Ordered Level Planarity and Geodesic Planarity

Boris Klemz* Günter Rote*

Abstract

We introduce and study the problem Ordered Level Planarity which asks for a planar drawing of a graph such that vertices are placed at prescribed positions in the plane and such that every edge is realized as a y-monotone curve. This can be interpreted as a variant of Level Planarity in which the vertices on each level appear in a prescribed total order. We show $\mathcal{NP}$-completeness even for the special case that the number of vertices on each level is bounded by $\lambda = 2$ and that the maximum degree is $\Delta = 2$. This establishes a tight border of tractability since for $\lambda = 1$ the problem is in $\mathcal{P}$. Our result is motivated by the following applications.

We establish a connection to geodesic drawings. Geodesic Planarity asks for a planar drawing of a graph such that vertices are placed at prescribed positions in the plane and such that every edge e is realized as a polygonal path p composed of line segments with two adjacent directions from a given set S of directions symmetric with respect to the origin. Our results on Ordered Level Planarity imply $\mathcal{NP}$-hardness for any S with $|S| \geq 4$ even if the given graph is a matching. Katz, Krug, Rutter and Wolff claimed that for matchings Manhattan Geodesic Planarity is in $\mathcal{P}$ [GD'09]. Our results imply that this is incorrect unless $\mathcal{P} = \mathcal{NP}$. Further, our results imply the $\mathcal{NP}$-hardness of the Bi-Monotonicity problem.

We narrow the gap between tractability and $\mathcal{NP}$-hardness in the established hierarchy of Level Planarity variants. To this end, we provide reductions to T-Level Planarity, Clustered Level Planarity and Constrained Planarity. As a by-product, we strengthen previous $\mathcal{NP}$-hardness results. In particular, our reduction to Clustered Level Planarity generates instances with $\Delta = 2$, $\lambda = 2$ and only two non-trivial clusters.

1 Introduction

An *upward* planar drawing of a directed graph is a plane drawing where every edge $e = (u, v)$ is realized as a y-monotone curve that goes upward from u to v. Upward planar drawings provide a very natural way of visualizing a partial order on a set of items. The problem Upward Planarity of testing whether a directed graph has an upward planar drawing is $\mathcal{NP}$-complete [5]. However, if the y-coordinate of each vertex is prescribed, the problem can be solved in polynomial time [6]. Formally, this is captured by the notion of level graphs. A *level graph* $\mathcal{G} = (G, \gamma)$ is a directed graph $G = (V, E)$ together with a *level assignment* $\gamma : V \to \{0, \ldots, h\}$ for G where γ is a surjective map with $\gamma(u) < \gamma(v)$ for every edge $(u, v) \in E$. Value h is the *height* of $\mathcal{G}$. The vertex set $V_i = \{v \mid \gamma(v) = i\}$ is called the i-th *level* of $\mathcal{G}$. Value $\lambda_i = |V_i|$ is the *width* of level i and the *level-width* λ of $\mathcal{G}$ is the maximum width of any level in $\mathcal{G}$. A *level* planar drawing of $\mathcal{G}$ is an upward planar drawing of G where the y-coordinate of each vertex v is $\gamma(v)$. The problem Level Planarity of testing whether a given level graph has a level planar drawing is solvable in linear time [6].

We introduce a natural variant of Level Planarity that takes into account a total order for the vertices on each level. An *ordered* level graph $\mathcal{G}$ is a triple $(G = (V, E), \gamma, \chi)$ where (G, γ) is a level graph and $\chi : V \to \{0, \ldots, \lambda - 1\}$ is a *level ordering* for G. We require that χ restricted to domain V_i bijectively maps to $\{0, \ldots, \lambda_i - 1\}$. An *ordered* level planar drawing of an ordered level graph $\mathcal{G}$ is a level planar drawing of (G, γ) where for every $v \in V$ the x-coordinate of v is $\chi(v)$. Thus, the position of every vertex is fixed. The problem Ordered Level Planarity asks whether a given ordered level graph has an ordered level planar drawing. We remark that in the above definitions, the x- and y-coordinates assigned via χ and γ merely act as a convenient way to encode total and partial orders respectively. In terms of realizability, the problems are equivalent to generalized versions were χ and γ map to the reals. In other words, the fixed vertex positions can be any points in the plane. All reductions and algorithms in this paper carry over to these generalized versions, if we pay the cost for presorting the vertices according to their coordinates.

We establish a connection between ordered level planar drawings and geodesic drawings. Let $S \subset \mathbb{R}^2$ be a finite set of directions symmetric with respect to the origin, i.e. for each direction $s \in S$, the reverse direction $-s$ is also contained in S. A plane drawing of a graph is *geodesic* with respect to S if every edge is realized as a polygonal path p composed of line segments with two adjacent directions from S.

*Institute of Computer Science, Freie Universität Berlin, Germany

Such path p is a geodesic with respect to the polygonal norm that corresponds to S. An instance of the decision problem Geodesic Planarity is a 4-tuple $\mathcal{G} = (G = (V, E), x, y, S)$ where G is a graph, x and y map from V to the reals and S is a set of directions as stated above. The task is to decide whether $\mathcal{G}$ has a *geodesic drawing*, that is, G has a geodesic drawing with respect to S in which every vertex $v \in V$ is placed at $(x(v), y(v))$.

Katz, Krug, Rutter and Wolff [7] study Manhattan Geodesic Planarity, which is the special case of Geodesic Planarity where set S consists of the two horizontal and the two vertical directions. Geodesic drawings with respect to this set of directions are also referred to as orthogeodesic drawings. Katz et al. [7] show that a variant of Manhattan Geodesic Planarity in which the drawings are restricted to the integer grid is $\mathcal{NP}$-hard even if G is a perfect matching. The proof is by reduction from 3-Partition and makes use of the fact the number of edges that can pass between two vertices on a grid line is bounded. In contrast, they claim that the standard version of Manhattan Geodesic Planarity is polynomial-time solvable for perfect matchings. To this end, they sketch a plane sweep algorithm that maintains a linear order among the edges that cross the sweep line. When a new edge is encountered it is inserted as low as possible subject to the constraints implied by the prescribed vertex positions. When asked for more details, the authors informed us that they are no longer convinced of the correctness of their approach. Unless $\mathcal{P} = \mathcal{NP}$, one of the results of our paper implies that their approach is indeed incorrect.

The results and layout of this abstract are as follows. In Section 3 we study the complexity of Ordered Level Planarity. While Upward Planarity is $\mathcal{NP}$-complete [5] in general but becomes polynomial-time solvable [6] for prescribed y-coordinates, we show that prescribing both x and y-coordinates renders the problem $\mathcal{NP}$-complete. Precisely, our results are summarized in the following theorem.

Theorem 1 Ordered Level Planarity *is $\mathcal{NP}$-complete, even for maximum degree $\Delta = 2$ and level-width $\lambda = 2$. For level-width $\lambda = 1$,* Ordered Level Planarity *can be solved in linear time.*

Theorem 1 states an explicit gap between tractability and $\mathcal{NP}$-hardness. We motivate this result with the multiple applications. In Section 2 we study the complexity of Geodesic Planarity. We utilize Theorem 1 to obtain the following:

Theorem 2 Geodesic Planarity *is $\mathcal{NP}$-hard for any set of directions S with $|S| \geq 4$ even for perfect matchings in general position.*

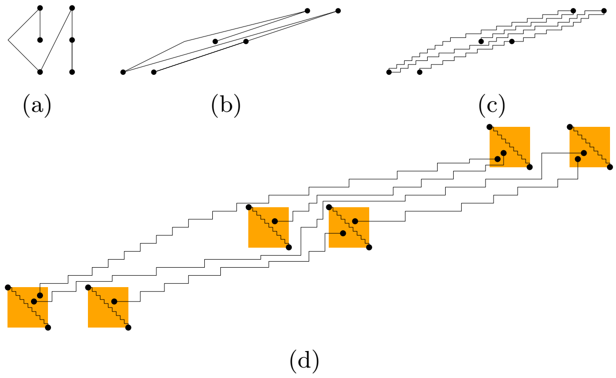


Figure 1: Reduction to Geodesic Planarity.

In the full version we provide reductions which establish Ordered Level Planarity as a special case of T-Level Planarity [1], Clustered Level Planarity [1] and Constrained Planarity [3]. As a by-product, we strengthen previous $\mathcal{NP}$-hardness results. In particular, we show that Clustered Level Planarity is $\mathcal{NP}$-hard even for instances with $\lambda = 2$, $\Delta = 2$ and only two non-trivial clusters. We observe that Geodesic Planarity restricted to instances as stated in Theorem 2 reduces immediately to Bi-Monotonicty [4] if S contains precisely the horizontal and vertical directions. Thus, we settle the latter problem's complexity.

2 Geodesic Planarity

In this section we sketch the proof of Theorem 2. To this end, we transform an Ordered Level Planarity instance $\mathcal{G}_o = (G_o = (V, E), \gamma, \chi)$ with maximum degree $\Delta = 2$ and level-width $\lambda = 2$ into a Geodesic Planarity instance $\mathcal{G}_g = (G_g, x, y, S)$ where G_g is a perfect matching. In this abstract we describe the reduction specifically for the case that the set S consists of precisely the horizontal and vertical directions. However, the construction is invariant under shearing and, thus, works for any prescribed set S of directions with $|S| \geq 4$. The reduction is carried out in two steps.

First, we transform $\mathcal{G}_o$ into a Geodesic Planarity instance $\mathcal{G}'_g = (G_o, x', \gamma, S)$ by translating the vertices of level V_i by $3i$ units to the right, see Fig.1a and Fig.1b. Clearly, every geodesic drawing of $\mathcal{G}'_g$ can be turned into an ordered level planar drawing of $\mathcal{G}_o$. On the other hand, consider an ordered level planar drawing of $\mathcal{G}_o$. W.l.o.g. all edges are realized as polygonal paths in which bend points occur only on the horizontal lines L_i through the levels V_i of $\mathcal{G}_o$, see Fig.1a. Further, assume that all bend points have an x-coordinate in the interval $[-1, 2]$. We translate all bend points on L_i by $3i$ units to the right, see Fig.1b. In the resulting drawing all edge-segments have a slope from interval $(0, \infty)$. Thus, since the

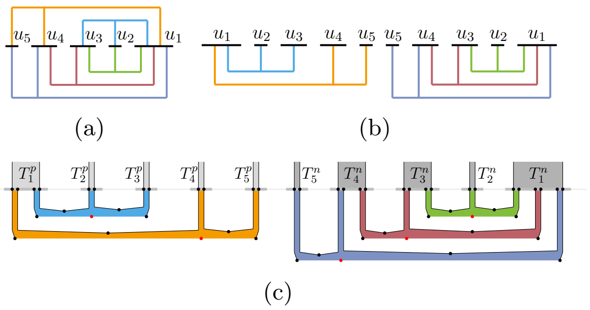


Figure 2: Representation of a planar monotone 3SAT formula and its usage in the reduction to Ordered Level Planarity.

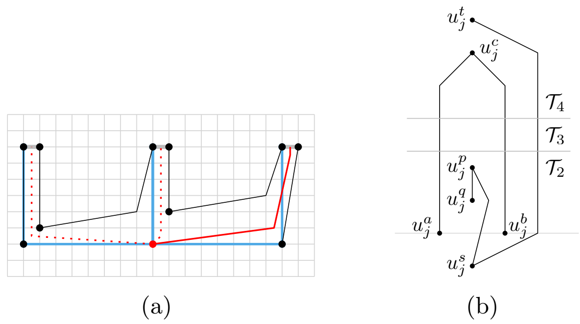


Figure 3: **(a)** The subgraph (black/red) created for every E-shape (blue) and the three respective gates (grey). **(b)** The variable gadget for u_j.

maximum degree is $\Delta = 2$ we can be replace all edge-segments by geodesic staircases that closely trace the segments, see Fig.1c.

In the second step we turn G_o into a perfect matching in order to obtain $\mathcal{G}_g$. To this end, we essentially split each vertex v by replacing it with a small gadget that fits inside a $1/4 \times 1/4$ square centered on v, see Fig.1d. The gadget contains a degree-1 vertex for every edge incident to v. In order to maintain equivalence we have to prevent non-incident edges from being drawn through the gadget square. To this end, we create a *blocking* edge between vertices in the top left and bottom right corners of the gadget square.

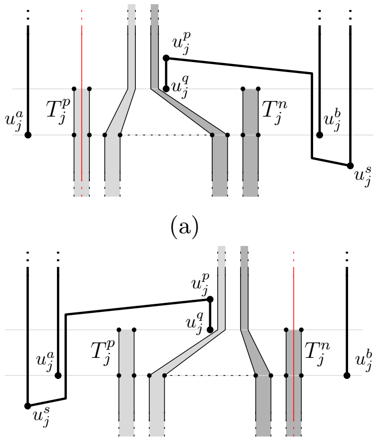
Note that all x-coordinates are distinct. Only the up to 8 vertices of the gadgets on each level may have duplicate y-coordinates. Thus, by placing these vertices more carefully we can guarantee that the assigned vertex positions are in *general position*, that is, no two vertices lie on a line with a direction from S.

3 Ordered Level Planarity

To obtain $\mathcal{NP}$-hardness of Ordered Level Planarity we perform a reduction from Planar Monotone 3-Satisfiability. In this $\mathcal{NP}$-hard [2] special case of 3SAT the input is a 3SAT formula φ together with a contact representation $\mathcal{R}$ of φ, see Fig. 2a. All variables are represented as line segments arranged on a line. Each clause c is represented as an E-shape that touches precisely the variables contained in c. Furthermore, all clauses are either *positive* or *negative*, i.e. they contain exclusively positive or negative literals, respectively. In $\mathcal{R}$ all negative clauses are below the line of variables and all positive clauses are above the line.

Recall that in terms of realizability Ordered Level Planarity is equivalent to the generalized version where γ and χ map to the reals. For the sake of convenience we will describe our construction in this generalized setting. We create an ordered level graph whose level assignment is partitioned into four tiers $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$. Each clause c_i is associated with a *clause edge* starting in $\mathcal{T}_1$ and ending in $\mathcal{T}_3$. In tier $\mathcal{T}_1$ we do the following. We take the top part of $\mathcal{R}$, rotate it by 180° and place it to the left of the bottom part, see Fig. 2b. For each E-shape we create a 11-vertex subgraph as illustrated in Fig. 3a. The red vertex is the bottom vertex of the clause edge belonging to the clause that corresponds to the E-shape. Observe that drawings of this subgraph are *unique* in the sense that the left-to-right order of vertices and edges intersected by a horizontal line through any of the vertices is the identical in every ordered level planar drawing of the subgraph. This is due to the fact that the order of the top 6 vertices is fixed since they are placed on the same level. As a consequence, the clause edge starting at the red vertex has to intersect one of the thick gray regions, which we call *gates*. Fig. 2c illustrates the entirety of $\mathcal{T}_1$. The subgraph induced by $\mathcal{T}_1$ has a unique drawing. Further, note that each gate is located in the line segment of one of the variables of $\mathcal{R}$. We bundle all gates that are located in the line segment of the same literal together by creating *tunnels* as depicted in the top of Fig. 2c. Observe that the clause edge of clause c_i has to be drawn inside the tunnel of one of the literals of c_i. This corresponds to the fact that in a satisfying truth assignment every clause has at least one satisfied literal.

We need to ensure that for each variable u_j either its positive tunnel T_j^p or negative tunnel T_j^n can be used, but not both. To this end, we create a *variable gadget* for each variable u_j, see Figure 3b. These gadgets start in $\mathcal{T}_2$ and end in $\mathcal{T}_4$. In $\mathcal{T}_2$ the gadget for u_j starts above all the gadgets of variables with smaller index. In $\mathcal{T}_4$ the gadget for u_j ends below all the gadgets of variables with smaller index. The tunnels T_j^p and T_j^n end inside the gadget of variable u_j on level $\gamma(u_j^q)$. We force all tunnels with index at least n to be drawn between u_j^a and u_j^b by subdividing the tunnel edges appropriately, see Fig. 4a. The *long* edge (u_j^s, u_j^t) has to be drawn left or right of u_j^c in $\mathcal{T}_4$. Accordingly it has to be drawn left of u_j^a or right of u_j^b in $\mathcal{T}_2$ and, thus, left or right of all the



(a)

(b)

Figure 4: The two states of a variable gadet.

tunnels that are drawn between u_j^a and u_j^b. As a consequence, the *blocking* edge (u_j^s, u_j^p) is also drawn left (Fig. 4b) or right (Fig. 4a) of all tunnels. Together with the edge (u_j^q, u_j^p) it prevents clause edges from being drawn in T_j^p or T_j^n depending on whether the long edge and the blocking edge are drawn left or right.

We summarize the reduction. If there exists an ordered level planar drawing, then the clause edge of each clause is drawn inside of a tunnel that corresponds to one of its literals. Due to the variable gadgets, edges can only be drawn either inside the positive tunnel or inside the negative tunnel of a variable. Thus, we obtain a satisfying truth assignment. On the other hand, given a satisfying truth assignment we can create a drawing by placing the long edges of the variable gadgets according to the assignment. Fig. 5 illustrates how variable gadgets can be nested and clause edges can be drawn.

The resulting Ordered Level Planarity instance has maximum degree $\Delta = 2$. The level-width λ is linear in the input size, however, it can be decreased to $\lambda = 2$ by replacing a level with width $\lambda_i > 2$ with $\lambda_i - 1$ levels containing exactly two vertices each. For more details, we refer to the full version. For $\lambda = 1$ Ordered Level Planarity is solvable in linear time since Level Planarity can be solved in linear time [6].

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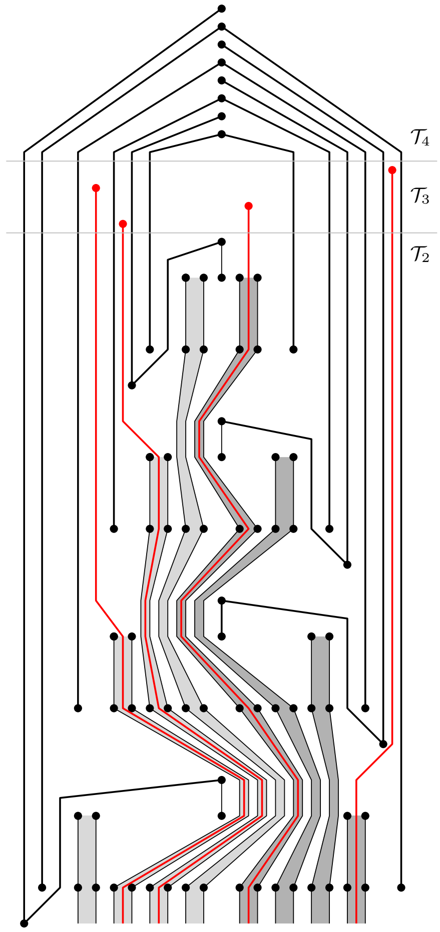

Figure 5: Nesting structure of the variable gadgets.

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