QUASI-PARALLEL SEGMENTS AND CHARACTERIZATIONS OF UNIQUE BICHROMATIC MATCHINGS

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Abstract. Given a set of n blue and n red points in general position in the plane, it is well-known that there is at least one bichromatic perfect matching realized by non-crossing straight line segments. We characterize the situation in which such a point set has *exactly* one matching M of this kind. In this case, we say that M is a *unique matching*. We find several geometric descriptions of unique matchings and give an algorithm that checks in $O(n \log n)$ time whether a given set of n blue and n red points has a unique matching. On the way to these results, we characterize and classify the larger class of bichromatic perfect matchings without so-called chromatic cuts.

1 Introduction

1.1 Basic notation and definitions

Let F be a set of n blue points and n red points in the plane, such that the whole set is in general position (that is, no three points of F lie on the same line). Throughout the paper, such sets will be referred to as *bichromatic sets*. A *perfect bichromatic non-crossing straight-line matching* of F is a perfect matching of points of F realized by non-crossing straight line segments, where each segment connects points of different colors. In many sources, such matchings are referred to as *BR-matchings*. In order to simplify the notation and the drawings, we shall instead color the points of F white and black and denote them by $\circ$ and $\bullet$. We emphasize that we only deal with *perfect* matchings, thus "a matching" stands for "a perfect matching" throughout the paper.

It is well known that any bichromatic set has at least one BR-matching. One easy way to see this is to use recursively the Ham-Sandwich Theorem; another way is to show that the bichromatic matching that minimizes the total length of segments is necessarily non-crossing. The main goal of our work is to characterize bichromatic sets with *exactly one* BR-matching. On the way to answering this modest-looking question, we will study several related issues. For our main characterization of unique BR-matchings (Theorem 12) we give, besides an elementary geometric proof by contradiction, another proof that puts

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more topological structure on the problem in the form of the so-called Fishnet Lemma (Lemma 14), which might be of independent interest (Section 3.2).

In what follows, M usually denotes a BR-matching of some bichromatic point set F.

Definition. A BR-matching M of some bichromatic set F is a *unique matching* if it is the only BR-matching of F.

The convex hull of F will be denoted by $\mathrm{CH}(F)$, and its boundary by $\partial\mathrm{CH}(F)$. Consider the circular sequence of colors of the points of F that lie on $\partial\mathrm{CH}(F)$; a *color interval* is a maximal subsequence of this circular sequence that consists of points of the same color. For example, in Figure 1(a), $\partial\mathrm{CH}(F)$ has four color intervals: two $\circ$-intervals (of sizes 1 and 2) and two $\bullet$-intervals (of sizes 2 and 3).

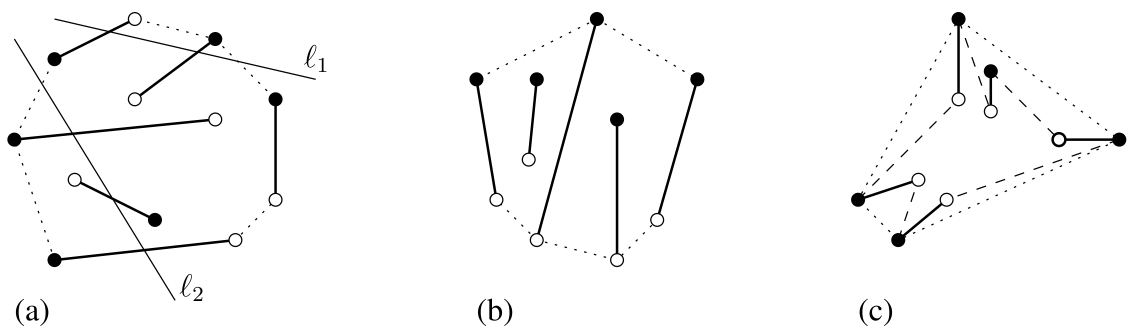


Figure 1: (a) A matching with chromatic cuts. (b) A linear matching. (c) A circular matching. Another matching for the same point set is indicated by dashed lines.

In order to state our main results, we need the notion of a chromatic cut.

Definition. Let M be a BR-matching. A *chromatic cut* of M is a line ℓ that crosses two segments of M so that their $\bullet$-ends are on different sides of ℓ (ℓ can cross other segments of M as well).

For example, the lines ℓ_1 and ℓ_2 in Figure 1(a) are chromatic cuts of the matching shown in this figure. The matching in Figure 1(b) and the matching shown by solid segments in Figure 1(c) have no chromatic cuts. Aloupis, Barba, Langerman, and Souvaine [4, Lemma 9] proved that a BR-matching M that has a chromatic cut cannot be unique. (They actually proved a stronger statement: in such a case there is a BR-matching $M' \neq M$ such that M' is *compatible* to M, which means that the union of M and M' is non-crossing.) Thus, having no chromatic cut is a *necessary condition* for a unique BR-matching. However, it is *not sufficient*, as shown by the example in Figure 1(c).

We will give a thorough treatment of BR-matchings without chromatic cuts. We shall prove in Lemma 7 that BR-matchings without chromatic cuts can be classified into the following two types. A *matching of linear type* (or, for shortness, *linear matching*) is a BR-matching without a chromatic cut such that $\partial\mathrm{CH}(F)$ consists of exactly two color intervals, both necessarily of size at least 2. A *matching of circular type* (or *circular matching*) is a BR-matching without a chromatic cut such that all points of $\partial\mathrm{CH}(F)$ have the same color. The matching shown in Figure 1(b) is a linear matching, and the matching shown in Figure 1(c) by solid segments is a circular matching. We will show that for matchings without chromatic cuts no other possibilities are possible. The reason for the terms "linear type" and "circular type" will be clarified as well. We shall prove that the unique BR-matchings are precisely the linear matchings. This will be a part of our main result, Theorem 2 below.

The segments in M are considered directed from the $\circ$-end to the $\bullet$-end. For $A \in M$, the line that contains A is denoted by $g(A)$, and it is considered directed consistently with A. For two directed segments A and B such that the lines $g(A)$ and $g(B)$ do not cross, we say that the segments (respectively, the lines) are *parallel* if they have the same orientation; otherwise we call them *antiparallel*. The set $g(A) \setminus \mathrm{int}(A)$ consists of two *outer rays*: the $\circ$-*ray* and the $\bullet$-*ray*, according to the color of the respective initial points.

Definition. For two (directed) segments A and B, the *sidedness relation* $\multimap$ is defined as follows: $A \multimap B$ if B lies strictly to the right of $g(A)$ *and* A lies strictly to the left of $g(B)$.

The definition implies directly that the relation $\multimap$ is asymmetric: $A \multimap B$ and $B \multimap A$ cannot hold simultaneously. However, it is not necessarily transitive, as the example in Figure 2(c) shows: We have $A \multimap B$ and $B \multimap C$ (written more compactly as $A \multimap B \multimap C$) but not $A \multimap C$. In Figure 2(a–b), the two edges are incomparable by the $\multimap$ relation.

1.2 The main results

Our main results are the following three theorems.

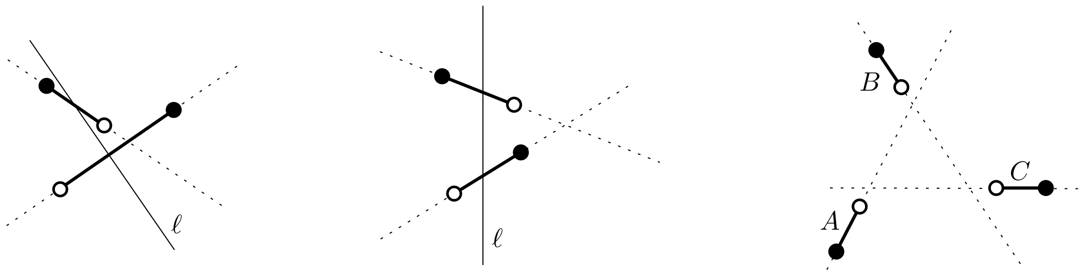
Theorem 1. *Let M be a BR-matching without a chromatic cut. Then M is either of linear or circular type.*

Theorem 2 presents several equivalent characterizations of unique BR-matchings, or, equivalently (in view of $1 \Leftrightarrow 2$), those of linear matchings. The definition of *bichromatic quasi-parallel matchings* in condition 5 will be given later (see Definition 3.1 and Figure 5). They are a variation of (monochromatic) quasi-parallel matchings, introduced in [17, 18].

Theorem 2 (Characterization of unique BR-matchings)**.** *Let M be a BR-matching of F. Then the following conditions are equivalent:*

1. *M is a unique matching.*
2. *M is a linear matching.*
3. *The relation $\multimap$ is a linear order on M.*
4. *No subset of segments forms one of the three patterns in Figure 2.*
5. *M is a bichromatic quasi-parallel matching.*

Remark. If M satisfies any of the conditions of Theorem 2, then any submatching of M satisfies the conditions as well. Indeed, it is obvious that conditions 3 and 4 directly imply that they hold for all subsets. For the other conditions, it follows from the equivalence stated in the theorem.



(a) (b) (c) 3-star

Figure 2: Forbidden patterns for quasi-parallel matchings. All patterns should be understood up to reversal of the colors and reflection of the plane. The pattern (b) includes the case of antiparallel segments, and the pattern (c) includes the case where three lines go through a common point. In cases (a) and (b), a chromatic cut ℓ is shown, see Lemma 4.

Theorem 3 (Properties of circular matchings)**.** *Let M be a BR-matching of F. Then the following conditions are equivalent:*

1. *M is a circular matching.*
2. *The sidedness relation $\multimap$ is a total relation but not a linear order.*
3. *No two segments from M form one of the patterns in Figure 2(a, b), but there are three segments in M that form the 3-star pattern in Figure 2(c).*

Furthermore, if these conditions hold, then:

p1. The sidedness relation $\multimap$ induces naturally a circular order, as explained in Section 4.

p2. M is not unique: There are at least two BR-matchings M' and M'' of F so that each of them is disjoint (and moreover, compatible) to M.

The following table compares linear and circular matchings with respect to the properties mentioned in Theorems 2 and 3.

	Linear Type	Circular Type
Uniqueness	M is unique	M is not unique
Patterns from Figure 2	(a), (b) and (c) are avoided	(a) and (b) are avoided; (c) is present
Relation $\multimap$	Linear order	Total, not linear; induces a circular order

1.3 Related work

Our work belongs to the study of straight-line graph drawings. One of the directions intensively studied in the recent years is that of straight-line matchings, both monochromatic and bichromatic.

Given a bichromatic set F, one can consider the *bichromatic compatible matching graph*, whose nodes correspond to the BR-matchings of F, and where two nodes are connected by an edge if and only if the corresponding matchings are compatible, in the sense that their union is crossing-free. Aloupis, Barba, Langerman, and Souvaine [4] proved that the bichromatic compatible matching graph is always connected. Aichholzer, Barba, Hackl, Pilz, and Vogtenhuber [1] proved that the diameter of this graph is at most $2n$, and this is asymptotically tight. In our work we study the situation when this graph is as small as possible—namely, when it consists of a single node. As for the *maximum* number of BR-matchings that a set of n blue and n red points can admit, Sharir and Welzl [19] established a bound of $O(7.61^{2n})$.

Analogous questions were also studied for *non-colored* ("monochromatic") point sets of size $n = 2m$. García, Noy, and Tejel [8] showed that number of matchings in n-point sets is minimized when the points are in convex position (such sets have $\Theta(2^n/n^{3/2})$ matchings). As for the maximum, Sharir and Welzl [19] proved that any monochromatic set of n points has $O(10.04^n)$ matchings, and on the other hand Asinowski and Rote [5] have recently constructed monochromatic sets with $\Omega(3.09^n)$ matchings. Aichholzer et al. proved that the *(monochromatic) compatible matching graph* for a set of $2m$ points has diameter $O(\log m)$ [2]. Ishaque, Souvaine, and Tóth [13] showed that for any monochromatic matching, there is even a *disjoint* monochromatic compatible matching.

A related direction of research is that of geometric augmentation, see Hurtado and Tóth [12] for a survey. The general pattern of problems can be described as follows. Given a geometric graph, one wants to determine whether it is possible to add edges (segments) in order to get a bigger graph with a certain property, under what conditions this can be done, how many segments one has to add, etc. Hurtado, Kano, Rappaport, and Tóth [11] proved that any BR-matching can be augmented to a non-crossing spanning tree in $O(n \log n)$ time.

1.4 Outline

In Section 2 we prove several preliminary results about chromatic cuts and the sidedness relation $\twoheadrightarrow$. In particular, we give a simple proof of the fact that a BR-matching which has a chromatic cut is not unique. Section 3 is devoted to linear matchings. We give several characterizations of them, and we give two proofs that a linear matching is unique. One proof, via the Fishnet Lemma, requires more effort to set up some additional geometric structure, but it makes the argument more transparent (Section 3.2). Section 4 analyzes circular matchings in depth, and we prove that they are never unique. Then we complete the proof of the main theorem about unique BR-matchings, Theorem 2. In Section 5 we discuss some additional issues: Can every linear matching be realized by *parallel* segments on a point set with the same order type? An example with six segments shows that the answer is no (Section 5.1). What are the sidedness relations of circular matchings on n elements? We characterize them and show that their number is $2^{n-1} - n$ (Section 5.2). We also discuss an apparent defect of our characterization in Theorem 2: all conditions involve the point set *together with its matching*. We argue why this is necessarily so, and why some "local" condition in terms of the point set alone is impossible (Section 5.3).

In Section 6 we turn to algorithmic issues. We describe an algorithm that recognizes point sets F which admit a unique matching, an algorithm that recognizes circular matchings, and an algorithm that detects the existence of a chromatic cut by computing a so-called *balanced line*. All these algorithms run in $O(n \log n)$ time.

We conclude with some open problems and directions for future research in Section 7.

2 Preliminary results

2.1 Chromatic cuts

We start with a simple geometric description of BR-matchings that admit a chromatic cut.

Lemma 4. *Let M be a BR-matching of F. M admits a chromatic cut if and only if contains two segments A, B forming the pattern in Figure 2 (a) or (b), or more explicitly, if an outer ray of one segment crosses the second segment (a), or the intersection point of $g(A)$ and $g(B)$ belongs to outer rays of different colors (b), or A and B are antiparallel, which is a special case of pattern (b).*

Proof. [$\Leftarrow$] If an outer ray of the segment A crosses the second segment B, then, if we rotate $g(A)$ around an inner point of A by a small angle in one of two possible directions, depending on the orientation of A and B, then a chromatic cut is obtained, see Figure 2 (a). If the $\bullet$-ray of one segment and the $\circ$-ray of the second segment cross each other, then any line through inner points of A and B is a chromatic cut, see Figure 2 (b). The same is true if A and B are antiparallel.

[$\Rightarrow$] Let ℓ be a chromatic cut of M, and let A and B be two segments that have their $\bullet$-ends on the opposite sides of ℓ. Consider the lines $g(A)$ and $g(B)$. If $g(A)$ and $g(B)$ do not cross, they clearly must be antiparallel. If they cross, then it is not possible that the two outer rays of the same color meet, because they are on opposite sides of ℓ. □

A line ℓ is a *balanced line* if in each open half-plane determined by ℓ, the number of $\bullet$-points is equal to the number of $\circ$-points. The next lemma reveals a relation between chromatic cuts and balanced lines.

Lemma 5. *A BR-matching M has a chromatic cut if and only if there exists a balanced line that crosses a segment of M.*

Proof. [$\Leftarrow$] Let ℓ be a balanced line that crosses a segment A of M. We can assume that ℓ does not contain points from F: it cannot contain exactly one point of F; and if it contains two points of F of different colors, we can translate it slightly, obtaining a balanced line that still crosses A but does not contain points of F. If it contains two points of the same color, we rotate it slightly about the midpoint between these two points.

Now, A has a $\bullet$-end in one half-plane of ℓ and a $\circ$-end in the other half-plane. Since ℓ is balanced, there must be another segment B that crosses ℓ in such a way that ℓ is a chromatic cut.

[⇒] First, let A be a segment in M, and let p be an inner point of A that does not belong to any line determined by two points of F, other than the endpoints of A. We claim that if there is no balanced line that crosses A at p, then $g(A)$ is a balanced line.

Assume that there is no balanced line that crosses A at p. We use a continuity argument. Let $m = m_0$ be any directed line that crosses A at p. Rotate m around p counterclockwise until it makes a half-turn. Denote by m_α the line obtained from m after rotation by the angle α; so, we rotate it until we get m_π. Let φ $(0 < \varphi < \pi)$ be the angle such that m_φ coincides with $g(A)$ (as a line, ignoring the orientations). Assume without loss of generality that the right half-plane bounded by m is *dominated* by $\bullet$, in the sense that it contains more $\bullet$-points than $\circ$-points. Then the right half-plane bounded by m_π is dominated by $\circ$. As we rotate m, the points of F change sides *one by one*, except at $\alpha = \varphi$. When one point changes sides, m_α cannot change from $\bullet$-dominance to $\circ$-dominance without becoming a balanced line. Therefore, for each $0 \leq \alpha < \varphi$, the right side of m_α is dominated by $\bullet$, and for each $\varphi < \alpha \leq \pi$, the right side of m_α is dominated by $\circ$. At $\alpha = \varphi$, exactly two points of different colors change sides. The only possibility is that the $\bullet$-end of A passes from from the right side to the left side and the $\circ$-end of A passes from the left side to the right side of the rotated line. It follows that at this moment the value of $\#(\bullet) - \#(\circ)$ in the right half-plane changes from 1 to -1, and that $m_\varphi = g(A)$ is a balanced line.

Now, let ℓ be a chromatic cut that crosses $A, B \in M$ so that the $\bullet$-end of A and the $\circ$-end of B are in the same half-plane bounded by ℓ. Denote by p and q the points of intersection of ℓ with A and B, respectively. We assume without loss of generality that p and q do not belong to any line determined by points of F.

If there is a balanced line that crosses A at p, or a balanced line that crosses B at q, we are done. By the above claim, it remains to consider the case when the lines $g(A)$ and $g(B)$ are balanced. Assume without loss of generality that ℓ is horizontal, p is left of q, and the $\bullet$-end of A is above ℓ, see Figure 3 for an illustration.

We start with the line $k = g(A)$, directed upwards, rotate it clockwise around p until it coincides with ℓ, and then continue to rotate it clockwise around q until it coincides with $g(B)$, directed down. As above, we monitor $\#(\bullet) - \#(\circ)$ on the right side of the line k: this quantity is 0 in the initial and the final position. Just after the initial position it is -1, and just before the final position it is $+1$. In between, it makes only ± 1 jumps, since the points of F change sides of the rotated line k one by one. It follows that for some intermediate position it is 0, and thus we have a balanced line crossing one of the edges. □

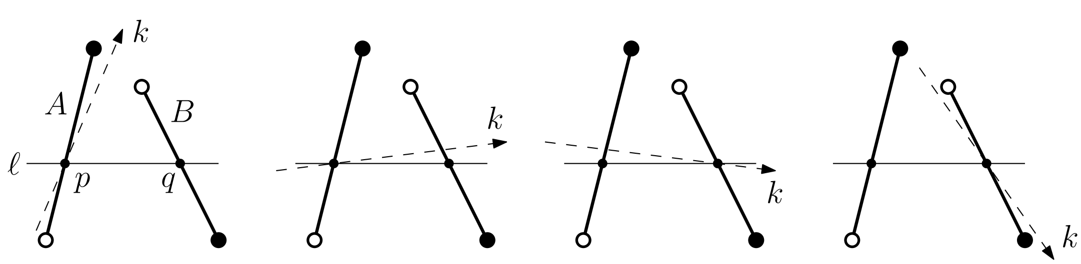


Figure 3: Finding a balanced line in the proof of Lemma 5.

In Section 6.3, we discuss the algorithmic implementation of this proof.

Corollary 6. *Let M be a BR-matching of F with a chromatic cut. Then M is not unique.*

Proof. By Lemma 5, there is a balanced line ℓ crossing a segment $A \in M$. We construct matchings on both sides of ℓ, and denote their union by M'. Then M' is a matching of F, and we have $M' \neq M$ since M' does not use A. □

Remark. As mentioned in the introduction, Corollary 6 follows from the stronger statement of [4, Lemma 9]: the existence of a compatible matching $M' \neq M$. We have given a simpler alternative proof.

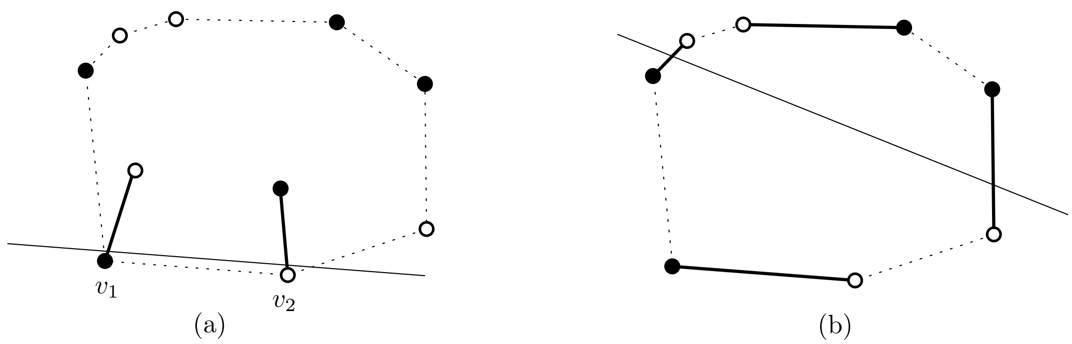


Figure 4: Chromatic cuts in the proof of Lemma 7.

Lemma 7. *Let M be a BR-matching of F that has no chromatic cut. Then*

- *either all points of $\partial CH(F)$ have the same color,*
- *or the points of $\partial CH(F)$ form precisely two color intervals, each of which must have size at least* 2.

In the latter case, the two boundary segments connecting points of different color necessarily belong to M.

Proof. Assume that $\partial\mathrm{CH}(F)$ has points of both colors.

If v_1 and v_2 are two neighboring points on $\partial\mathrm{CH}(F)$ with different colors, then they are matched by a segment of M. Indeed, let ℓ' be the line through v_1 and v_2. If v_1 and v_2 are not matched by a segment of M, then each of them is an endpoint of some segment of M. When we shift ℓ' slightly so that it crosses these two segments, a chromatic cut is obtained, see Figure 4(a).

Therefore, if the points of $\partial\mathrm{CH}(F)$ form more than two color intervals, then at least four segments of M have both ends on $\partial\mathrm{CH}(F)$. At least two among them have the $\bullet$-end before the $\circ$-end, with respect to their circular order. Any line that crosses these two segments will be then a chromatic cut, see Figure 4(b).

Thus, we have exactly two color intervals. If one of them consists of one point, then this point has two neighbors of another color of $\partial\mathrm{CH}(F)$. As observed above, this point must be matched by M to both of them, which is clearly impossible. □

We recall the definition from the introduction: a *linear matching* is a BR-matching without a chromatic cut such that $\partial\mathrm{CH}(F)$ consists of exactly two color intervals, both of size at least 2; a *circular matching* is a BR-matching without a chromatic cut such that all points of $\partial\mathrm{CH}(F)$ have the same color. So, we have established that a BR-matching without a chromatic cut necessarily belongs to one of these two types, and we have thus completed the proof of Theorem 1. In the next sections we study these types in more detail.

2.2 The sidedness relation between segments

Now we show how the presence or absence of a chromatic cut affects some properties of the sidedness relation $\multimap$.

Lemma 8. *Let M be a BR-matching. M has no chromatic cut if and only if the sidedness relation $\multimap$ is a total relation, that is, for any two segments $A, B \in M$, $A \neq B$, we have $A \multimap B$ or $B \multimap A$.*

Proof. If two segments A and B have a chromatic cut, then the lines $g(A)$ and $g(B)$ must intersect as in Figure 2 (a) or (b), and the segments are not comparable by $\multimap$; otherwise, the lines $g(A)$ and $g(B)$ are parallel or intersect in the outer rays of the same color, and then the segments are comparable. □

Recall that the relation $\multimap$ is asymmetric by definition: we never have $A \multimap B$ and $B \multimap A$. Moreover, we will see that if M has no chromatic cut, then, in order to prove $A \multimap B$, it suffices to prove only one condition from the definition of $\multimap$:

Lemma 9. *Let M be a BR-matching without chromatic cut, and let $A, B \in M$ ($A \neq B$). If B lies to the right of $g(A)$, or if A lies to the left of $g(B)$, then $A \multimap B$.*

Proof. If M has no chromatic cut, then we have either $A \multimap B$ or $B \multimap A$ by Lemma 8. Given one of the above conditions, $B \multimap A$ is ruled out. □

3 Quasi-parallel, or linear, matchings

3.1 Characterizations of linear matchings

In this section, we give several characterizations of linear matchings and prove that such matchings are unique for their point sets.

Lemma 10. *Let M be a linear matching. There exist $A_1, A_n \in M$, the "minimum" and the "maximum" element, such that for every $B \in M \setminus \{A_1\}$ we have $A_1 \multimap B$, and for every $C \in M \setminus \{A_n\}$ we have $C \multimap A_n$.*

Proof. By Lemma 7, the two boundary segments connecting points of different color belong to M. For one of them, to be denoted by A_1, all other segments of M belong to the right half-plane bounded by $g(A_1)$; for the second, to be denoted by A_n, all other segments of M belong to the left half-plane bounded by $g(A_n)$. Since M has no chromatic cut, the claim follows directly from Lemma 9. □

Definition. A BR-matching M is *(bichromatic) quasi-parallel* if there exists a directed *reference line* ℓ such that the following conditions hold:

(i) No segment is perpendicular to ℓ.

(ii) For every $A \in M$, the direction of its projection on ℓ (as usual, from ∘ to •) coincides with the direction of ℓ.

(iii) For every non-parallel $A, B \in M$, the projection of the intersection point of $g(A)$ and $g(B)$ on ℓ lies outside the convex hull of the projections of A and B on ℓ.

Figure 5 shows an example of quasi-parallel matching, with horizontal ℓ.

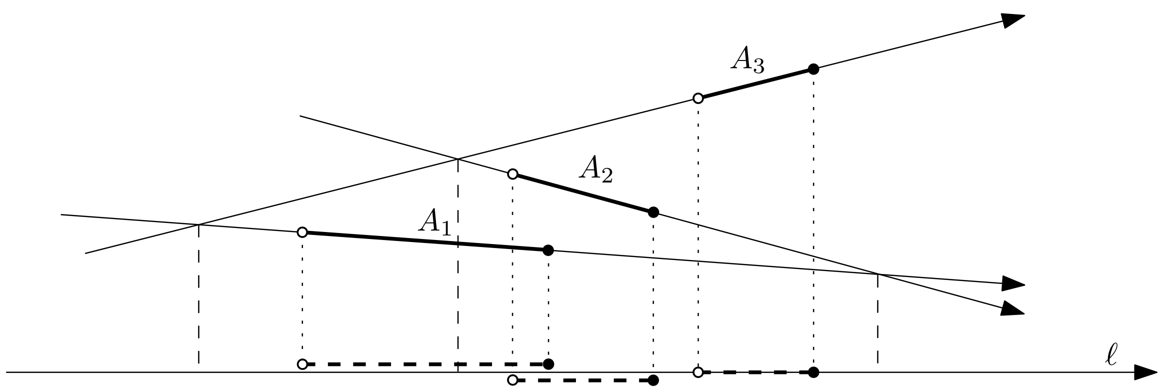


Figure 5: A quasi-parallel matching.

Remark. In the monochromatic setting, the notion of quasi-parallel segments was introduced by Rote [17, 18] as a generalization of parallel segments, in the context of a dynamic programming algorithm for some instances of the traveling salesman problem. His definition can be obtained from our one by dropping condition (ii).

Lemma 11. *Let M be a BR-matching of a bichromatic set F. Then the following conditions are equivalent:*

1. *M is a linear matching.*
2. *The relation $\multimap$ in M is a strict linear order.*
3. *M has no patterns of the three kinds in Figure 2.*
4. *M is a quasi-parallel matching.*

Proof. [1 ⇒ 2] By definition, the relation $\twoheadrightarrow$ is asymmetric, and according to Lemma 8, it is total.

It remains to establish transitivity. By Lemma 10, there exist $A_1, A_n \in M$ (the "minimum" and the "maximum" elements) such that for every $B \in M \setminus \{A_1\}$ we have $A_1 \twoheadrightarrow B$, and for every $C \in M \setminus \{A_n\}$ we have $C \twoheadrightarrow A_n$. We define inductively $A_2, \ldots, A_{n-1}$ as follows. Assume $A_1, \ldots, A_{i-1}$ are already defined. Let $M_i = M \setminus \{A_1, A_2, \ldots, A_{i-1}\}$. Then M_i is also a linear matching: indeed, it has no chromatic cut and has both colors on the boundary of the convex hull because A_n belongs to it. Denote the "minimum" element of M_i by A_i and repeat until all labels are assigned. (Note that we never label A_n as A_i with $i < n$.)

It follows from the construction that for all $i < j$, A_j lies to the right of $g(A_i)$. Thus, by Lemma 9, we have $i < j \;\Rightarrow\; A_i \twoheadrightarrow A_j$. This implies that $\twoheadrightarrow$ is a linear order.

[2 ⇒ 3] It is easy to check that none of the configurations in Figure 2 is ordered linearly by $\twoheadrightarrow$.

[3 ⇒ 4] In this proof, we follow the idea from [18]. As a preparation, one can establish by case distinction that any two or three segments that contain none of the patterns from Figure 2 are quasi-parallel. (We omit the details.)

Now, let M be a BR-matching without the forbidden patterns from Figure 2. For each $A \in M$, let $a(A)$ be the arc on the circle of directions corresponding to positive directions of lines m such that the angle between A and m is acute. (These are the lines that can play the role of a reference line ℓ in the definition of quasi-parallel matching, with respect to A.) Each $a(A)$ is an open half-circle, see Figure 6.

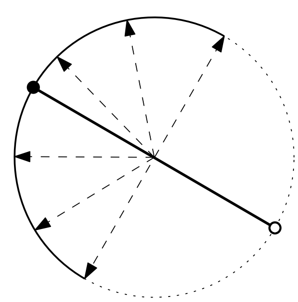


Figure 6: The open arc $a(A)$ for a matching segment A, used to prove 3 ⇒ 4 in Lemma 11.

We fix some segment $S \in M$. For any segment $A \in M$, $\{S, A\}$ is a quasi-parallel matching, and hence the intersection of the corresponding arcs $a(S) \cap a(A)$ is a non-empty sub-arc of $a(S)$, which we denote by $a'(A)$. Now, for any two segments $A, B \in M$, $\{S, A, B\}$ is a quasi-parallel matching, and hence the intersection of the corresponding arcs is non-empty. In other words, $a'(A) \cap a'(B) \neq \emptyset$. We apply Helly's Theorem to the arcs $a'(A)$ (considering them as sub-arcs of $a(S)$) and conclude that there exists a direction in the intersection of the arcs corresponding to all segments of M. A line ℓ in this direction will satisfy conditions (i) and (ii) of the definition of quasi-parallel matching. Finally, the absence of forbidden patterns implies that condition (iii) is satisfied as well.

[4 ⇒ 1] Condition (iii) in the definition of quasi-parallel matchings implies that for any $A, B \in M$, $A \neq B$, the lines $g(A)$ and $g(B)$ are either parallel, or the outer rays of the same color cross. It follows from Lemma 4 that there is no chromatic cut.

A lowest point and a highest point of F with respect to ℓ belong to the boundary of the convex hull, and they have different colors. Therefore, M is of linear type. □

Remark. A similar characterization of *monochromatic* quasi-parallel matchings by seven forbidden patterns was given by Rote [17]. (In the journal version [18], one of the forbidden patterns has been inadvertently omitted.) We have fewer forbidden patterns because avoiding certain monochromatic patterns becomes equivalent to the single pattern from Figure 2(b) once colors are added.

Lemma 11 proves the equivalence of conditions 2, 3, 4, and 5 in Theorem 2. Condition 3 justifies the term "matching of linear type". Now we prove that they imply the uniqueness of M.

Theorem 12. *Let M be a linear matching on the point set F. Then M is unique, that is, M is the only matching of F.*

Proof. By Lemma 11, the matching M is quasi-parallel, with reference line ℓ. We assume without loss of generality that ℓ is vertical.

Assume for contradiction that another matching M' exists. (In the figures below, the segments of M are denoted by solid lines, and the segments of M' by dashed lines.) The symmetric difference of M and M' is the union of alternating cycles. We now claim that *an alternating cycle must intersect itself.*

Consider the alternating cycle $\Pi = p_1q_1p_2q_2p_3q_3 \dots p_mq_mp_1$ that consists of segments $p_iq_i \in M$ and $q_ip_{i+1}; q_mp_1 \in M'$. We assume that p_i are ∘-vertices and q_i are •-vertices. Let B be the minimum (with respect to ⇾) segment and let C be the maximum segment of M that belongs to Π. Then no points of Π lie left of $g(B)$ or right of $g(C)$. Since for both B and C the •-end is higher than the ∘-end, the path Π must cross itself at least once, establishing the claim, see Figure 7.

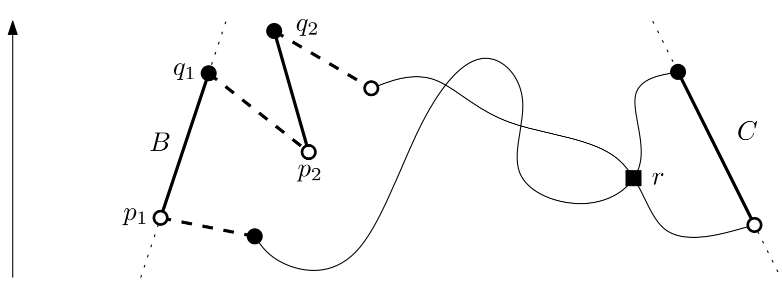


Figure 7: Illustration for the proof of Theorem 12: an alternating path for M crosses itself.

We now traverse the path Π, starting from $p_1q_1p_2q_2\dots$, until we reach some point r for the second time. The path between these two occurrences of r forms a simple polygon without self-crossings. The two segments that cross at r cannot be two segments of M or two segments of M'. Hence, the first occurrence of r on Π is on a segment p_iq_i of M, and the second is on a segment q_jp_{j+1} of M', or vice versa. We consider only the first case, the other being similar. In this case, we consider the matching N that consists of segments $rq_i, p_{i+1}q_{i+1}, p_{i+2}q_{i+2}, \dots p_jq_j$ (that is, N consists of the segments of M that occur on Π between the two times that it visits r, and the part of segment of M that contains r). It is clear that N is also quasi-parallel, with respect to the same reference line ℓ.

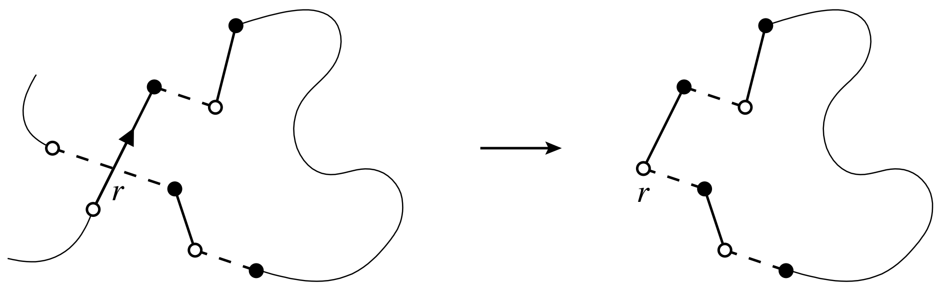


Figure 8: Illustration to the proof of Theorem 12: an alternating path for N.

The closed path $rq_ip_{i+1}q_{i+1}p_{i+2}q_{i+2}\dots p_jq_jr$ is an alternating path for N. By the choice of r, this path does not intersect itself, see Figure 8, which contradicts the claim proved above that an alternating path of a quasi-parallel matching always intersects itself. This contradicts the existence of M' and finishes the proof. □

The proof of Theorem 12 tells us that a closed alternating path cannot exist. In contrast, it is always possible to construct at least two *open* alternating paths from the minimum to the maximum element of M. We formalize this observation for later usage.

Observation 13. *Let M be a linear matching. Then there exist two alternating paths containing all segments of M in the order $\multimap$.*

Proof. Let $A_1, \dots, A_n$ be the segments of M, ordered by $\multimap$. We proceed by induction, see Figure 9. Let R_k be a path from A_1 to A_k in which the segments of M appear according to $\multimap$. We obtain R_{k+1} by taking R_k and adding a color-conforming segment from A_k to A_{k+1}. This is possible because there is no other segment of M between A_k and A_{k+1}. The color of the starting point can be chosen and thus we have two such paths. □

3.2 Proof of Theorem 12 by the Fishnet Lemma

Let us return to the proof of Theorem 12 for the following restricted case. Assume that all segments of the given matching M are vertical, and all segments of a supposed matching M' are horizontal. Then the alternating path as described in the proof will be weakly y-monotone. This is essentially the reason why in this case we never get a *closed* alternating cycle, see Figure 10.

This argument can be extended for the general case if we replace the horizontal and vertical lines (that contain the segments in the special case) by an appropriately constructed

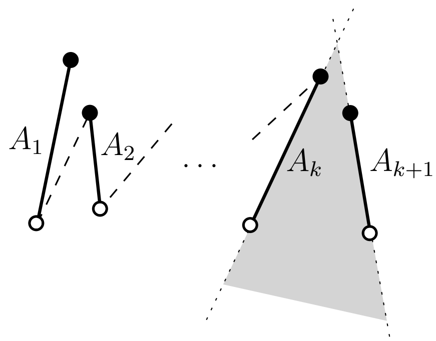


Figure 9: Extending an alternating path in the proof of Observation 13.

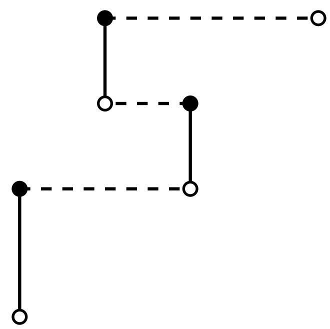


Figure 10: The path Π as in Theorem 12 is weakly y-monotone if all segments of M are vertical and all segments of M' are horizontal.

topological grid. Thus we obtain another proof of Theorem 12, which shows in a more clear and intuitive light why one cannot close the path. This approach will be made precise with the following *Fishnet Lemma*. We will apply it only to polygonal curves, but we formulate it for arbitrary curves, see Figure 11.

Consider a set $V = \{v_1, \dots, v_n\}$ of pairwise non-crossing unbounded Jordan curves ("ropes") such that the plane is partitioned into $n+1$ connected regions $R_0, \dots, R_n$: R_0 is bounded only by v_1; R_i, $1 \le i \le n-1$, is bounded by v_i and v_{i+1}; and R_n bounded only by v_n. These curves will be called the *vertical* curves. In the illustrations they will be black.

Consider another set $G = \{g_1, \dots, g_m\}$ of pairwise non-crossing Jordan arcs, called the *horizontal* arcs. They are drawn in green, and they have the following properties: every curve g_k has its endpoints on two different vertical curves v_i and v_j $(i < j)$, it has exactly one intersection point with each vertical curve $v_i, v_{i+1}, v_{i+2}, \dots, v_j$, and it has no intersection with the other curves. See Figure 11(a) for an example. We say that the curves $V \cup G$ form a *partial (topological) grid.*

Lemma 14 (The Fishnet Lemma)**.** *The horizontal arcs g_k of a partial topological grid $V \cup G$ can be extended to pairwise non-crossing unbounded Jordan arcs h_k in such a way that the curves $H = \{h_1, \dots, h_m\}$ together with V form a complete topological grid $V \cup H$: each horizontal curve h_k crosses each vertical curve v_i exactly once. See Figure 11(b).*

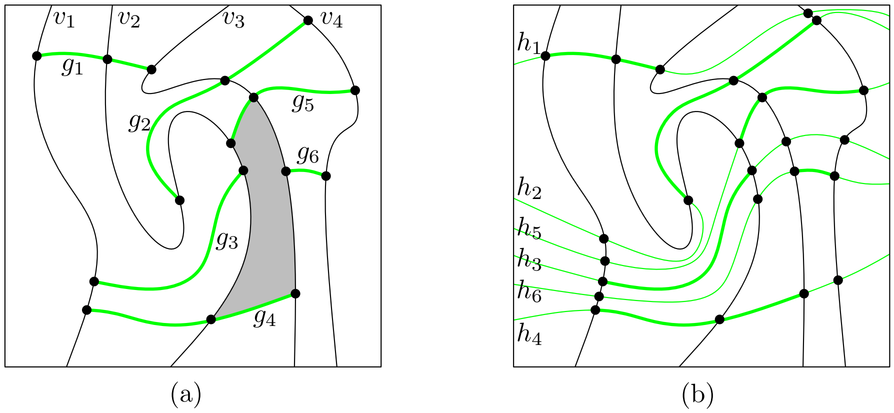


Figure 11: (a) A partial grid. (b) Extension to a complete grid of ropes.

Proof. We prove the lemma by a construction which incrementally extends the horizontal segments until a complete topological grid is obtained.

The bounded faces of the given curve arrangement $V \cup G$ are topological quadrilaterals: they are bounded by two consecutive vertical curves and two horizontal curves. The bounded faces of the desired final curve arrangement $V \cup H$ are also such quadrilaterals, with the additional property that they have no extra vertices on their boundary besides the four corner intersections. In $V \cup G$, such extra vertices arise as the endpoints of the segments g_k.

Let us take such a bounded face, between two vertical curves v_i and v_{i+1}, with an endpoint of g_k on one of its vertical sides, see Figure 12(a, b). We can extend g_k to some point on the opposite vertical side, chosen to be distinct from all other endpoints, splitting the face into two and creating a new intersection point. (The existence of such an extension follows from the Jordan–Schoenflies Theorem, by which the bounded face is homeomorphic to a disc.) An unbounded face between two successive vertical curves v_i and v_{i+1} that has

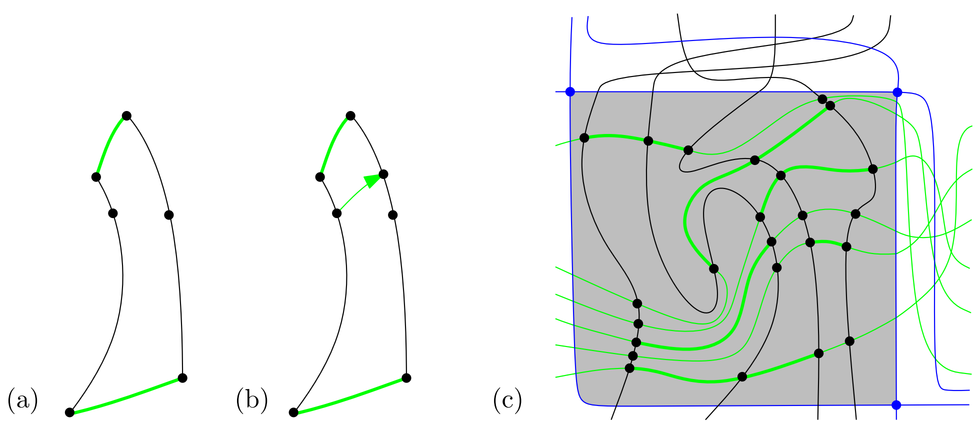


Figure 12: (a) A quadrilateral face with extra vertices; the shaded face from Figure 11(a). (b) Adding an edge. (c) Embedding the grid into a pseudoline arrangement.

an extra vertex on a vertical side can be treated similarly.

We continue the above extension procedure as long as possible. Since we are adding new intersection points, but no two curves can intersect twice, this must terminate. Now we are almost done: each horizontal curve extends from v_1 to v_n and crosses each vertical curve exactly once. We just extend the horizontal curves to infinity, into R_0 and R_n, without crossings. □

This lemma can be interpreted in the context of pseudoline arrangements. In an arrangement of pseudolines, each pseudoline is an unbounded Jordan curve, and every pair of pseudolines has to cross *exactly* once. The grid construction can be embedded in a true pseudoline arrangement, see Figure 12(c): simply enclose all crossings in a bounded region formed by three new (blue) pseudolines and let the crossings between vertical lines and between horizontal lines occur outside this region.

We return to the proof of Theorem 12.

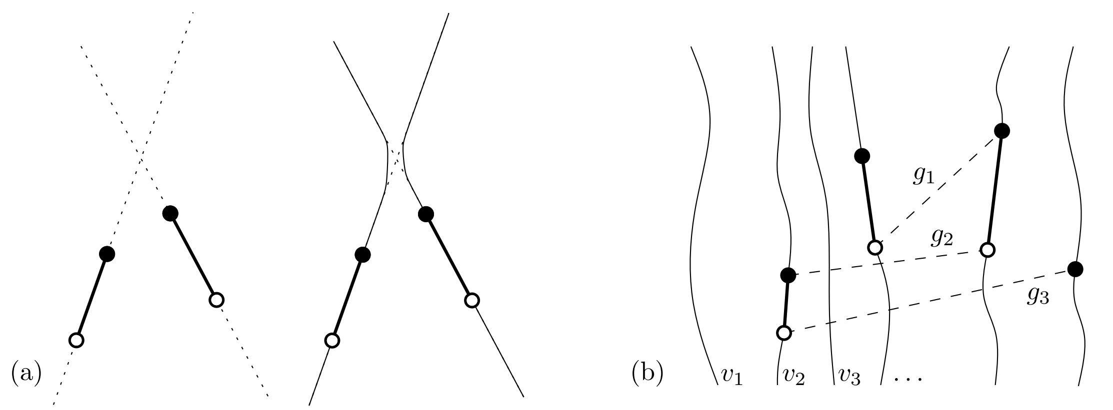


Figure 13: Applying the Fishnet Lemma.

Proof. Given a quasi-parallel matching M, we construct a set of Jordan curves V as in Lemma 14 by considering the line arrangement formed by the segments $s_1 \rightarrow \cdots \rightarrow s_n$ with the corresponding lines $g(s_1), \ldots, g(s_n)$. We construct curve v_i by starting from s_i and going along $g(s_i)$. At each intersection, the curves switch from one line to the other, and after a slight deformation in the vicinity of the intersections, they become non-crossing, see Figure 13(a). These crossings lie outside the parts of the lines where the segments lie; therefore the switchings have no influence on the left-to-right order of the segments s_i. The setup of these n non-crossing "vertical" curves gives us the possibility to establish a common orientation and speak about "above" and "below" on each curve in a consistent way.

Now assume there is another matching M'. Then M and M' form at least one closed alternating path. Let $G = \{g_1, \ldots, g_m\}$ be the segments of M' on such a cycle in the order in which they are traversed. V and G satisfy the condition of the Fishnet Lemma and thus can be extended to a complete topological grid. Assume without loss of generality that g_1 lies above g_2 on the common incident edge of M, see Figure 13(b). Since the relative order of ◦-vertices and •-vertices is the same on all vertical curves, this property carries over to

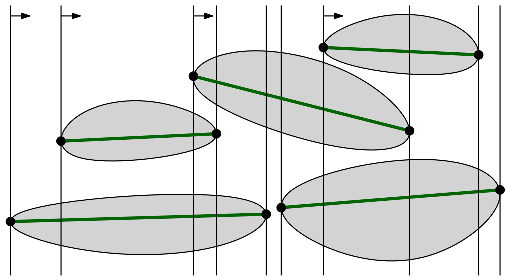


Figure 14: Separability by vertical translation

successive edges: g_i lies above g_{i+1} on the vertical curve containing their common segment of M, for $i = 1, \ldots, k-1$. Since the extended horizontal curves $h_1, \ldots, h_k$ intersect each vertical curve in the same order, we conclude that every vertical curve intersects h_i above h_{i+1}. But then g_m cannot reach the starting point above g_1 on the common incident edge of M, a contradiction. □

We mention separability by translation [7] as another easy consequence of the Fishnet Lemma: in any family of n disjoint convex (or even just x-monotone) sets in the plane, one can find one set that can be translated vertically upward to infinity without colliding with the others (Figure 14): Just draw a "horizontal" segment g_i between the leftmost and the rightmost point of each set, and vertical lines through all segment endpoints. The Fishnet Lemma will identify a horizontal curve h_i lying above all other curves, and the corresponding set can be translated to infinity. (There is, however, an easy direct proof of vertical separability, see Guibas and Yao [9, 10]: Among the sets whose left endpoint is visible from above, as marked by arrows in Figure 14, the one with the rightmost left endpoint can be translated to infinity.)

4 Circular matchings

4.1 The structure of circular matchings

In this section, we study circular matchings in more detail. Recall that such a matching is a BR-matching without a chromatic cut for which all points on the convex hull have the same color. We assume without loss of generality that this color is •.

We prove that if M is of circular type, then its point set has at least two other matchings. Moreover, we show that for a circular matching, the relation $\multimap$ induces a *circular order* (this will justify the term "matching of circular type"), and describe such matchings in terms of forbidden patterns.

Lemma 15. *A BR-matching M is of circular type if and only if it has no patterns (a) and (b) from Figure 2, and has at least one pattern (c) (a 3-star).*

Proof. We saw in Lemma 4 that a BR-matching has no chromatic cut if and only if it avoids the patterns (a) and (b). By Lemma 7, a BR-matching without chromatic cut is either of linear or of circular type. By Lemma 11, a BR-matching is of linear type if and only if it avoids (a), (b) and (c). Therefore, a BR-matching is of circular type if and only if it avoids (a) and (b), but contains (c). □

Theorem 16. *Let M be a matching of circular type on the point set F. Then there are at least two disjoint BR-matchings on F, compatible to M.*

Proof. By Lemma 15, there are three segments that form a 3-star. They split the plane into three convex regions Q_1, Q_2 and Q_3 and a triangle as in Figure 15(a). The triangle is bounded (without loss of generality) by three ∘-rays, and no points of F lie in the interior of this triangle. Otherwise, if a whole segment A of M lies inside the triangle, then the •-ray determined by A crosses a ∘-ray, and we have pattern (b). And if only one endpoint of A lies inside the triangle, then A itself crosses a ∘-ray, and we have pattern (a). Both cases contradict Lemma 15.

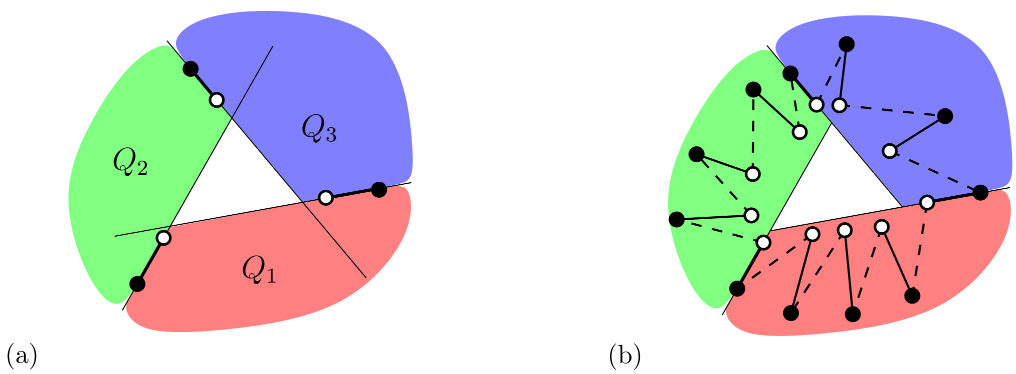


Figure 15: The three regions Q_1, Q_2, Q_3 and an alternating cycle in the proof of Theorem 16.

All segments in a region Q_i together with the two defining segments are of linear type (indeed, they have no chromatic cut but have both colors on the boundary of the convex hull). Thus, by Observation 13, in each region there is an alternating path from the ∘-point of the left bounding segment to the •-point of the right bounding segment (or vice versa), see Figure 15(b). The union of the three paths forms an alternating polygon and thus we have found a different compatible BR-matching M'. If we choose the paths in the other direction (from the •-point of the left bounding segment to the ∘-point of the right bounding segment), we get another BR-matching M''. □

Remark. If M is a matching of circular type on the point set F, then F can in fact have exponentially many BR-matchings, as the following construction shows. Let A_1, A_2, A_3 be three segments so that $g(A_1), g(A_2), g(A_3)$ intersect at one point O that belongs to their ∘-rays, and so that $\{A_1, A_2, A_3\}$ is a circular matching. Repeat this construction inside the triangle bounded by the ∘-ends of A_1, A_2, A_3, using the same point O and only taking care of general position, see Figure 16. This can be repeated an arbitrary number of times. Then in each "layer" we have three BR-matchings; hence, at least $3^{n/3}$ BR-matchings for the whole point set (n denotes the number of segments).

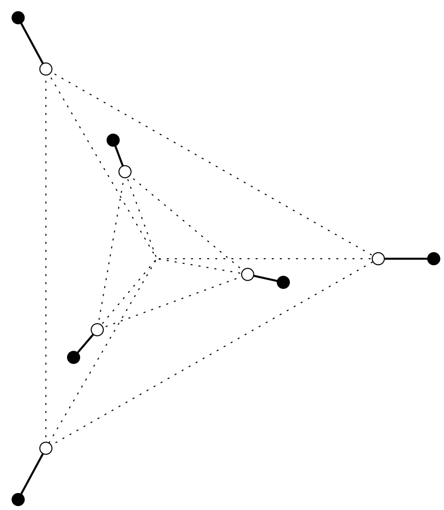


Figure 16: A circular matching whose point set has exponentially many BR-matchings.

Now we study in more detail the relation $\multimap$ for circular matchings. In the proof of Theorem 16 we saw that a circular matching is a union of three linear matchings, see Figure 15(b). In the next lemma we prove that in fact it is a union of *two* linear matchings.

Lemma 17. *Let M be a circular matching, and let B be a segment of M. The matchings*

$$M_B^R = \{X \in M : B \multimap X\},$$
$$M_B^{R+} = \{X \in M : B \multimap X\} \cup \{B\},$$
$$M_B^L = \{X \in M : X \multimap B\},$$
$$M_B^{L+} = \{X \in M : X \multimap B\} \cup \{B\}.$$

are not empty, and they are of linear type.

Proof. Consider first the matching M_B^{R+}. Since it contains B, it is non-empty. Since it is a submatching of M, it has no chromatic cut. Both the $\circ$-end and the $\bullet$-end of B belong to the boundary of its convex hull; therefore M_B^{R+} must be of linear type. Similarly, M_B^{L+} is of linear type.

If M_B^R is empty, then $M_B^{L+} = M$, which is impossible since M is of circular type, and M_B^{L+} of linear type. Now, since M_B^{R+} is of linear type, and M_B^R is a subset of this matching, M_B^R is of linear type as well (this follows from $1 \Leftrightarrow 3$ in Lemma 11). The proof for M_B^L is similar. $\square$

Corollary 18. *The relation $\multimap$ in a matching M of circular type has neither a "minimal" nor a "maximal" element:*

for every $B \in M$ there exists an $A \in M$ such that $A \multimap B$;
for every $B \in M$ there exists an $A \in M$ such that $B \multimap A$.

Proof. Otherwise, for such an element B, M_B^L or M_B^R would be empty. □

Lemma 19. *Let M be a circular matching. Let B be any segment of M. Let A be the minimum (with respect to $\twoheadrightarrow$) element of M_B^L, and let Z be the maximum element of M_B^R. Then the triple $\{A, B, Z\}$ is a circular matching (a 3-star).*

Proof. If M is of size 3, that is, $M = \{A, B, Z\}$, there is nothing to prove. Thus, we assume that there is at least one more segment in M. Assume without loss of generality that M_B^R contains at least one segment in addition to Z.

Let C be a segment of M such that $C \twoheadrightarrow A$. (Such a segment exists by Corollary 18.) Since A is the minimum element of M_B^L, we have $C \in M_B^R$, that is, $B \twoheadrightarrow C$.

If $C = Z$ then $Z \twoheadrightarrow A \twoheadrightarrow B \twoheadrightarrow Z$, that is, the relation $\twoheadrightarrow$ in the triple $\{A, B, Z\}$ is not linear; therefore $\{A, B, Z\}$ is of circular type.

Suppose now that $C \neq Z$, and consider the matching $\{A, B, C, Z\}$. We have $C \twoheadrightarrow A \twoheadrightarrow B \twoheadrightarrow C$. Thus, the relation $\twoheadrightarrow$ in the matching $\{A, B, C, Z\}$ is not linear; therefore, $\{A, B, C, Z\}$ is of circular type. Now, by Corollary 18, some segment in $\{A, B, C, Z\}$ must lie to the right of Z according to the relation $\twoheadrightarrow$. Since $B \twoheadrightarrow Z$ and $C \twoheadrightarrow Z$, we have $Z \twoheadrightarrow A$. Thus, $Z \twoheadrightarrow A \twoheadrightarrow B \twoheadrightarrow Z$, and this means that $\{A, B, Z\}$ is of circular type. □

We shall show that if M is a circular matching, then there exists a natural *circular order* of its edges. A circular (or cyclic) order is a ternary relation which models the "clockwise" relation among elements arranged on a cycle. A standard way of constructing a circular order from j linear orders $A_{11} \leq A_{12} \leq \cdots \leq A_{1i_1}$, $A_{21} \leq A_{22} \leq \cdots \leq A_{2i_2}$, ..., $A_{j1} \leq A_{j2} \leq \cdots \leq A_{ji_j}$ is their "gluing": we say that $[X, Y, Z]$ (and, equivalently, $[Y, Z, X]$ and $[Z, X, Y]$) if X, Y and Z appear in the order XYZ or YZX or ZXY in the sequence

$$A_{11}, A_{12}, \ldots, A_{1i_1},\ A_{21}, A_{22}, \ldots, A_{2i_2},\ \ldots,\ A_{j1}, A_{j2}, \ldots, A_{ji_j}$$

We fix $B \in M$ and apply this procedure on M_B^{L+} and M_B^R in which $\twoheadrightarrow$ is linear by Lemma 17. Let $A_1, A_2, \ldots, A_k$ be the segments of M_B^L labeled so that $A_1 \twoheadrightarrow A_2 \twoheadrightarrow \cdots \twoheadrightarrow A_k$, and let $C_1, C_2, \ldots, C_m$ be the segments of M_B^R labeled so that $C_1 \twoheadrightarrow C_2 \twoheadrightarrow \cdots \twoheadrightarrow C_m$. By Lemma 19 we have $C_m \twoheadrightarrow A_1$. Thus, we consider the circular order $[*, *, *]$ induced by

$$B \twoheadrightarrow C_1 \twoheadrightarrow C_2 \twoheadrightarrow \cdots \twoheadrightarrow C_m \twoheadrightarrow A_1 \twoheadrightarrow A_2 \twoheadrightarrow \cdots \twoheadrightarrow A_k \twoheadrightarrow B. \tag{1}$$

That is, for $X, Y, Z \in M$ we have $[X, Y, Z]$ (and, equivalently $[Y, Z, X]$ and $[Z, X, Y]$) if and only if we have in (1) $X \twoheadrightarrow \cdots \twoheadrightarrow Y \twoheadrightarrow \cdots \twoheadrightarrow Z$, or $Y \twoheadrightarrow \cdots \twoheadrightarrow Z \twoheadrightarrow \cdots \twoheadrightarrow X$, or $Z \twoheadrightarrow \cdots \twoheadrightarrow X \twoheadrightarrow \cdots \twoheadrightarrow Y$. We always have either $[X, Y, Z]$ or $[X, Z, Y]$, but never both.

The circular order $[*, *, *]$ will be referred to as the *canonical circular order on M*. The next results describe the geometric intuition beyond this definition: we shall see that $[X, Y, Z]$ means in fact that these segments appear in this order clockwise. Moreover, we shall see that the definition of $[*, *, *]$ does not depend on the choice of B.

Lemma 20. *Let M be a circular matching, and let $X, Y, Z \in M$. Then we have $[X, Y, Z]$ if and only if at least two among the following three conditions hold: $X \twoheadrightarrow Y$; $Y \twoheadrightarrow Z$; $Z \twoheadrightarrow X$.*

If all three conditions hold, then $\{X, Y, Z\}$ is a 3-star; and if exactly two among the statement hold, then $\{X, Y, Z\}$ is a linear matching. All possible situations for $[X, Y, Z]$ (with respect to $\multimap$) appear in Figure 17.

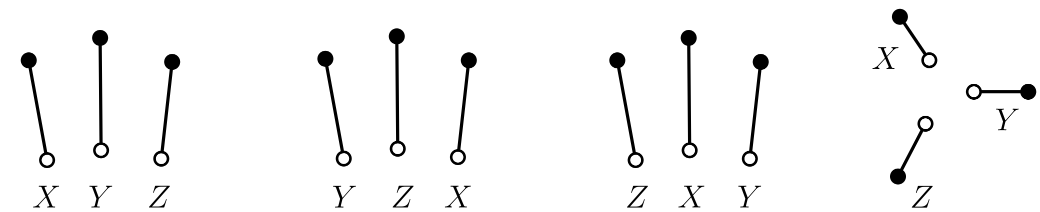


Figure 17: Possible configurations of three segments that satisfy $[X, Y, Z]$.

Proof. The segment B from the definition of $[*, *, *]$ is the maximum element of M_B^{L+}. Therefore, it is convenient to denote $A_{k+1} = B$. Now we have four cases.

- Case 1: $X, Y, Z \in M_B^{L+}$.

  In this case $\{X, Y, Z\}$ is of linear type (by Lemma 17). Therefore either one or two of the conditions hold. If exactly two conditions hold: assume without loss of generality that $X \multimap Y \multimap Z$. Since $A_1 \multimap \cdots \multimap A_{k+1}$ is a linear order in M_B^{L+}, we have $X = A_\alpha, Y = A_\beta, Z = A_\gamma$ for some $1 \leq \alpha < \beta < \gamma \leq k+1$. Now we have $[X, Y, Z]$ by definition. If exactly one condition holds: assume that it is $X \multimap Y$; then we have $X \multimap Z \multimap Y$, which implies "not $[X, Y, Z]$".

- Case 2: two edges of $\{X, Y, Z\}$ belong to M_B^{L+}, and one to M_B^R. Assume without loss of generality that $X, Y \in M_B^{L+}, Z \in M_B^R$ and that $X \multimap Y$.

  Then we have $X = A_\alpha, Y = A_\beta$ for some $\alpha < \beta$ and $Z = C_\gamma$ for some γ, and, therefore, $[X, Y, Z]$.

  At the same time in this case at least two of the conditions hold: indeed, assume $X \multimap Z \multimap Y$. Then B is distinct from X, Y, Z (in particular, $B \neq Y$ because $B \multimap Z$). Now, in the matching $\{X, Y, Z, B\}$ there is a minimum element, X, but there is no maximum element. Therefore, $\{X, Y, Z, B\}$ is neither of linear nor of circular type—a contradiction.

- Case 3: one edge of $\{X, Y, Z\}$ belongs to M_B^{L+}, and two to M_B^R, and Case 4: all edges of $\{X, Y, Z\}$ belong to M_B^R, are similar to cases 2 and 1. Therefore, we omit their proofs. □

Corollary 21. *The canonical circular order does not depend on the choice of B.*

Proof. By Lemma 20, we have an equivalent definition of the circular order that only depends on relations between triples of segments. □

Lemma 22. *Let M be a circular matching, and let $X \in M$. Then the immediate successor of X in the canonical circular order is the minimum element of M_X^R.*

Proof. This is immediate for B (as in definition of $[*,*,*]$), and, since we saw in Corollary 21 that the circular order $[*,*,*]$ does not depend on the choice of B, this is true for all segments. □

Lemmas 20 and 22 show that the canonical circular order describes the combinatorial structure of circular matchings in a natural way, similarly to the way in which $\twoheadrightarrow$ describes the structure of linear matchings. In Section 5.2 we will provide a finer classification of the relations $\twoheadrightarrow$ that are realizable in circular matchings.

4.2 Proofs of Theorems 2 and 3

At this point we are ready to complete the proofs of Theorems 2 and 3.

Proof of Theorem 2. Equivalence of conditions $2, 3, 4, 5$ is proven in Lemma 11. Finally, $2 \Rightarrow 1$ (if a BR-matching M is of linear type, then it is unique) is proven in Theorem 12; and $1 \Leftarrow 2$ (if a BR-matching M is unique, then it is of linear type) follows from Corollary 6 (if M is unique, then it has no chromatic cut), Lemma 7 (if M has no chromatic cut, then it is either of linear or circular type), and Theorem 16 (if M is of circular type, then it is not unique). □

Proof of Theorem 3. First we observe that the following statements are equivalent:

$1'$. M contains no chromatic cut.

$2'$. The sidedness relation $\twoheadrightarrow$ is a total relation.

$3'$. No two segments from M form one of the patterns in Figure 2 (a, b).

The equivalence $1' \Leftrightarrow 2'$ is Lemma 8, and the equivalence $1' \Leftrightarrow 3'$ is Lemma 4. Each of the three conditions $1', 2', 3'$ is equivalent to the corresponding condition $1, 2, 3$ in the theorem with the additional constraint that M is not of linear type, by Theorem 2 (conditions $2, 3, 4$, respectively). This establishes that the three first conditions of the theorem are equivalent.

Property p1 is explained in Lemmas 20 and 22, and p2 is proved by Theorem 16. □

5 Miscellaneous questions

A *parallel matching* is a BR-matching that consists of parallel segments. As we saw in Theorem 2, quasi-parallel matchings generalize parallel matchings in the sense that they are exactly the BR-matchings for which the relation $\twoheadrightarrow$ is a linear order. Similarly, circular matchings generalize *radial matchings*—BR-matchings whose edges lie on distinct rays with a common endpoint O and oriented away from O.

In this section we study how far quasi-parallel (respectively, circular) matchings generalize parallel (respectively, radial) matchings, in two aspects. In Section 5.1 we consider

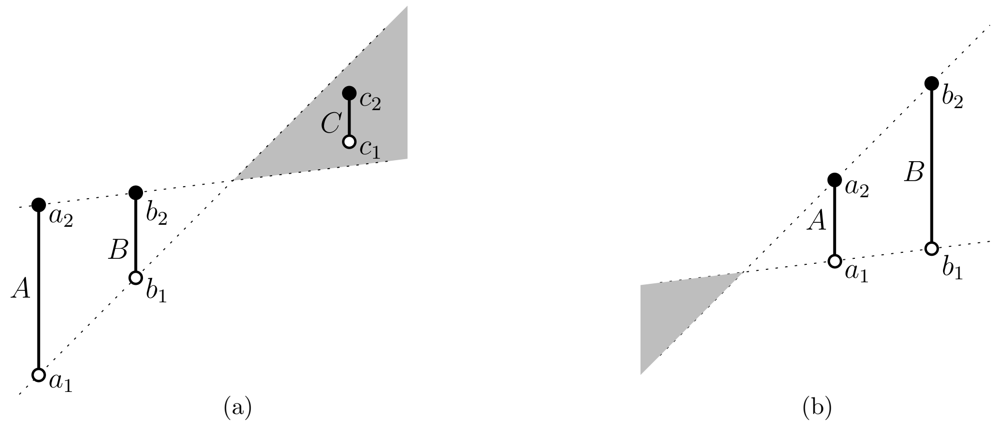


Figure 18: Illustration to Observation 23.

order types, and in Section 5.2 we study the class of sidedness relations $\multimap$ realizable in circular matchings.

Finally, in Section 5.3, we discuss the possibility of characterizing point sets F with unique matchings in terms of F alone, without reference to the matching.

5.1 Order types in parallel versus quasi-parallel matchings

Since, as mentioned above, quasi-parallel matchings generalize parallel matchings, it is natural to ask whether all order types (determined by orientations of triples of points) of bichromatic point sets with a unique BR-matching are realizable by corresponding endpoints of a parallel matching.

We construct an example that shows that the answer to this question is negative. The construction is based on the following observation.

Observation 23. *Let A, B, C be three parallel vertical segments such that $A \multimap B \multimap C$. Denote by a_1, b_1, c_1 the lower ends, and by a_2, b_2, c_2 the upper ends of the corresponding segments. If the triple $[a_1, b_1, c_2]$ is oriented clockwise, and the triple $[a_2, b_2, c_1]$ counter-clockwise, then B is shorter than A.*

Proof. See Figure 18 for illustration. The conditions mean that c_2 is situated below the line a_1b_1, and c_1 above the line a_2b_2. However, if B is not shorter than A, then the wedge that should contain C is situated to the left of A, see Figure 18(b). Thus, in this case $A \multimap C$ is impossible. □

Now, the construction goes as follows. Consider three pairs of parallel (auxiliary) lines with slopes, say, 0°, 60°, and 120°, and three vertical segments A_0, B_0, C_0, as shown in Figure 19(a). Change slightly the slopes of the auxiliary lines so that each pair will intersect as indicated schematically in the right part, and so that the new segments A, B, C whose endpoints are intersection points of the modified lines are almost vertical. Add vertical segments in the wedges formed by the auxiliary lines, as shown in Figure 19(b). This can be done so that the new matching (consisting of six segments) is quasi-parallel; denote it by M.

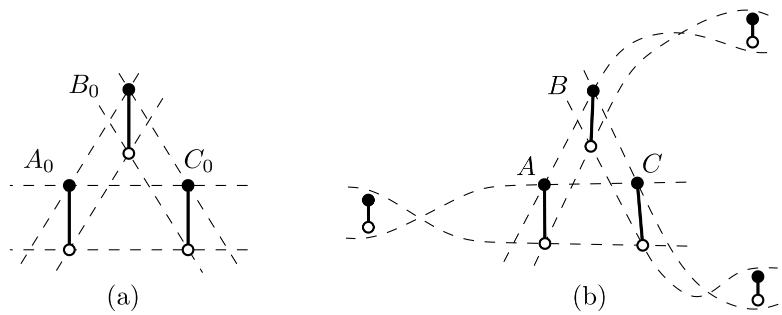


Figure 19: The construction of a “non-parallelizable” quasi-parallel matching.

Now, assume that there is a parallel matching M' with endpoints of the same order type, and denote by A', B', C' the segments that correspond in M' to A, B, C. Then, according to Observation 23, A' is longer than B', B' is longer than C', and C' is longer than A'. This is a contradiction.

One can extend this example to show that radial matchings do not capture all order types realized by circular matching. To see this, let M be a non-parallelizable linear BR-matching as explained above. One can augment it by two edges so that a circular matching M_1 is obtained, see Figure 20. Assume that the reference line ℓ is vertical. Let R be an axis-aligned rectangle such that all edges of M lie in the interior of R, and such that for each $A \in M$, the line $g(A)$ intersects the upper and the lower sides of R. Then one can add two suitably oriented almost horizontal segments B, C slightly below the line that contains the lower side of R (one on the right and another on the left) so that $M_1 = M \cup \{B, C\}$ is a circular matching. Now, if the order type of M_1 could be realized by a radial matching M_1' with center O, it would be possible to apply to M_1' a projective transformation which moves O to infinity. The segments of M would be then parallel—a contradiction.

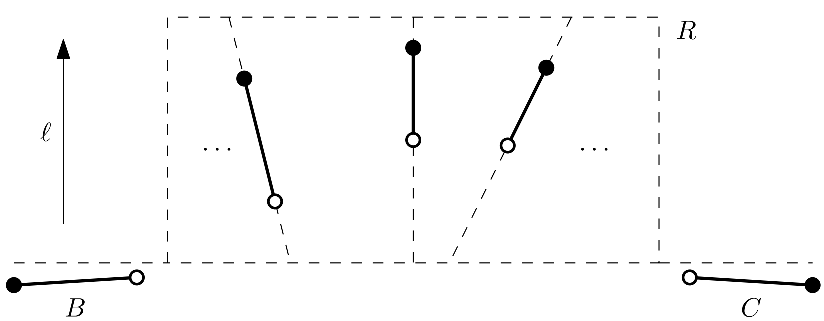


Figure 20: Augmenting a linear matching with two segments to obtain a circular matching.

5.2 Sidedness relations in circular matchings

If $M = \{A_1, A_2, \ldots, A_n\}$ is a linear matching, and we know that $A_1 \multimap A_2 \multimap \cdots \multimap A_n$, then the relation $\multimap$ is completely determined, since it is linear by Lemma 11. In contrast, for matchings of n segments of circular type, there are several relations $\multimap$ that satisfy $A_1 \multimap A_2 \multimap A_3 \multimap \cdots \multimap A_n \multimap A_1$. In Theorem 26 we enumerate such relations; its proof also provides us with a classification of circular matchings in the sense of the relations $\multimap$ realizable in such matchings. In this section, all index arithmetic will be done modulo n, that is, A_{n+1} denotes the same point as A_1, etc.

Definition. Let M be a circular matching, and let $A_1 \multimap A_2 \multimap A_3 \multimap \cdots \multimap A_n \multimap A_1$ be the canonical circular order of its elements.

1. Two consecutive segments, A_i and A_{i+1} are *twins* if there is no $A_\ell \in M \setminus \{A_i, A_{i+1}\}$ so that $g(A_\ell)$ separates A_i from A_{i+1}.
2. A *T-set* is a maximal set of consecutive segments, $T = \{A_i, A_{i+1}, A_{i+2}, \ldots, A_j\}$ such that every two consecutive elements of T are twins.

The matchings in Figure 21 have five T-sets: $\{10, 1, 2\}$, $\{3\}$, $\{4, 5, 6\}$, $\{7, 8\}$, and $\{9\}$. By Lemma 15, any circular matching M contains a 3-star $S = \{A, B, C\}$. It is easy to see that A, B and C belong to different T-sets. Therefore any circular matching has at least three T-sets; and any T-set itself is a linear matching. Further, we prove the following.

Lemma 24. *The number of T-sets in a circular matching is odd.*

Proof. If $|M| = 3$, that is, M is a 3-star, then it has three T-sets of size 1.

Suppose $|M| > 3$. Let S be a 3-star in M. We show that upon erasing any element A_ℓ of $M \setminus S$ the number of T-sets does not change its parity. We have two cases.

1. Case 1: A_ℓ has no twin, that is, $\{A_\ell\}$ is a T-set. There is a unique i so that $g(A_\ell)$ separates A_i and A_{i+1}, or, more precisely, $A_{i+1} \multimap A_\ell$ and $A_\ell \multimap A_i$. (A_i is the maximum element of $M^R_{A_\ell}$, and A_{i+1} is the minimum element of $M^L_{A_\ell}$.) Thus, before erasing A_ℓ, the segments A_i and A_{i+1} belong to different T-sets. We claim that upon erasing A_ℓ, these T-sets are united into one T-set.

   First we show that $g(A_{i+1})$ separates A_ℓ and $A_{\ell+1}$. Since A_ℓ and $A_{\ell+1}$ belong to different T-sets, there is *some* $A_k \in M$ so that $g(A_k)$ separates these two segments. If $A_k = A_{i+1}$, we are done. Otherwise, if $A_k \neq A_{i+1}$, we proceed as follows. First, $A_k \multimap A_\ell$: indeed, since $M^{R+}_{A_\ell}$ is a linear matching, none of its elements can separate A_ℓ and $A_{\ell+1}$. Moreover, $A_{i+1} \multimap A_k$ (since A_{i+1} is the minimum element of $M^L_{A_\ell}$), and also $A_{\ell+1} \multimap A_k$ (since $g(A_k)$ separates A_ℓ and $A_{\ell+1}$). Now, if $A_{i+1} \multimap A_{\ell+1}$, then the matching $\{A_\ell, A_{\ell+1}, A_{i+1}, A_k\}$ has a minimum element (A_{i+1}) but no maximum element—a contradiction. Therefore, $A_{\ell+1} \multimap A_{i+1}$, and it follows that $g(A_{i+1})$ separates A_ℓ and $A_{\ell+1}$.

   This means that the segment $A_{\ell+1}$ is the minimum element of the linear matching $M^R_{A_{i+1}}$. Thus, none of the segments $A_{\ell+1}, A_{\ell+2}, \ldots, A_{i-1}$ can separate A_i and A_{i+1}.

Similarly one shows that none of the segments $A_{i+2}, A_{\ell+3}, \ldots, A_{\ell-1}$ can separate A_i and A_{i+1}. Therefore, A_ℓ is the only segment that separates A_i and A_{i+1}. Thus, after erasing A_ℓ, the T-sets of A_i and of A_{i+1} are united into one T-set. The T-sets of $A_{\ell-1}$ and of $A_{\ell+1}$ are not united since $g(A_i)$ and $g(A_{i+1})$ separate these segments. No other T-sets are united since $g(A_\ell)$ separates only one pair of consecutive segments.

The T-set $\{A_\ell\}$ itself disappears, and, thus, the number of T-sets decreases by 2.

2. Case 2: A_ℓ has a twin. Without loss of generality, $A_{\ell+1}$ belongs to the same T-set. Since for each $A_j \in M \setminus \{A_\ell, A_{\ell+1}\}$ the line $g(A_j)$ does not separate A_ℓ and $A_{\ell+1}$, the lines $g(A_\ell)$ and $g(A_{\ell+1})$ separate exactly the same pair of consecutive elements of M. Thus, after erasing A_ℓ, no two T-sets are united, and the T-set that contains it does not disappear. Therefore, the number of T-sets does not change in this case.

We continue erasing the elements of $M \setminus S$ one by one until S remains. Since at this stage we have 3 T-sets, the number of T-sets of M is necessarily odd. □

Lemma 25. *Let M be a circular matching, and assume that its canonical circular order is induced by $A_1 \twoheadrightarrow A_2 \twoheadrightarrow A_3 \twoheadrightarrow \cdots \twoheadrightarrow A_n \twoheadrightarrow A_1$. The relation $\twoheadrightarrow$ is uniquely determined by partition of M into T-sets.*

Proof. We show that the relation $\twoheadrightarrow$ is determined between any edge $A_\ell \in M$ and all other edges. It is clear that every T-set that does not contain A_ℓ is contained either in $M^R_{A_\ell}$ or in $M^L_{A_\ell}$. We claim that the number of T-sets contained in $M^R_{A_\ell}$ is the same as the number of T-sets contained in $M^L_{A_\ell}$. From this, we can conclude that the T-sets determine uniquely the relation $\twoheadrightarrow$ between A_ℓ and the other elements of M, and the lemma is proved.

The claim, which is a strengthening of the previous lemma, is proved by induction on n. The base case ($n = 3$) is trivial. For the induction step, we follow the proof of Lemma 24. In Case 1, we erase A_ℓ, and we apply the inductive hypothesis to A_i in the resulting matching. The matchings $M^R_{A_i}$ and $M^L_{A_i}$ after erasing A_ℓ coincide with $M^L_{A_\ell}$ and $M^R_{A_\ell}$ before erasing A_ℓ, except for the element A_i. Let us look at the T-sets contained in these intervals. Before erasing A_ℓ, the T-set of A_{i+1} belongs to $M^L_{A_\ell}$ and the T-set of A_i belongs to $M^R_{A_\ell}$. After erasing A_ℓ, these two T-sets are joined, and thus they are not counted in the number of T-sets contained in $M^R_{A_i}$ or $M^L_{A_i}$. The number of T-sets goes down by one in each interval. All other T-sets remain unchanged, and they remain part of their respective interval.

Case 2 is easy: we apply the inductive hypothesis to $A_{\ell+1}$. The number of T-sets in M^R and M^L is the same, no matter whether we look from A_ℓ or $A_{\ell+1}$, and it is unchanged if we erase A_ℓ. □

Theorem 26. *The number of sidedness relations $\twoheadrightarrow$ realizable by circular matchings $\{A_1, A_2, A_3, \ldots, A_n\}$ that satisfy $A_1 \twoheadrightarrow A_2 \twoheadrightarrow A_3 \twoheadrightarrow \cdots \twoheadrightarrow A_n \twoheadrightarrow A_1$ is $2^{n-1} - n$.*

Proof. If a circular matching M of size n has k T-sets, then one can specify its partition into T-sets by marking, say, the maximum element of each T-set with respect to $\twoheadrightarrow$. There are $\binom{n}{k}$ ways to do this, and thus, the number of orderings is

$$\sum_{\substack{3\le k\le n\\ k \text{ odd}}} \binom{n}{k} = \sum_{\substack{0\le k\le n\\ k \text{ odd}}} \binom{n}{k} - n = 2^{n-1} - n. \qquad \square$$

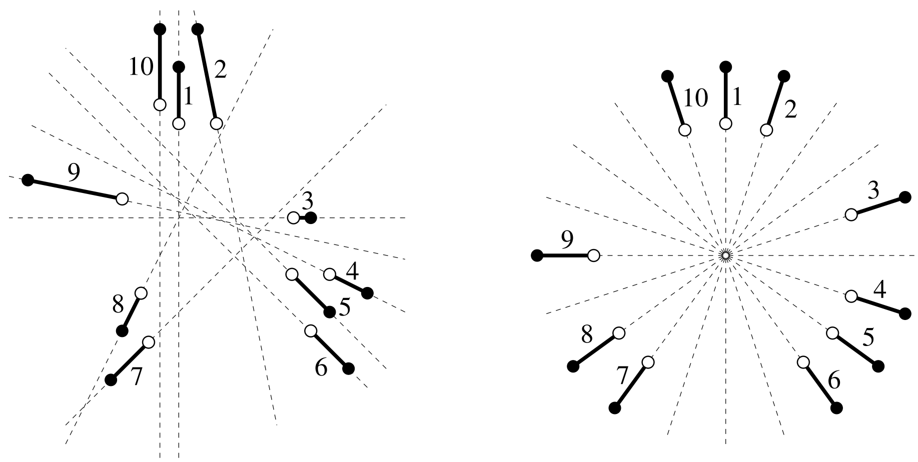


Figure 21: A circular matching and its standardization.

It is easy to see that any $\multimap$ relation that is realizable with circular matchings can be realized by a radial matching; furthermore, it is possible to "standardize" the geometry so that the endpoints of A_k lie on the line through the origin with polar angle $\tau(k)\frac{\pi}{n}$, where τ is a suitable permutation of $\{1, 2, \ldots, n\}$, at two fixed distances from the origin, see Figure 21. Now the formula $2^{n-1} - n$ becomes clear: consider n lines passing through a common point O. For each line, there are two possibilities to choose on which ray we put a segment (except the fixed segment A_1). Thus we have 2^{n-1} matchings: n of them are of linear type, and the others are of circular type.

5.3 Characterization in terms of point sets

We described point sets with unique matchings in terms of a given matching M rather than in terms of the set F itself. It would be nice to characterize the points sets F directly, for example by forbidden patterns *of points*. However, such a characterization is impossible.

Suppose that there exists a collection of patterns of points (of two colors) such that F has unique matching if and only if F avoids these patterns. Equivalently, F has several matchings if and only if F contains any of these patterns. However, in such a case we can duplicate the elements of F: for each $p_i \in F$ we add a point p'_i so that p_i and p'_i are of opposite colors, all segments $p_i p'_i$ are parallel (including orientation), and the new set is in general position (see Figure 22). Then the matching that consists of the segments $p_i p'_i$ is a (quasi-)parallel linear matching, and thus is a unique matching of the new set, while it contains the assumed patterns.

We can actually move the additional points as far away as we like. Thus, even a more "local" characterization, that a certain convex region should contain some pattern and no other points, is impossible.

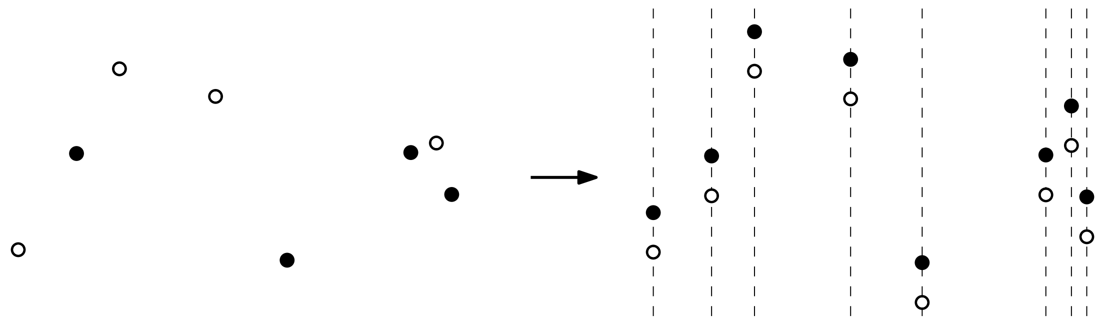


Figure 22: Generating a unique matching by duplicating the point set.

On the other hand, suppose that there is a collection of point patterns such that F has several matchings if and only if F avoids these patterns. Equivalently, F has a unique matching if and only if F contains any of these patterns. However, in such a case we can take this matching and add one more segment to obtain a BR-matching with a chromatic cut. Thus, the new point set will have more than one matching while it contains the assumed patterns. As above, the additional segment can be placed arbitrarily far away.

6 Algorithms

In this section, we describe several algorithms. The first checks whether a given point set F has a unique BR-matching. This algorithm is based on yet another characterization of unique BR-matchings. The second checks if a given BR-matching is circular. Applying these algorithms together, we can check if a given matching has a chromatic cut. The third algorithm finds a balanced line through one of the segments involved in a chromatic cut (Lemma 5).

6.1 Testing a matching for uniqueness

Definition. A BR-matching M has the *drum property* with respect to the segments $A, B \in M$ $(A \neq B)$ if A and B are the only segments from M on $\partial\mathrm{CH}(F)$.

Theorem 27. *Let $M = \{A_1, A_2, \dots, A_n\}$ be a BR-matching such that $A_1 \twoheadrightarrow A_2 \twoheadrightarrow \cdots \twoheadrightarrow A_n$. Then the following conditions are equivalent:*

1. *M is the unique BR-matching.*
2. *For every $i < j$, every subset $S \subseteq \{A_i, A_{i+1}, \dots, A_j\}$ with $A_i, A_j \in S$ has the drum property for A_i and A_j.*
3. *For every $j > 1$, the set $\{A_1, A_2, \dots, A_j\}$ has the drum property for A_1 and A_j; and for every $i < n$, the set $\{A_i, A_{i+1}, \dots, A_n\}$ has the drum property for A_i and A_n.*

Recall that the relation $\multimap$ is not necessarily transitive. Thus, the assumption of the theorem does not imply $A_i \multimap A_j$ for $i < j$.

Proof. [1 ⇒ 2] By Condition 3 of Theorem 2, $\multimap$ is a linear order on M, and by Condition 2, the convex hull has only two color intervals. These properties carry over from M to the subset S, as mentioned in the remark after the theorem. In particular, S has a unique matching, the induced order $\multimap$ on S is linear, and the convex hull of S has two color intervals. The minimal and maximal elements of S are A_i and A_j. In particular, $A_i \multimap B$ for every $B \in S \setminus \{A_i\}$, and thus, A_i lies on the convex hull. Similarly, A_j lies on the convex hull. Since there are only two color intervals on the convex hull of S, there can be no other matching edges on the convex hull. Thus, S has the drum property for A_i and A_j.

[2 ⇒ 3] is clear since 3 is a special case of 2.

[3 ⇒ 1]: Since $\{A_1, A_2, \dots, A_j\}$ has the drum property for A_1 and A_j, all segments $A_1, \dots, A_{j-1}$ lie on the same side of $g(A_j)$. Since $A_{j-1} \multimap A_j$ by assumption, we know that the segment A_{j-1} lies left of A_j, and hence we conclude that all segments A_i lie left of $g(A_j)$, for $i < j$. Similarly, from the drum property for $\{A_i, A_{i+1}, \dots, A_n\}$ we conclude that the segments A_j lie right of $g(A_i)$, for $j > i$. These two conditions together mean that $A_i \multimap A_j$ for $i < j$. Therefore, Condition 3 of Theorem 2 holds, and M is unique. □

From Property 3 of Theorem 27 we can derive a linear-time algorithm for testing whether M is unique, once an ordering with $A_1 \multimap A_2 \multimap \cdots \multimap A_n$ has been computed: We incrementally compute $P_j := \mathrm{CH}(\{A_1, A_2, \dots, A_j\})$ for $j = 2, \dots, n$ and check the drum property as we go.

The algorithm that we shall describe proceeds similarly as the folklore linear-time algorithm for computing the convex hull of points that are given in sorted order by x-coordinate, but it is valid for a different reason. We start from the general paradigm for computing convex hulls incrementally (see, for example, [14]), which is the basis for more elaborate randomized incremental algorithms that also work in higher dimensions, see [6, Chapter 11]. The current convex hull H is extended by a new point p as follows:

C1. Check whether $p \in H$. If this is the case, stop.

C2. If not, find a boundary point $q \in \partial H$ that is visible from p.

C3. Walk from q in both directions to find the tangents pq_1 and pq_2 from p to H.

C4. Update the convex hull: remove the part between q_1 and q_2 that has been walked over, and replace it with $q_1 p q_2$.

If H is maintained as a linked list, Steps C3 and C4 take only linear time overall, because everything that is walked over is deleted. The "expensive" steps that are responsible for the superlinear running time of convex hull algorithms are C1 and C2. However, just as for the linear-time hull computation of sorted points, we will see that these steps are trivial in our case. (We extend the convex hull by inserting not a single point but two points of A_{j+1} at a time.)

We want to check the drum property for $\{A_1, A_2, \ldots, A_{j+1}\}$. If it holds, then we *know* that the new points of A_{j+1} do not lie in P_j; and since A_j lies on the boundary of P_j but not of P_{j+1}, we know that A_j is visible from at least one point of A_{j+1}. We can start the search in Step C3 from there. This visibility assumption can be checked in constant time, and it guarantees that A_j disappears from the boundary of P_{j+1}. The other part of the drum property, that A_1 and A_{j+1} lie on the boundary of P_{j+1}, is trivial to check after Step C4 is completed. The overall running time is linear.

In a second symmetric step, we start from the end and compute $\mathrm{CH}(\{A_i, A_{i+1}, \ldots, A_n\})$ for $i = n-1, \ldots, 1$.

Theorem 28. *It can be checked in $O(n \log n)$ time whether a bichromatic set has a unique non-crossing BR-matching.*

Proof. First we have to compute some BR-matching $M = \{A_1, A_2, \ldots, A_n\}$. It is well-known that this can be done by recursive ham-sandwich cuts in $O(n \log n)$ time. A ham-sandwich cut is a line ℓ that partitions a bichromatic set such that each open half-plane contains at most $\lfloor \frac{n}{2} \rfloor$ points of each color. If n is odd, ℓ must go through a red and a blue point. We can match these points to each other, and recursively find a BR-matching in the $\left(\frac{n-1}{2} + \frac{n-1}{2}\right)$-sets in each half-plane. If n is even, ℓ may go through one or two points, but by shifting ℓ slightly we can push these points to the correct side such that each half-plane contains an $\left(\frac{n}{2} + \frac{n}{2}\right)$-set. We recurse as above. A ham-sandwich cut can be found in linear time [16]. Hence this procedure leads to a running time of $T(n) = O(n) + 2 \cdot T(n/2)$, which gives $T(n) = O(n \log n)$.

Next, we compute an ordering such that

$$A_1 \multimap A_2 \multimap \cdots \multimap A_n. \tag{2}$$

We do this by a standard sorting algorithm in $O(n \log n)$ time, as if the relation $\multimap$ were a linear order. If, at any time during the sort, we find two segments that are not comparable by $\multimap$, we quit. Finally, we check condition (2) in $O(n)$ time. (This final check is not necessary, if, for example, merge sort is used as the sorting algorithm.) This step is guaranteed to find an ordering (2) if the matching is unique. If $\multimap$ is not a linear order, it may succeed or fail.

As the last step, we check Property 3 of Theorem 27 in linear time, as outlined above. □

6.2 Testing for a circular matching

It is also possible to determine in $O(n \log n)$ if a BR-matching M is circular, by an easy divide-and-conquer algorithm. Let A and B be two arbitrary segments in M. Let $M_1 = M_A^{L+}$ and $M_2 = M_A^{R+}$ (that is, the segments that lie to the left or to the right of A, including A itself) and likewise $N_1 = M_B^{L+}$ and $N_2 = M_B^{R+}$ (see Figure 23; recall that the segments are implicitly directed from white to black). M_i and N_i are linear matchings by Lemma 17. Finally, define $Q_1 := (M_2 \cap N_2) \cup (M_1 \cap N_1)$ and $Q_2 := (M_1 \cap N_2) \cup (M_2 \cap N_1)$.

Observation 29. *A BR-matching M has a chromatic cut if and only if at least one of the six matchings defined above has a chromatic cut.*

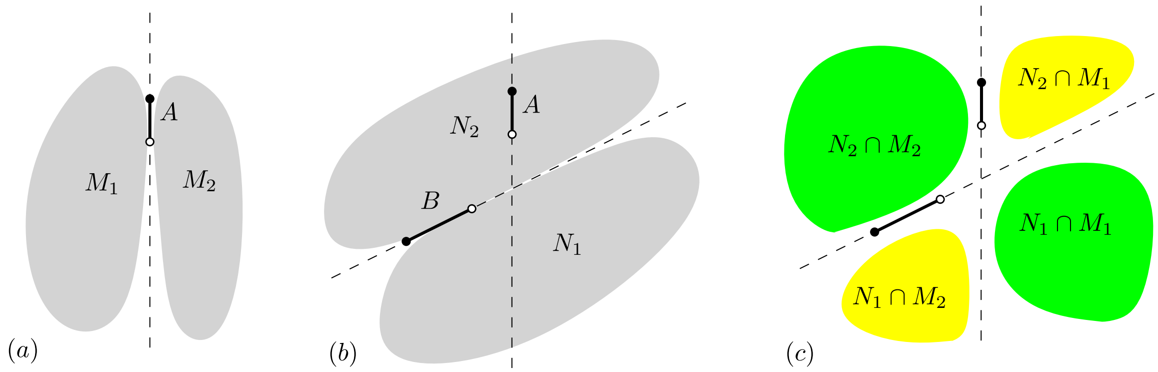


Figure 23: Separation of M into 6 overlapping BR-Matchings

Proof. Consider two segments in M. Then they must be both in one of the matchings M' as defined above. If they have a chromatic cut, then M' has a chromatic cut. The other direction is obvious. □

Theorem 30. *It can be checked in $O(n \log n)$ time whether a BR-matching M is of circular type.*

Proof. The algorithm starts to compute the convex hull of M. If all points on $\partial CH(M)$ are of the same color we know it is not a linear matching and it remains to check if M has no forbidden pattern as in Figure 2(a, b).

We pick any segment A_0 and split M along $g(A_0)$. We compute the linear order of both parts. This gives a potential circular order. We remember this order for the remaining part.

The rest of the algorithm works recursively. We start by defining M_1 and M_2 as above for any segment A. Let B be the median of the larger of the M_i with respect to $\rightarrow$. The BR-matchings N_i and Q_i are also defined as above. For the BR-matchings M_i and N_i, it can be checked in linear time if they are of linear type, because we have already precomputed the order. As B is the median of the larger of the M_i, $n/4 \leq |Q_i| \leq 3n/4$ for every i. We check recursively if Q_1 and Q_2 has no chromatic cut. For the running time $T(n)$, we have $T(n) \leq O(n) + \max_{1/4 \leq \alpha \leq 3/4}[T(\alpha n) + T((1-\alpha)n)]$. Thus $T(n) = O(n \log n)$.

If any of these steps in the algorithm fails, a forbidden configuration is present. In this case we just stop and return that M has a chromatic cut. Otherwise we return the correct circular order. □

6.3 Finding a balanced line

The last algorithm we want to present computes a balanced line as in Lemma 5. As a preprocessing step we need to find a point on a segment in general position with respect to the remaining points F.

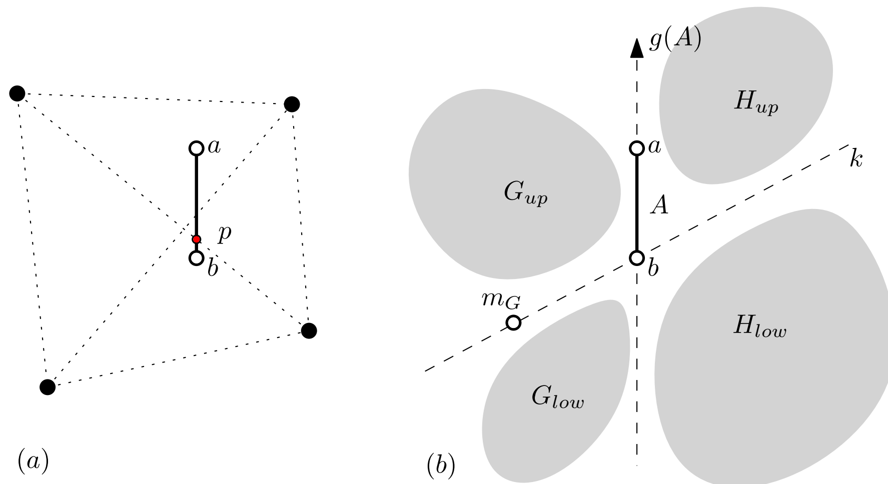


Figure 24: The line $g(A)$ splits the point set F into G and H.

Lemma 31. *Let F be a point set in the plane and $A = (a, b)$ be a vertical segment such that $F \cup \{a, b\}$ lies in general position, that is, no three points lie on a line. Then the lowest intersection p of A with a segment formed by two points in F can be computed in deterministic $O(n \log n)$ time.*

Proof. Consider the point sets G and H left and right of $g(A)$. Let m_G be the median of the larger set G, with respect to the order defined by a ray rotating around b. The line k through m_G and b defines the four sets G_{up}, G_{low}, H_{up} and H_{low}, as in Figure 24(b). Now any two points defining the lowest intersection with C are either in G_{up} and H_{up}, or in two opposite sets (that is, G_{up} and H_{low} or G_{low} and H_{up}). The lowest intersecting segment of G_{up} and H_{up} is the convex hull edge of G_{up} and H_{up} intersecting the line $g(A)$. It can be found in linear deterministic time with a subroutine of the convex hull algorithm by Kirkpatrick and Seidel [15] or by an algorithm by Aichholzer, Miltzow and Pilz [3]. The second algorithm only uses order type information. The two opposite sets are treated recursively. Note that $n/4 \leq \#(G_{\mathrm{low}} \cup H_{\mathrm{up}}) \leq 3n/4$ and likewise $n/4 \leq \#(G_{\mathrm{up}} \cup H_{\mathrm{low}}) \leq 3n/4$. Therefore, for the running time we get $T(n) = O(n) + \max_{1/4 \leq \alpha \leq 3/4}[T(\alpha n) + T((1-\alpha)n)]$, which gives $T(n) = O(n \log n)$. □

For the next Lemma we refer to Figure 25.

Lemma 32. *Let A be a segment and F a point set right of $g(A)$ in general position. Then the lowest intersection p of A with a line through two points in F can be computed in deterministic $O(n \log n)$ time.*

Proof. First consider the points $q, r \in F$ which form the lowest crossing with A. We show they are neighbors in the radial order around b. Consider the area swept by a ray from q to r. If it contained any point s then either the line through q and s or r and s would have a lower intersection with A.

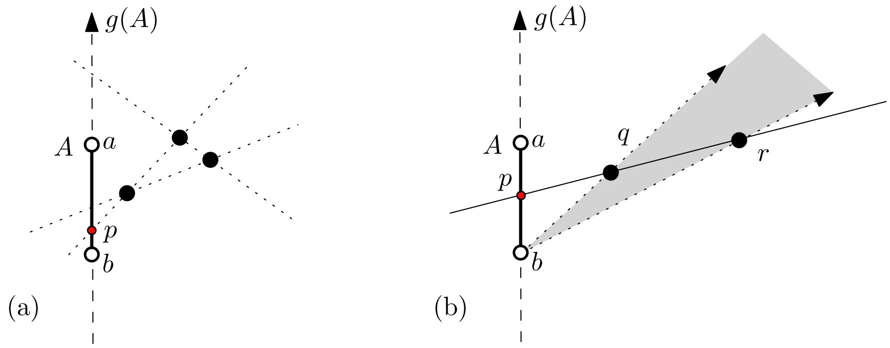


Figure 25: (a) the line arrangement formed by the points in F and the lowest crossing with A; (b) the cone with apex b spanned by the minimal pair of points is empty of points of F

Thus we merely compute the radial order around b and for any neighboring pair the intersection point with A. The running time $T(n) = O(n \log n)$ is dominated by the sorting procedure. □

Theorem 33. *Let F be a point set in the plane and $A = (a, b)$ be a vertical segment such that $F \cup \{a, b\}$ lies in general position (that is, no three points lie on a line). Then the lowest intersection of A with a line through two points in F can be computed in deterministic $O(n \log n)$ time.*

Proof. Compute the lowest intersection point with a line separately for the points left and right of A according to Lemma 32 and all possible intersections with A by pairs of points on opposite sites of A according to Lemma 31. □

Corollary 34. *Given a point set F and a segment A without three points on a line, a point on A in general position with respect to F can be computed in $O(n \log n)$ time.*

Proof. Any point between the lowest intersection and the lower endpoint of A is in general position with respect to F. □

Lemma 35. *Let M be a BR-matching of a point set F in general position and A, B be two segments as in Figure 2(a, b). Then we can compute a balanced line through the interior of A or B in $O(n \log n)$ time.*

Proof. Let $p \in A$ and $q \in B$ be points as in Corollary 34. We know by the proof of Lemma 5 that a balanced line through p or q exists. The algorithm in [3] can be adapted to find the desired balanced line through p or q in $O(n)$ time. □

Remark. Once we have an $O(n \log n)$ time algorithm to test whether a BR-matching is linear or circular we automatically get an algorithm to test if a BR-matching has a chromatic cut in $O(n \log n)$ time. Note that both algorithms above can be executed until they find a forbidden configuration. Thus we are able to compute a forbidden configuration also in

$O(n \log n)$ time. In the case of linear matchings we compute the linear order and for circular matchings the circular order.

The remaining notions that were introduced in this paper can also be computed efficiently: It is easy to construct a reference line in linear time, as in the Definition 3.1 of quasi-parallel segments. Given a forbidden configuration, it is possible to compute in constant time a chromatic cut (that is, the actual line). Finally, given a forbidden configuration, we can compute a balanced line intersecting one of the segments.

7 Open questions

Our method for testing whether a point set F has a unique non-crossing BR-matching starts by finding such a BR-matching M, in $O(n \log n)$ time, by repeated ham-sandwich cuts. This algorithm does not care whether M is unique, and it is in fact the fastest known algorithm for finding *any* non-crossing BR-matching in an arbitrary point set. Is there a faster algorithm for checking whether M is unique (without necessarily constructing M)?

Our paper can also be seen as the study of sets of segments with certain forbidden patterns. These particular segment sets have a lot of nice geometric structure. We wonder whether other forbidden patterns also lead to interesting geometric properties.

Consider n blue, n red and n green points in $\mathbb{R}^3$. By repeatedly applying ham-sandwich cuts we know that there exists a non-crossing colorful 3-uniform geometric matching: Each hyperedge is represented by the convex hull of its vertices. Thus we ask for a geometric characterization of point sets with just one such matching.

Acknowledgments.

We thank Michael Payne, Lothar Narins and Veit Wiechert for helpful discussions, and we thank Stefan Felsner for a hint to the literature. We also thank the anonymous referees for valuable comments. Research of Andrei Asinowski was supported by the ESF EUROCORES programme EuroGIGA, CRP ComPoSe, Deutsche Forschungsgemeinschaft (DFG), under grant FE 340/9-1. Tillmann Miltzow is supported by ERC Starting Grant PARAMTIGHT (No. 280152).

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