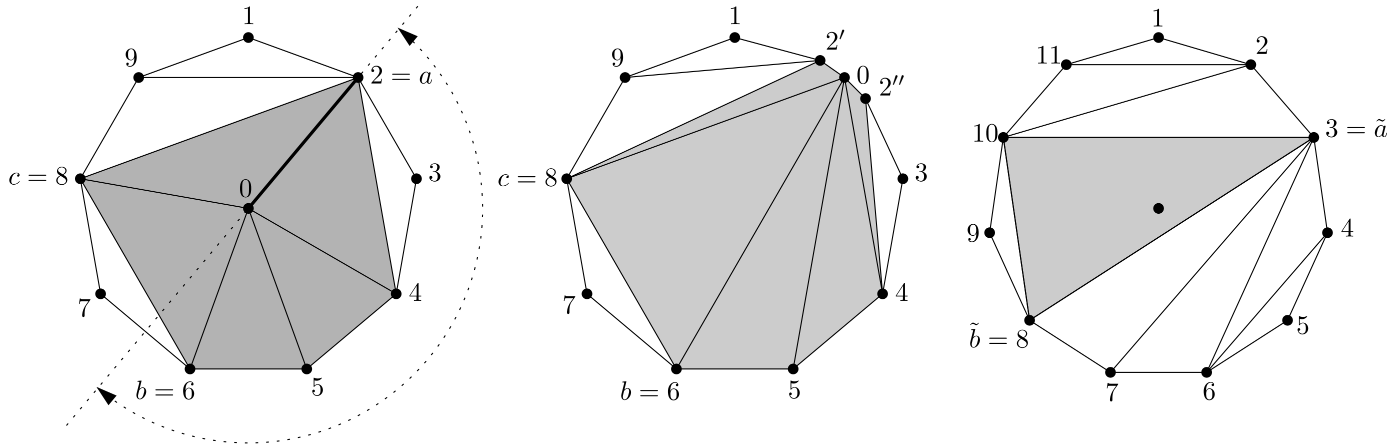

Figure 4: Converting a triangulation of the $(2m-1)$-wheel to a triangulation of a regular $(2m+1)$-gon.

The formula $\#T = \frac{1}{2}(C_{2m+1} - (2m-1)C^2_{m+1})$ implies that the number of triangulations of the regular $(2m-1)$-wheel is approximately one half of the number of triangulations of the regular $(2m+1)$-gon, since the second term in the formula is small. In the rest of this section, we prove this by establishing a bijection using a probabilistic argument.

Let T be a triangulation of the regular $(2m-1)$-wheel. We denote the central vertex by 0. Let us choose a distinguished vertex of the given $(2m-1)$-gon and number the vertices from 1 to $2m-1$ in clockwise order, beginning from the distinguished vertex. let a be the *first vertex* in this sequence which is adjacent to the central vertex. As in Figure 4, we cut the polygon along the edge from a to the central vertex, *expanding* a into two copies a' and a''. We get a non-simple $(2m+1)$-gon with vertex sequence

$$1, 2, \ldots, a', 0, a'', a+1, \ldots, 2m-1.$$

We can draw this polygon as a regular $(2m+1)$-gon, opening up the angle $a'0a''$, and we get a triangulation $\tilde{T}$ of that convex $(2m+1)$-gon. If we renumber the vertices from 1 to $2m+1$, starting from the same distinguished point, the old central vertex will be numbered $\tilde{a} := a+1$.

We can reverse this process: All vertices adjacent to $\tilde{a}$ become neighbors of the central vertex 0, and the two neighbors $\tilde{a}-1$ and $\tilde{a}-1$ are identified. However, if we start from an arbitrary triangulation $\tilde{T}$ and select an arbitrary vertex $\tilde{a}$, this procedure might fail, for two reasons: (a) we might not be able to draw the central point in the center because its neighbors do not completely "surround" it. (b) The vertex a resulting from the identification of $\tilde{a}-1$ and $\tilde{a}-1$ might not be the first neighbor of 0 in the sequence $1, 2, \ldots$.

We now describe the conditions that are necessary and sufficient for reversing the process.

In $\tilde{T}$, there is one *central triangle* which contains the center of the regular polygon.

(i) The vertex $\tilde{a}$ must be a vertex of the central triangle, and it must be the first vertex in the sequence $2, 3, \ldots$. Let $\tilde{b}$ and $\tilde{c}$ be the other two vertices (in increasing order.)

(i) Let e be the third vertex of the triangle incident to the side $(\tilde{a}-1, \tilde{a})$. (The possible range of e extends between $\tilde{c}$ and $\tilde{a}-2$, inclusive, where these indices are taken modulo $2m+1$.) Then e must lie between $\tilde{c}$ and $2m+1$.

(iii) The difference $\tilde{c} - \tilde{b}$ (modulo $2m+1$) is less than m. (This difference could otherwise possibly be as large m.)

Conditions (i) and (ii) ensure that a is indeed the first neighbor of 0. Condition (iii) ensures that the center lies in the convex hull of its neighbors.

Lemma 7 *Conditions (i)–(iii) ensure that conversion of a triangulation T of the $(2m-1)$-wheel to a triangulation $\tilde{T}$ of a regular $(2m+1)$-gon by the operation of Figure 4 is reversible.*

Proof: Consider a triangulation T of the $(2m-1)$-wheel, with a, b, and c identified according to Figure 4. By the definition of a we must have $1 \le a \le m-1$. Let b be the largest neighbor of 0 which is $\le a+m$, and let c be next neighbor of 0 after b, which satisfies $a+m+1 \le c \le 2m-1$. These three neighbors exist, since 0 is contained in the convex hull of its neighbors.

Then $\tilde{a} := a+1$, $\tilde{b} := b+2$, and $\tilde{c} := c+2$ will form the central triangle of $\tilde{T}$. This is easy to check. The vertex e is the first neighbor of 0 in counterclockwise order from a, and thus condition (ii) ensures that a is indeed the vertex selected according to the rule. If we perform the contraction of the two neightbors of $\tilde{a}$, the new angular distance (when seen from the center) between a and b or c is smaller than the old angular distance between $\tilde{a}$ and $\tilde{b}$ or $\tilde{c}$, respectively, which is less than π by the definition of the central triangle. The only thing that can happen is that the difference $\tilde{c}-\tilde{b}$ achieves its maximum possible value of m. This would lead to $c-b=m$, and since there are no other neighbors of 0 between b and c, the neighbors of 0 would not contain the center in T. Whenever $\tilde{c}-\tilde{b}<m$, the angle between b and c is less than π, and thus 0 is contained in the convex hull of its neighbors a, b, and c. ∎

Now suppose that we are given an arbitrary triangulation $\tilde{T}$ of the regular $(2m+1)$-gon, with vertices numbered clockwise from some distinguished vertex 1. The number of these triangulations is C_{2m+1}. Condition (i) allows us to uniquely identify the three vertices $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$. We will show that precisely one half of all triangulations fulfill condition (ii). A small fraction of them will violate condition (iii), and they account for the term that is subtracted from $C_{2m+1}/2$ in formula for $\#T$.

Let us consider a *random* rotation of a given triangulation $\tilde{T}$. In other words, we select the distinguished starting vertex randomly from the $2m+1$ vertices of the polygon. Let us try to find out, which are the good choices for the starting vertex. Consider the i_1 vertices in the range $\tilde{c},\ldots,\tilde{a}-1$, where $i_1 \equiv \tilde{a}-\tilde{c} \pmod{2m+1}$. These are the vertices whose choice results in the given distribution of labels $\tilde{a}$, $\tilde{b}$, $\tilde{c}$. Now, if e is the third vertex of the triangle incident to the side $(\tilde{a}-1,\tilde{a})$, then the *good* choices for the starting point are $e, e+1, \ldots, \tilde{a}-1$.

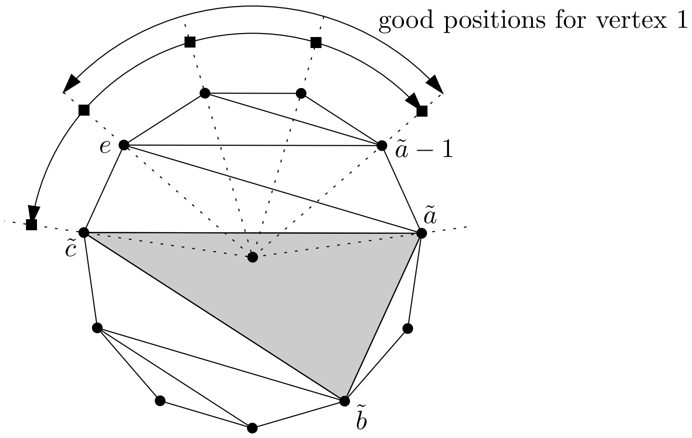

Figure 5: The squares shown are the conceivable positions of the distinguished vertex, assuming that $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ are correctly labeled.

To analyze the total number of good choices, let us look at a *fixed* central triangle $\tilde{a}\tilde{b}\tilde{c}$ together with *random* triangulations of the (i_1+1)-gon between $\tilde{c}$ and $\tilde{a}$, of the (i_2+1)-gon between $\tilde{a}$ and $\tilde{b}$, and of the (i_3+1)-gon between $\tilde{b}$ and $\tilde{b}$, where $i_1+i_2+i_3=2m+1$. Let us first concentrate on the (i_1+1)-gon between $\tilde{c}$ and $\tilde{a}$. The triangle with edge $(\tilde{a}-1,\tilde{a})$ splits this (i_1+1)-gon into a j_1-gon between e and $\tilde{a}-1$ and a j_1'-gon between $\tilde{c}$ and e (including the vertex $\tilde{a}$), with $j_1+j_1'=i_1+2$. the number of good choices between $\tilde{c}$ and $\tilde{a}$ is then j_1-1. By symmetry, the *expected* value of j_1 is $(i_1+2)/2$, since this is just a random triangulation of an (i_1+1)-gon. It follows that the expected number of good points is $i_1/2$. The same argument applies to the other two sides of the central triangle (except that the vertices would be labeled differently), and hence the total expected number of good choices is $(i_1 + i_2 + i_3)/2 = (2m + 1)/2$. This is precisely the one half of the number of all possible choices. This explains the term $C_{2m+1}/2$.

Now let us consider the triangulation which violated condition (iii). They must contain a *long edge* $\tilde{b}\tilde{c}$ with $\tilde{c} - \tilde{b} \equiv m \pmod{2m+1}$. Such a long side splits the $(2m + 1)$-gon into an $(m + 1)$-gon and an $(m + 2)$-gon, and hence the number of triangulations containing a long edge is

$$C_{m+1}C_{m+2} = C_{m+1}^2 \frac{2(2m-1)}{m+1} \tag{1}$$

This expression consideres a *fixed* long edge. To get all triangulations that contain a long edge, we would have to multiply this by $2m+1$ to account for the possible rotations. But then we would count triangulations with two long edges doubly; we will actually make use of this overcounting below.

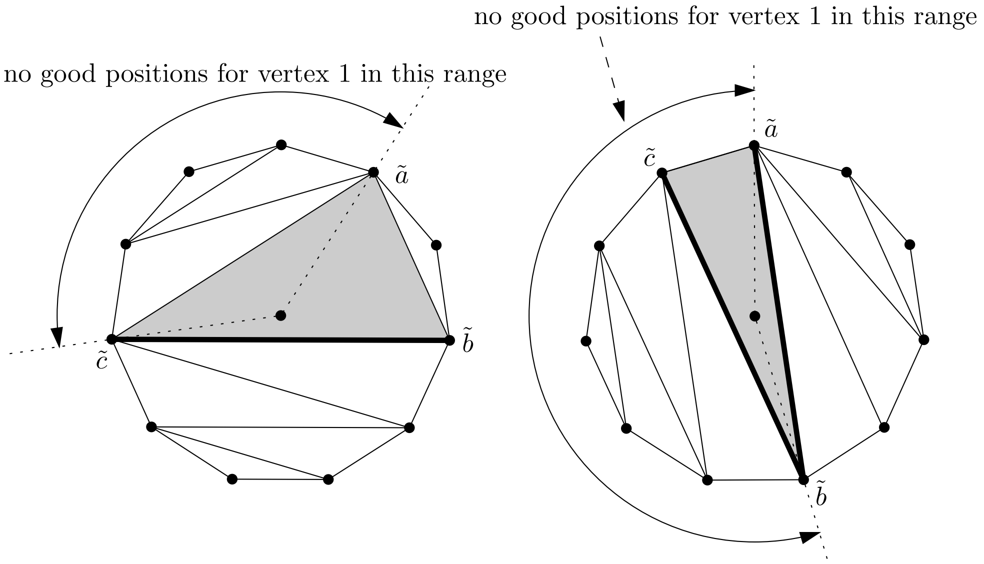

Figure 6: A long edge $\tilde{b}\tilde{c}$ excludes some choices for the distinguished vertex.

If $\tilde{b}\tilde{c}$ is a long edge, then there are no good choices between $\tilde{c}$ and $\tilde{a}$ (assuming a clockwise order of $\tilde{a}$, $\tilde{b}$, $\tilde{c}$). So, for each such triangulation, we must subtract $i_1/2$ from our count of good choices, where $i_1 \equiv \tilde{a} - \tilde{c} \pmod{2m+1}$. The central triangle is always incident to the long edge, and $\tilde{a}$ is the third vertex of that triangle. If we keep the edge $\tilde{b}\tilde{c}$ fixed and consider again a random triangulation of the $(m+2)$-gon between $\tilde{c}$ and $\tilde{b}$, the expected value of i_1 is $(m + 1)/2$. We have to subtract the expected value of $i_1/2$, which is $(m + 1)/4$.

Now there are also triangulations with *two* long edges. These two edges form the central triangle together with an edge between two adjacent vertices, for example $(\tilde{a}, \tilde{b}, \tilde{c}) = (1, m + 1, 2m + 1)$. Then there are no good choices at all in the range from $\tilde{b}$ to $\tilde{a}$. The expression $(i_1 + i_2)/2 = (m + 1)/2$ which was accounted for this range must be subtracted.

Summarizing, let us look at the class of all triangulations that can be obtained from a given triangulation with a long edge by rotating it in all $2m+1$ possible ways. we must subtract an average of $(m+1)/4$ for each rotation class of triangulations with a single long edge, and an average of $(m + 1)/2$ for each class with two long edges. The expression (1) already counts rotation classes with two long edges twice, so we can simply multiply it by $(m + 1)/4$ to get the correct amount that we have to subtract.

$$C_{m+1}^2 \frac{2(2m-1)}{m+1} \cdot \frac{m+1}{4} = C_{m+1}^2 \frac{(2m-1)}{2}.$$

3 A general bound

Let now P be an arbitrary point configuration in general position. Let P_i be the subset of interior points of P. For any subset $W \subseteq P_i$ let PT_W denote the set of pseudo-triangulations whose set of pointed interior vertices is W. In the notation of the previous section, $\#PT_{P_i} = \#PT$ and $\#PT_\emptyset = \#T$.

Theorem 8 *For every $W \subseteq P_i$ and $v \in W$, $\#PT_W \leq 3\#PT_{W\setminus\{v\}}$. In particular, $\#PT \leq 3^i \#T$, where $i = \#P_i$.*

Let any vertex in a pseudo-triangulation which is not pointed be called *cyclic.*

Lemma 9 *Given any pseudo-triangulation with pointed vertices W and any interior vertex $v \in W$, one can add an edge to form a new pseudo-triangulation with pointed vertices $W \setminus \{v\}$.*

Proof: Since v is pointed, there is a (unique) pseudo-triangle with v in its boundary and of which v is not a corner. Let the corners of this pseudo-triangle be x, y, z and assume v lies on the concave chain between x and y. If the edge vz lies on the interior of this pseudotriangle, then its addition makes v cylic and z remains pointed (in case it was pointed before). If vz intersects the pseudo-triangle, then there must be a vertex w lying on the concave chain from x to z, or from y to z, such that vw is tangent to this chain. It is easy to check that the addition of vw makes v cyclic and it does not make w cyclic. ■

Given a pseudo-triangulation and a cyclic vertex v, call any edge incident to v *critical for v* if its removal makes v pointed.

Lemma 10 *for any cyclic vertex v there are at most three critical edges.*

Proof: An edge vw is critical for v only if the two angles incident to vw at v together add more than π. Adding this for all the critical edges we are counting each angle incident to v at most twice, giving a total of at most 4π. Hence, #critical edges$< 4\pi/\pi$, as stated. ■

Proof: (of Theorem 8) Adding the numbers of edges which are critical for v over all the pseudo-triangulations in $PT_{W\setminus\{v\}}$ we are counting the elements of $PT_{W\setminus\{v\}}$ at most three times by Lemma 10 and those of PT_W at least once by Lemma 9. ■

4 Acknowledgments

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