An Upper Bound on the Number of Facets of a 0-1-Polytope

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Theorem 1 *A 0-1-polytope in d dimensions has at most $6.4\, d!/\sqrt{d}$ facets.*

Ziegler [3, Exercise 0.15*, p. 25] mentions an upper bound of $d! + 2d$, which follows from a simple volume argument due to Imre Bárány. By slightly refining this argument and trading the volume inside the polytope against the volume outside, several people [2] could improve this bound to $d! - (d-1)!$.

The known lower bounds are exponential, the current record being 3.6^d for $d \to \infty$ [1]. By a construction of [2], any d-dimensional 0-1-polytope with C^d vertices, for some constant C, gives rise to an infinite family of such polytopes for infinitely many d. Thus any particular 0-1-polytope gives rise to an exponential lower bound, and the best lower bound follows from a random polytopes in 13 dimensions whose facets were enumerated by computer.

Lemma 1 *The surface area of a 0-1-polytope in d dimensions is at most $2d$.*

Proof. If a convex polytope A contains another convex polytope B then the surface area of A is bigger than that of B. This follows for example from Cauchy's surface-area formula, which represents the surface area of A as an integral of the $(d-1)$-dimensional volumes of the orthogonal projections of A onto all $(d-1)$-dimensional hyperplanes. Any 0-1-polytope is contained in the d-dimensional unit hypercube, which has surface area $2d$. □

We assume that the given polytope is full-dimensional. Then each facet has a defining inequality $\sum_{i=1}^{d} a_i x_i \le b$ which is unique up to multiplication with a positive constant.

Definition 1 *A k-*facet *is a facet with exactly k nonzero coefficients a_i in the defining inequality.* $(1 \le k \le d)$

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(Distinguish this from the notion of a k-face, which is usually taken to be a k-dimensional face.)

Lemma 2 *The area of a k-facet of a d-dimensional 0-1-polytope is at least*

$$\frac{\sqrt{k}}{(d-1)!}.$$

Proof. Suppose without loss of generality that $|a_1| \le |a_2| \le \cdots \le |a_k|$ and $a_{k+1} = \cdots = a_d = 0$. If we project the facet ortogonally onto the hyperplace $x_1 = 0$ we get a $(d-1)$-dimensional polytope, since $a_1 \neq 0$. It is a 0-1-polytope, and hence its volume is at least $1/(d-1)!$. When projecting from a hyperplane $\sum_{i=1}^{d} a_i x_i = b$ with defining vector a orthogonally to the hyperplane $x_1 = 0$ with defining vector $e_1 = (1, 0, \ldots, 0)^T$, the volume of each set is reduced by a constant factor, which is the cosine between the two normal vectors,

$$\frac{\langle a, e_1 \rangle}{\|a\| \cdot \|e_1\|} = \frac{a_1}{\sqrt{a_1^2 + a_2^2 + \cdots + a_d^2}} \le \frac{1}{\sqrt{k}}.$$

Hence the area of the original facet was at least $\sqrt{k}/(d-1)!$. □

Let us denote by B_k the number of k-facets of the given polytope.

Lemma 3 *The number B_d of d-facets in a d-dimensional 0-1-polytope is at most*

$$\frac{2\,d!}{\sqrt{d}}.$$

Proof. This follows from Lemmas 2 and 1. □

Lemma 4 *The number B_k of k-facets in a d-dimensional 0-1-polytope is at most*

$$\frac{2\,d!}{(d-k)!\sqrt{k}}.$$

Proof. Consider a k-facet with defining inequality $\sum_{i=1}^{d} a_i x_i \le b$, with $a_1, a_2, \ldots, a_k \neq 0$ and $a_{k+1} = \cdots = a_d = 0$. When we project the whole polytope onto the subspace spanned by the first k coordinate directions, we get a k-dimensional 0-1-polytope, for which the inequality $\sum_{i=1}^{k} a_i x_i \le b$ defines a facet. By Lemma 3, the number of these facets is at most $2\,k!/\sqrt{k}$. This is the maximum number of k-facets with nonzero coefficients in the first k positions, and also the maximum number of k-facets with nonzero coefficients in any specified set of k positions. There are $\binom{d}{k}$ choices for the positions of the nonzero coefficients, and this gives a total of

$$\binom{d}{k} \cdot \frac{2\,k!}{\sqrt{k}} = \frac{2\,d!}{(d-k)!\sqrt{k}}.$$

□

Now we can complete the proof of the theorem. The total number of facets is

$$B_d + B_{d-1} + \cdots + B_2 + B_1$$

$$\leq \sum_{i=0}^{d-1} \frac{2\, d!}{i! \sqrt{d-i}} = \frac{2\, d!}{\sqrt{d}} \cdot \left(\sum_{i=0}^{d-1} \frac{\sqrt{1 + i/(d-i)}}{i!} \right)$$

The last sum is monotonically decreasing for $d \geq 4$, as can be easily checked. In fact, it takes its maximum of approximately 3.19514 at $d = 4$ and it converges to e. (This maximum is a remarkably close approximation to π, except that the digits are not quite in order.) This gives the bound of the theorem.

Remark. By bounding the *total* number of all k-facets with $k \geq d - \log d$ by $2d!/\sqrt{d - \log d}$ as in Lemma 3, and using Lemma 4 for the k-facets with $k < d - \log d$, one can improve the constant in Theorem 1 from 6.4 to $2 + O(\frac{\log d}{d})$.

References

[1] Thomas Christof and Gerhard Reinelt, Efficient parallel facet enumeration for 0/1-polytopes. Preprint IWR Heidelberg, March 1997.

[2] Ulrich H. Kortenkamp, Jürgen Richter-Gebert, A. Sanangarajan, and Günter Ziegler, Extremal properties of 0/1-polytopes. Preprint TU Berlin, June 1996. to appear in *Discrete and Computational Geometry* **16** (no. 4) (1997).

[3] Günter Ziegler, *Lectures on Polytopes*. Springer-Verlag, 1995.