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On-Line q-adic Covering by the Method of the n-th Segment and its Application to On-Line Covering by Cubes

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Abstract. We prove that in Euclidean d-space every sequence of cubes with total volume $2^d + 3$ is able to cover on-line the unit cube. The proof is based on an on-line q-adic method of covering the unit segment by segments of lengths of the form q^{-r}, where $q \geq 2$ and $r \geq 1$ are integers. The fact that this method is q-adic means that every segment has to be placed in such a way that both end-points are at points that are multiples of the length of the segment.

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We say that a sequence $Q_1, Q_2, \ldots$ of subsets of Euclidean space E^d *permits a covering* of ae set $C \subset E^d$ if there exist rigid motions $\tau_1, \tau_2, \ldots$ such that C is contained in the union of sets $\tau_1 Q_1, \tau_2 Q_2, \ldots$. A survey of results about covering of convex bodies by sequences of convex bodies is provided by Groemer [2].

An *on-line* covering problem consists in finding the motions $\tau_1, \tau_2, \ldots$, under the condition that we are given every set Q_i, with $i > 1$, only *after* the motion τ_{i-1} has been provided. We learn the set Q_1 at the beginning. Problems of the on-line covering by convex bodies are considered in a number of papers. The first result about on-line covering the unit cube by cubes was obtained by Kuperberg [5]. This result was improved in [3,4] and the present paper provides a further improvement. The reader may compare the above on-line assumption with that in the area of bin-packing problems (see the survey paper [1]).

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In this paper we first deal with a specially restricted one-dimensional covering problem and then apply it to higher-dimensional problems.

By $[x, y]$ we denote the closed segment with end-points x and y, where $x < y$. The symbol (x, y) denotes the corresponding open segment.

Let q be an integer greater than or equal to 2. By the on-line *q-adic* covering problem of the unit segment $[0, 1]$ we mean the problem of on-line covering the segment $[0, 1]$ by a sequence of closed segments S_i of lengths δ_i, where $\delta_i \in \{q^{-1}, q^{-2}, \ldots\}$, and where the motions τ_i are restricted so that every segment $\tau_i S_i$ is of the form $[c_i\delta_i, (c_i + 1)\delta_i]$ with $c_i \in \{0, \ldots, \delta_i^{-1} - 1\}$ for $i = 1, 2, \ldots$. The 2-adic covering problem was posed by Kuperberg [6].

1. The method of the n-th segment

Let $n \geq 2$ be an integer. At every moment of the covering process we determine the greatest number $b \in [0, 1]$ such that the whole segment $[0, b]$ is covered. Assume that we are given a segment S from our sequence and let q^{-r} be the length of this segment. Denote by a the greatest number among $0, \ldots, q^r - 1$ such that $aq^{-r} \leq b$. If the interval $[(a + k - 1)q^{-r}, (a + k)q^{-r}]$, where $k \in \{1, 2, \ldots\}$, is a subset of $[0, 1]$, then we call it the *k-th segment*. Let c be the greatest number among $a, \ldots, a + n - 1$ such that the segment $T = [cq^{-r}, (c + 1)q^{-r}]$ is contained in $[0, 1]$ and such that T is not yet totally covered. The segment S is put on T, i. e., we provide a motion τ such that $\tau S = T$. (In other words, we successively try the n-th segment, the $(n - 1)$-st segment, and so on, until we find an interval not totally covered.) We finish the process when the whole segment $[0, 1]$ is covered. The above described on-line algorithm for q-adic covering is called the *n-th segment method*.

We will mainly be interested in the cases $n = q+1$, $n = q+2$, and $n = q+3$, because these three values give the best performance ratio for all values of q.

Let us first introduce some notation that will be used in the sequel. We imagine the segment $[0, 1]$ laid out horizontally with 0 on the left end. For every integer $i > 1$ we denote by b_i the greatest number in the segment $[0, 1]$ such that the segment $[0, b_i]$ is covered by the union of segments $\tau_h S_h$ for $h < i$. Moreover, let $b_1 = 0$. We call b_i the *current bottom*; we choose this name because our notion serves here a similar purpose as the notion of the current bottom in some other papers where a cube or a box is filled from bottom to top.

We start with a somewhat technical lemma.

Lemma 1. *Consider the n-th segment method for some $n > q$. Let p be a multiple of q^{-w} for some w such that $0 < p < 1$. Assume that at some point of the algorithm, the interval $[0, p]$ has not been completely covered yet. For $j \geq 0$ let ν_j be the number of segments of length q^{-w-j} which the algorithm has placed to the right of p. If $\nu_0, \ldots, \nu_\ell \geq n - 2$ for some $\ell \geq 0$, then at most one of these $\ell + 1$ numbers is equal to $n - 1$. If, in addition, one of the numbers $\nu_0, \ldots, \nu_\ell$ equals $n - 1$, then the interval $[p, p + q^{-w+1}]$ is completely covered.*

Proof. Assume that $\nu_j = n - 1$ for some j and $\nu_0, \ldots, \nu_{j-1} \geq n - 2$. We want to show that $\nu_0 = \cdots = \nu_{j-1} = n - 2$ and that the whole interval $[p, p + q^{-w+1}]$ is covered. Since p is a multiple of q^{-w-j}, the equation $\nu_j = n - 1$ can only hold if the $n - 1$ segments of length

q^{-w-j} to the right of p are the adjacent intervals $[p, p+q^{-w-j}], \ldots, [p+(n-2)q^{-w-j}, p+(n-1)q^{-w-j}]$. Since $n-1 \geq q$, this means that the interval $[p, p+q^{-w-j+1}]$ is completely covered. If we now look at ν_{j-1}, we see that the possible positions for the segments of length q^{-w-j+1} are the $n-1$ intervals $[p, p+q^{-w-j+1}], \ldots, [p+(n-2)q^{-w-j+1}, p+(n-1)q^{-w-j+1}]$. Among them, the first interval is not possible because it has been covered completely by smaller segments. Thus we conclude that $\nu_{j-1} = n-2$, and we know that $[p, p+q^{-w-j+2}]$ is completely covered. In this way we proceed inductively to $\nu_{j-2}, \ldots, \nu_0$, and the lemma is proved. □

During the course of the algorithm, the algorithm makes progress by gradually advancing the current bottom b_i. The following lemma relates this progress to the sum of the lengths of segments which have been used to cover the ground to the left of b_i.

Lemma 2. *Consider the n-th segment method for some n with $q < n \leq 2q$. Assume that $i \geq 1$ and $b_i < b_{i+1} < 1$ and put $\Delta b = b_{i+1} - b_i$. Denote by Δl the total length of those among the segments $\tau_1 S_1, \ldots, \tau_i S_i$ which have non-empty intersection with (b_i, b_{i+1}). We have*

$$\Delta l < \left(1 + \frac{1}{q-1} + \frac{1}{n-1} - \frac{1}{(q-1)(n-1)}\right)\Delta b. \tag{1}$$

Proof. Denote by w the smallest positive integer such that a segment of length q^{-w} has been used for the covering of the segment (b_i, b_{i+1}). From the description of the method we see that

$$q^{-w} < \Delta b \leq nq^{-w}.$$

Thus we can represent Δb in the form

$$\Delta b = \sum_{j=0}^{m} \lambda_j q^{-w-j},$$

where $\lambda_0 \in \{1, \ldots, n\}$. Moreover, if $m \geq 1$, then $\lambda_1, \ldots, \lambda_m \in \{0, \ldots, q-1\}$, and $\lambda_m \geq 1$.

For $j = 0, 1, 2, \ldots$, denote by μ_j the number of segments of length q^{-w-j} among $\tau_1 S_1, \ldots, \tau_{i-1} S_{i-1}$ which are used for covering (b_i, b_{i+1}). Let the length of the last segment S_i be $q^{-r} = \delta_i$. Of course, $0 \leq \mu_j \leq n-1$ for $j = 0, 1, 2, \ldots$, and $\mu_0 > 0$ unless $r = w$. We can write Δl as follows:

$$\Delta l = q^{-r} + \sum_{j=0}^{\infty} \mu_j q^{-w-j}$$

The interval (b_i, b_{i+1}) can be covered in different ways depending on the size of Δb. We distinguish four cases.

Case 1, when $\Delta b = nq^{-w}$. We have $\mu_j \leq n-1$ for $j = 0, 1, 2, \ldots$. Thus

$$\Delta l < q^{-r} + (n-1)\sum_{j=w}^{\infty} q^{-j} \leq q^{-w} + (n-1)\sum_{j=w}^{\infty} q^{-j} = \frac{nq-1}{q-1} q^{-w}$$

which implies (1).

Case 2, when $(n-1)q^{-w} \le \Delta b < nq^{-w}$. We have $\lambda_0 = n-1$. It follows that b_{i+1} must be a multiple of q^{-w}: The region immediately left of b_{i+1} must be covered by a segment of length q^{-w} because shorter segments cannot "reach" far enough to the right from b_i or from the left of b_i, since $n \le 2q$. Let us first assume $\Delta b > (n-1)q^{-w}$. In this case, let k denote the smallest number among $1, \ldots, m$ such that $\lambda_k > 0$. Then we have

$$\Delta b \ge (n-1)q^{-w} + \lambda_k q^{-w-k} \ge (n-1)q^{-w} + q^{-w-k}. \tag{2}$$

Since b_{i+1} is a multiple of q^{-w} and $b_i = b_{i+1} - \Delta b$, the point b_i is close to a multiple p of q^{-w}. More precisely,

$$p - q^{-w-k+1} < b_i \le p - q^{-w-k}.$$

It follows that all segments of length between q^{-w-k+1} and q^{-w} among $\tau_1 S_1, \ldots, \tau_{i-1} S_{i-1}$ which have been used to cover (b_i, b_{i+1}) must lie to the right of p. Since none of the segments of these lengths to the right of p can lie to the right of b_{i+1}, Lemma 1 can be applied, with $\nu_i = \mu_i$ for $i = 0, \ldots, k-1$. We distinguish two subcases: either all numbers $\mu_0, \ldots, \mu_{k-1}$ are at least $n-2$, or there is some first element μ_z in the sequence $\mu_0, \ldots, \mu_{k-1}$ which is less than $n-2$. In the first subcase at most one μ_j with $0 \le j < k$ equals $n-1$, and

$$\Delta l \le q^{-r} + (n-2) \sum_{j=w}^{w+k-1} q^{-j} + q^{-w} + (n-1) \sum_{j=w+k}^{\infty} q^{-j}. \tag{3}$$

The term q^{-w} accounts for the μ_j with $0 \le j < k$ which is possibly equal to $n-1$. In the second subcase we have

$$\Delta l \le q^{-r} + (n-2) \sum_{j=w}^{w+z-1} q^{-j} + q^{-w} + (n-3)q^{-w-z} + (n-1) \sum_{j=w+z+1}^{\infty} q^{-j}.$$

(For $z = 0$, the "empty sum" $\sum_{j=w}^{w+z-1}$ is taken as zero.) The term q^{-w} stands for the same reason as above. One checks that this last bound is weakly increasing in z, and for the maximum $z = k-1$ it is smaller than the bound of (3). Hence (3) holds in both subcases, and we get from (3), using the relation $q^{-r} \le q^{-w}$,

$$\begin{aligned} \Delta l &< 2q^{-w} + (n-2) \sum_{j=w}^{\infty} q^{-j} + \sum_{j=w+k}^{\infty} q^{-j} = q^{-w} \left(2 + \frac{(n-2)q}{q-1} \right) + q^{-w-k} \cdot \frac{q}{q-1} \\ &\le q^{-w} \cdot \frac{nq-2}{q-1} + q^{-w-k} \cdot \frac{nq-2}{(q-1)(n-1)} \end{aligned}$$

which together with (2) shows (1).

By letting k grow one can conclude that in this case the bound is tight. In the next section we construct examples where the above bound is the true expression for the ratio between Δl and Δb, thus showing that the ratio in the lemma cannot be improved.

The above argument worked under the assumption $\Delta b > (n-1)q^{-w}$. For the case $\Delta b = (n-1)q^{-w}$ we may modify the proof by "symbolically" setting $k = \infty$. Correspondingly we set $q^{-w-k} = 0$, etc. As the reader may check, this leads to a valid proof of (1).

Case 3, when $2q^{-w} \le \Delta b < (n-1)q^{-w}$. It is easy to see that $2 \le \lambda_0 \le n-2$.

In this case the μ_0 segments of length q^{-w} in (b_i, b_{i+1}) must have been placed before b_i was reached. In fact, they must have been placed at time t when b_t was less than $b_{i+1} - (n-1)q^{-w}$ because otherwise such a segment would have been placed to the right of b_{i+1}. By the same reason, the last segment q^{-r} cannot be q^{-w}, and thus $q^{-r} \le q^{-w-1}$. Clearly $\mu_0 \le \lambda_0$.

Subcase 3.1; $\mu_0 = \lambda_0 = n-2$. In this case $\Delta b \ne \lambda_0 q^{-w}$ because otherwise the μ_0 disjoint segments of length q^{-w} would cover (b_i, b_{i+1}) completely, and the segment of length q^{-w} which covers $(b_i, b_i + q^{-w})$ would be the last segment, which we have excluded. Therefore we can define the smallest number k among $1, \ldots, m$ such that $\lambda_k > 0$. Thus

$$\Delta b \ge \lambda_0 q^{-w} + \lambda_k q^{-w-k}. \tag{4}$$

Let us show the inequality

$$\mu_1 \le (n-1-q) + \lambda_1. \tag{5}$$

To do this, look at the possible positions for the μ_1 segments of length q^{-w-1}. To the left of the point $p := b_{i+1} - \lambda_0 q^{-w}$ there are λ_1 possible positions which are contained in $[b_i, b_{i+1}]$. An interval of length q^{-w-1} can be placed to the right of p only if that position has not been previously covered by an interval of length q^{-w}. As we have seen above, the $\mu_0 = \lambda_0$ segments of length q^{-w} have been placed to fill the interval (p, b_{i+1}) completely at a time t when b_t was smaller than $b_{i+1} - (n-1)q^{-w} = b_{i+1} - (\lambda_0+1)q^{-w} = p - q^{-w}$. At such a time, the endpoint of the rightmost possible position for a segment of length q^{-w-1} is less than $b_t + nq^{-w-1} \le p - q^{-w} + nq^{-w-1} \le p + (n-q)q^{-w-1}$. Therefore we get at most $n-q-1$ possible positions to the right of p. Together with the λ_k positions left of p this gives the inequality (5).

By a similar argument we show that $\mu_j \le \lambda_j$ for $1 < j \le k$. In this case the rightmost possible position for a segment of length q^{-w-j} is less than $p - q^{-w} + nq^{-w-1} \le p$, since $n \le q^2$ and $j \ge 2$. For $j > k$ we clearly have $\mu_j \le n-1$.

As a consequence of these inequalities for μ_j we obtain

$$\begin{aligned}
\Delta l &< q^{-w-1} + \lambda_0 q^{-w} + (n-q-1)q^{-w-1} + \lambda_k q^{-w-k} + (n-1)\sum_{j=w+k+1}^{\infty} q^{-j} \\
&= q^{-w} \cdot (n-2+n/q-1) + q^{-w-k} \cdot \left(\lambda_k + \frac{n-1}{q-1}\right) \\
&< q^{-w} \cdot (n-2+n/q-2/q) + q^{-w-k}\lambda_k + q^{-w-1} \cdot \frac{n-1}{q-1} + \frac{q(q-2)(n-2)-(n-1)}{q(q-1)(n-1)} \\
&= q^{-w}(n-2) \cdot \left(1 + \frac{1}{q-1} + \frac{1}{n-1} - \frac{1}{(q-1)(n-1)}\right) + q^{-w-k}\lambda_k,
\end{aligned}$$

which together with (4) gives (1). The last term which has been added in the third line is positive only for $q \ge 3$. For $q = 2$, a different calculation must be used. Starting with the second line above, we can write

$$
\begin{aligned}
\Delta l &< 2^{-w}(n-2)\cdot 3/2 + 2^{-w-k}\left(\lambda_k+n-1\right)\\
&= 2^{-w}(n-2)\cdot 3/2 + 2^{-w-k}\left(n-1-\lambda_k\right) + 2^{-w-k}\cdot 2\lambda_k\\
&\le 2^{-w}(n-2)\cdot 3/2 + 2^{-w-1}(n-2) + 2^{-w-k}\cdot 2\lambda_k\\
&= \left(2^{-w}(n-2)+2^{-w-k}\lambda_k\right)\cdot 2 \le \Delta b\cdot 2 = \Delta b\cdot\left(1+\frac{1}{q-1}+\frac{1}{n-1}-\frac{1}{(q-1)(n-1)}\right).
\end{aligned}
$$

Subcase 3.2; $2 \le \mu_0 = \lambda_0 < n-2$. Again we must have $\Delta b \ne \lambda_0 q^{-w}$, and we can define k as in the previous subcase. In contrast to that case, we can show the stronger relation $\mu_1 \le \lambda_1$. Arguing as in Subcase 3.1, with the same definition of p, we conclude that $b_t < b_{i+1} - (n-1)q^{-w} \le b_{i+1} - (\lambda_0+2)q^{-w} = p - 2q^{-w}$. Since $n \le 2q$, there are no possible positions to the right of p for segments of size at most q^{-k-1}. Thus we have $\mu_j \le \lambda_j$ for $j = 1, \ldots, k$, and $\mu_j \le n-1$ for $j > k$. We obtain

$$
\begin{aligned}
\Delta l &< q^{-w-1} + \lambda_0 q^{-w} + (\lambda_k) q^{-w-k} + (n-1)\sum_{j=w+k+1}^{\infty} q^{-j}\\
&\le q^{-w}(\lambda_0 + 1/q) + q^{-w-k}\cdot\left(\lambda_k + \frac{n-1}{q-1}\right)\\
&= q^{-w}\left(\lambda_0 + \frac{1}{q}\cdot\left(1+\frac{n-2}{q-1}\right)\right) + q^{-w-k}\cdot\left(\lambda_k+\frac{1}{q-1}\right)\\
&\le q^{-w}\left(\lambda_0 + \frac{\lambda_0}{n-1}\cdot\left(1+\frac{n-2}{q-1}\right)\right) + q^{-w-k}\cdot\left(\lambda_k+\frac{\lambda_k}{q-1}\right)\\
&= q^{-w}\lambda_0\cdot\left(1+\frac{1}{q-1}+\frac{1}{n-1}-\frac{1}{(q-1)(n-1)}\right) + q^{-w-k}\lambda_k\cdot\left(1+\frac{1}{q-1}\right),
\end{aligned}
$$

which, combined with (4), proves (1).

Subcase 3.3; $1 \le \mu_0 \le \lambda_0 - 1 = n-3$. Since $n \ge 4$ we must have $q \ge 3$. We have $\mu_j \le n-1$ for $j \ge 1$ and we get

$$
\begin{aligned}
\Delta l &< q^{-w-1} + (\lambda_0-1)q^{-w} + (n-1)\sum_{j=w+1}^{\infty} q^{-j} = q^{-w}\left(\frac{1}{q}+n-3+\frac{n-1}{q-1}\right)\\
&= q^{-w}\cdot\left(n-2+\frac{n-2}{q-1}+\frac{1}{q}+\frac{1}{q-1}-1\right)\\
&< q^{-w}(n-2)\cdot\left(1+\frac{1}{q-1}\right) \le \Delta b\left(1+\frac{1}{q-1}\right),
\end{aligned}
$$

from which (1) follows.

Subcase 3.4; $1 \le \mu_0 \le \lambda_0 - 1 < n-3$. Again, $\lambda_0 \ge 2$. Let us denote $p = b_{i+1} - \lambda_0 q^{-k}$. We show that in fact $\mu_0 = \lambda_0 - 1$, the μ_0 segments of length q^{-k} completely cover the interval $[p + q^{-k}, b_{i+1}]$, and they lie disjoint from all other intervals. First of all, since $n \le 2q$, no segment shorter than q^{-w} reaches far enough to the right from b_i or from the left of b_i, in order to cover the interval $[p+q^{-w}, p+2q^{-w}]$ completely. This is a fortiori true for the $\lambda_0 - 2$ intervals which lie between $p + 2q^{-w}$ and b_{i+1}. Since we have at most $\lambda - 1$ segments of length q^{-k} available, all of them must be used to cover these intervals, because otherwise they would remain uncovered. Secondly, we have previously seen that, before the last segment of length q^{-k} was placed inside (p, b_{i+1}), the value of b_t must have been less than $b_{i+1} - (n-1)q^{-w}$. Therefore a segment of length q^{-w-k} for $k \geq 1$ cannot cover anything to the right of $b_{i+1} - (n-1)q^{-w} + nq^{-w-k} \leq p$ before all μ_0 segments of length q^{-k} are placed. Therefore these μ_0 segments of length q^{-k} lie disjoint from all other segments.

From this it follows that $\mu_0 q^{-k} + \mu_1 q^{-k-1} \leq \lambda_0 q^{-k} + \lambda_1 q^{-k-1}$. Thus we have

$$\Delta l < q^{-w-1} + \mu_0 q^{-k} + \mu_1 q^{-k-1} + (n-1) \sum_{j=w+2}^{\infty} q^{-j}$$

$$\leq \lambda_0 q^{-k} + \lambda_1 q^{-k-1} + \frac{n-1}{q} \frac{1}{q-1} q^{-k} \leq \Delta b + \lambda_0 q^{-k} \frac{1}{q-1} < \Delta b \left(1 + \frac{1}{q-1}\right),$$

and we are done.

Case 4, when $q^{-w} \leq \Delta b < 2q^{-w}$. In order to avoid splitting into further cases, we reduce this case to the proof of the preceding cases. As in Case 3, we conclude that the last segment was not of length q^{-w}, and therefore $\mu_0 = \lambda_0 = 1$. Let the segment of length q^{-w} which was placed inside $[b_i, b_{i+1}]$ be at position $[p, p + q^{-w}]$. We reduce the proof of this case to the other cases by modifying the sequence $S_1, \ldots, S_i$ of intervals. We replace the segment of length q^{-w} which the algorithm placed at position $[p, p + q^{-w}]$ by some smaller segments.

We will now describe how we modify the sequence $S_1, \ldots, S_i$. We start the algorithm with no segment placed and go through the given sequence of segments step by step. Before each segment (including the start of the algorithm), we try to determine the smallest index $k \geq 1$ such that the n-th segment position for a segment of length q^{-w-k} would fall into an uncovered portion of the interval $[p, b_{i+1}]$. If we find such a k, we generate segments of length q^{-w-k} and insert them into the sequence of segments at the current point. We insert the right number of these segments so that, when the algorithm has processed them, the interval between p and the n-th position for q^{-w-k} is completely covered. Then we proceed to the next segment in the original sequence. If it is the segment of length q^{-w} which would go into the position $[p, p + q^{-w}]$ or (in the case, when $b_{i+1} > p + q^{-w}$) if it is the segment of length q^{-w-1} which would go into the segment $[p + q^{-w}, b_{i+1}]$, then we skip it.

It can be seen that we will always have $k = 1$ or $k = 2$. If some choice of k is not possible because its n-th position is already covered or lies to the left of p, then no larger choice of k is possible. The choice of $k = 1$ may be excluded also for the reason that the n-th segment position for q^{-w-1} may lie to the right of b_{i+1}. But since $n \leq 2q$ and $\Delta b \geq q^{-w}$, the n-th segment position for q^{-w-2} lies always to the left of b_{i+1}. Thus, if neither $k = 1$ nor $k = 2$ is possible, then no larger choice of k is possible. Moreover, once $k = 2$ has been chosen, $k = 1$ can never be chosen again. It follows that the new segments which are inserted have disjoint interiors.

When the algorithm processes the modified sequence of segments, the successive values of b_j are unchanged, except that the final value is some b'_{i+1} with $p \leq b'_{i+1} \leq b_{i+1}$. (We use the subscript $i+1$ although i may no longer reflect the number of segments of the modified sequence which have been processed.) In particular $\Delta b'$ is $b'_{i+1} - b_i$ with the same value b_i as before.

For the modified sequence we have

$$\Delta b' = b'_{i+1} - b_i = \Delta b - (b_{i+1} - b'_{i+1}).$$

Instead of the interval $[p, b_{i+1}]$, which was previously covered without overlap, we have now the interval $[p, b'_{i+1}]$, which is covered without overlap. Moreover, to the right of b'_{i+1} no new segments are added. To the left of p and to the right of b_{i+1} nothing has changed. Consequently, for the modified value $\Delta l'$ of Δl we get

$$\Delta l' = \Delta l - (b_{i+1} - p) + (b'_{i+1} - p) = \Delta l - (b_{i+1} - b'_{i+1}),$$

If $\Delta b'$ is greater or equal than two times the length of the longest segment used for the covering of the segment (b_i, b'_{i+1}), then we apply the preceding cases of our lemma to the modified sequence of segments;. from $\Delta l' < \left(1 + \frac{1}{q-1} + \frac{1}{n-1} - \frac{1}{(q-1)(n-1)}\right)\Delta b'$ we conclude that $\Delta l < \left(1 + \frac{1}{q-1} + \frac{1}{n-1} - \frac{1}{(q-1)(n-1)}\right)\Delta b$. In the opposite case, we apply repeatedly the procedure of modifying the sequence of intervals until the corresponding $\Delta b'$ becomes at least two times the length of the corresponding longest segment used for the covering of the corresponding segment (b_i, b'_{i+1}). The repeating of this procedure ends after a finite number of times because $\Delta b' \geq p - b_i > 0$. □

The previous lemma has considered the intervals to the left of the current b_i; the next lemma deals with the intervals that remain to the right of b_i.

Lemma 3. *Consider the n-th segment method for some n with $q < n \leq 2q$. Assume that for some $i > 0$ we have $b_i < 1$. Put $\Delta b = 1 - b_i$. Denote by w the integer for which $q^{-w} < \Delta b \leq q^{-w+1}$. Then the total length Δl of the segments among $\tau_1 S_1, \tau_2 S_2, \ldots, \tau_{i-1} S_{i-1}$ having non-empty intersection with $(b_i, 1)$ is less than*

$$\Delta b + \left(\frac{n-2}{q-1}\right) q^{-w}.$$

Moreover, if $\Delta b < 2q^{-w}$, then we have the stronger bound

$$\Delta b + \left(\frac{n-2}{q-1} - \frac{n-q-1}{q}\right) q^{-w}.$$

Proof. Let us represent Δb in the form $\sum_{j=0}^{m} \lambda_j q^{-w-j}$, where $1 \leq \lambda_0 \leq q$, $0 \leq \lambda_j \leq q-1$ for $0 < j < m$, and $0 < \lambda_m$. Denote by $\mathcal{F}$ the family of those among the segments $\tau_1 S_1, \ldots, \tau_{i-1} S_{i-1}$ which have non-empty intersection with $(b_i, 1]$. We know that a segment from $\mathcal{F}$ cannot contain the point b_i because otherwise b_i would be bigger.

Denote by μ_j the number of segments of length q^{-w-j} in $\mathcal{F}$ for $j = 0, 1, 2, \ldots$. We may assume that $\mathcal{F}$ does not contain segments of length over q^{-w}, that $\mu_0 \leq \lambda_0$, and that

$\mu_j \le q$ for $j = 1, 2, \ldots$. If any of these assumptions is not satisfied, we immediately see that the point b_i is covered by segments from the family $\mathcal{F}$ and thus we get a contradiction.

If $\mu_j \le n-2$ for all $j \ge 1$ then we have

$$\Delta l = \sum_{j=0}^{\infty} \mu_j q^{-w-j} \le \mu_0 q^{-w} + (n-2)\sum_{j=1}^{\infty} q^{-w-j} \le \lambda_0 q^{-w} + q^{-w} \cdot \frac{n-2}{q-1},$$

and the proof is completed. Otherwise, let z be the smallest non-negative integer such that $\mu_z = n-1$. If $z = 0$, then $[b_i, 1]$ is covered, so assume that $z \ge 1$. Using the inequalities $\mu_1, \ldots, \mu_{z-1} \le n-2$ and $\mu_z, \mu_{z+1}, \ldots \le n-1$ we obtain

$$\begin{aligned}\Delta l = \sum_{j=0}^{\infty} \mu_j q^{-w-j} &\le \mu_0 q^{-w} + (n-2)\sum_{j=1}^{z-1} q^{-w-j} + (n-1)\sum_{j=z}^{\infty} q^{-w-j}\\ &= \mu_0 q^{-w} + \frac{n-2}{q-1} \cdot q^{-w} + \left(1 + \frac{1}{q-1}\right) q^{-w-z}.\end{aligned}$$

Similarly, using the inequalities $\lambda_i \ge 0$ for $i \ge 1$ we can write

$$\Delta b = \sum_{j=0}^{m} \lambda_j q^{-w-j} \ge \lambda_0 q^{-w}.$$

Putting these inequalities together and using $\mu_0 \le \lambda_0$ we obtain

$$\Delta l - (\Delta b + \frac{n-2}{q-1} \cdot q^{-w}) < q^{-w-z+1}.$$

Since we can bound the difference $\Delta l - (\Delta b + q^{-w})$, which we want to prove to be negative, by such a small quantity, we see that we are done if any of $\mu_1, \ldots, \mu_{z-1}$ decreases below $q-1$; if any of $\lambda_1, \ldots, \lambda_{z-1}$ increases above 0; or if the inequality $\mu_0 \le \lambda_0$ is strict. So we only have to deal with the case

$$\mu_1 = \cdots = \mu_{z-1} = n-2, \quad \lambda_1 = \cdots = \lambda_{z-1} = 0, \quad \text{and } \mu_0 = \lambda_0.$$

In this case we know that $b_i = 1 - \Delta b$ is close to a multiple p of q^{-w}:

$$p - q^{-w-z+1} < b_i < p.$$

As in the proof of Lemma 2, Case 2, this means that Lemma 1 can be applied. We use the sequence $\mu_1, \ldots, \mu_z$ for the sequence $\nu_0, \ldots, \nu_\ell$ in the statement of Lemma 1 and conclude that the interval $[p, p + q^{-w}]$ is completely covered by intervals of size less than q^{-w}. Consider the λ_0 intervals if size q^{-w} lying between p and 1: $[p, p+q^{-w}], \ldots, [p + (\lambda_0 - 1)q^{-w}, p + \lambda_0 \cdot q^{-w}]$. Among them only $\lambda_0 - 1$ intervals are available for the $\mu_0 = \lambda_0$ segments of length q^{-w}. Therefore at least one segment must go into the interval $[p - q^{-w}, p]$ covering b_i, a contradiction. This concludes the proof of the first statement of the lemma.

If $\Delta b < 2q^{-w}$, then $\lambda_0 = 1$ and we modify the argument as follows. The above upper bound on $\Delta l - \Delta b = \sum_{j=0}^{\infty} \mu_j q^{-w-j} - \sum_{j=0}^{m} \lambda_j q^{-w-j}$ was based on the inequalities $\mu_0 \le \lambda_0$, $\lambda_1 \ge 0$ and $\mu_1 \le n-2$. (In some cases it was possible that the weaker relation $\mu_1 \le n-1$ was used.) If we can improve the consequence of these inequalities,

$$(\mu_0 - \lambda_0) + \frac{\mu_1 - \lambda_1}{q} \le \frac{n-2}{q},$$

to

$$(\mu_0 - \lambda_0) + \frac{\mu_1 - \lambda_1}{q} \le \frac{q-1}{q}, \tag{6}$$

we get an additional term $-q^{-w-1}(n-q-1)$ in all expressions, and this gives the second bound of the theorem, as the reader may check.

To show the improved inequality (6), consider the possible positions for the μ_1 segments of length q^{-w-1} in $\mathcal{F}$. There are λ_1 positions in $[b_i, 1-q^{-w}]$ and q positions in $[1-q^{-w}]$. If $\mu_0 = \lambda_0 = 1$, then the interval $[1-q^{-w}]$ is covered by the one segment of length q^{-w} in $\mathcal{F}$, and this means that the μ_1 segments of length q^{-w-1} cannot cover all q possible positions in $[1-q^{-w}]$. We get $\mu_1 \le \lambda_1 + (q-1)$, which gives (6). On the other hand, if $\mu_0 = 0 < \lambda_0 = 1$, then from $n \le 2q$ we easily obtain

$$(\mu_0 - \lambda_0) + \frac{\mu_1 - \lambda_1}{q} \le -1 + \frac{n-1}{q} = \frac{n-1-q}{q} \le \frac{q-1}{q}.$$

$\square$

Theorem 1. *Let $q \ge 2$ be an integer. Consider a sequence of segments of lengths from the set $\{q^{-1}, q^{-2}, \ldots\}$. If their total length is at least*

$$1 + \frac{3}{q} - \frac{2}{q^2},$$

then the method of the $(q+1)$-st segment ensures an on-line covering of the unit interval. For $q \ge 3$ the method of the $(q+2)$-nd segment guarantees the covering of the unit interval if the total length is at least

$$1 + \frac{3}{q} - \frac{3q^2 - 4q - 1}{q^4 - q^2}.$$

For $q \ge 4$ the method of the $(q+3)$-rd segment ensures a covering of the unit interval if the total length is at least

$$1 + \frac{3}{q} - \frac{4q - 8}{q(q-1)(q+2)}.$$

For $q \ge 6$, the best of all the methods of the n-th segment is the method of the $(q+3)$-rd segment. For $q = 4$ and $q = 5$, the best method is the method of the $(q+2)$-nd segment. For $q = 2$ and $q = 3$, the best method is the method of the $(q+1)$-st segment.

Proof. We distinguish three cases:

Case 1. We have $b_i = 0$ throughout the algorithm. Then we can use Lemma 3 with $\Delta b = 1$ and $q^{-w} \le 1/q$, and a simple calculation shows that for each case the theorem is true.

Case 2. We have $\lim_{i\to\infty} b_i = 1$, for an infinite sequence of segments. During the course of the algorithm, b_i increases monotonically, starting at $b_1 = 0$. By applying Lemma 2 we see that the total length of segments among $\tau_1 S_1, \ldots, \tau_i S_i$ which have non-empty intersection with $[0, b_i]$ is less than $(1 + \frac{1}{q-1} + \frac{1}{n-1} - \frac{1}{(q-1)(n-1)})b_i$. It can be checked that this multiplicative factor is less than the number given in the theorem in all cases, and by letting i go to infinity we get a contradiction.

Case 3. We have $0 < b' < 1$, where b' is either $\lim_{i\to\infty} b_i$, in case of an infinite sequence of segments, or b' is the value of b_i after the given sequence of segments is exhausted. As in Case 3, we can conclude that the total length of segments $\tau_1 S_1, \tau_2 S_2, \ldots$ which have non-empty intersection with $[0, b_i]$ is less than $(1+\alpha)b'$, where we define the constant $\alpha := \frac{1}{q-1} + \frac{1}{n-1} - \frac{1}{(q-1)(n-1)}$. Lemma 3 gives a bound on the sum of lengths of the remaining segments: Let w be the smallest positive integer such that $q^{-w} < 1 - b'$. This means that $b' \le 1 - q^{-w}$ and $q^{-w} \le 1/q$. If $1 - b' < 2q^{-w}$, Lemma 3 gives the stronger bound $(1-b') + \gamma q^{-w}$, with $\gamma := \frac{n-2}{q-1} - \frac{n-q-1}{q}$. Thus the total length of all segments is bounded as follows:

$$\begin{aligned}\sum_i \delta_i &< (1+\alpha)b' + (1-b') + \gamma q^{-w} = 1 + \alpha b' + \gamma q^{-w}\\ &\le 1 + \alpha(1 - q^{-w}) + \gamma q^{-w} = 1 + \alpha + q^{-w}(\gamma - \alpha)\\ &\le 1 + \alpha + \frac{1}{q}(\gamma - \alpha) = 1 + \frac{2}{q} + \frac{1}{n-1}\left(1 - \frac{2}{q}\right) + \frac{n-q-1}{q^2(q-1)}.\end{aligned} \tag{7}$$

The inequality between the second and third line follows since $\alpha \le \gamma$.

If $1-b' \ge 2q^{-w}$, Lemma 3 gives only the weaker bound $(1-b') + \tilde{\gamma} q^{-w}$ with $\tilde{\gamma} := \frac{n-2}{q-1}$, but on the other hand, we can use the stronger inequality $b' \le 1 - 2q^{-w}$. An analogous chain of relations as above yields

$$\sum_i \delta_i < 1 + \alpha + q^{-w}(\tilde{\gamma} - 2\alpha).$$

For $n \ge 5$ we can check that $\tilde{\gamma} \ge 2\alpha$, and thus we get

$$\sum_i \delta_i < 1 + \alpha + \frac{1}{q}(\tilde{\gamma} - 2\alpha) = 1 + \frac{1}{q-1} \cdot \left(\left(1 - \frac{2}{q}\right)\left(1 + \frac{q-2}{n-1}\right) + \frac{n-2}{q}\right). \tag{8}$$

For $n \le 4$ (and $q \le n-1$) we have $\tilde{\gamma} \le 2\alpha$ and we therefore get

$$\sum_i \delta_i < 1 + \alpha. \tag{9}$$

We must use the weaker of the above two bounds (7) and (8) (or (9) respectively). For $n = q+1$ and $n = q+2$ this is the bound (7), whereas for $n = q+3$ the bound (8) is larger than the bound (7).

The comparison between the different methods for the last part of the theorem is a matter of straightforward computation.

It is possible to construct sequences of segments whose total lengths are arbitrarily close to the bounds of Theorem 1 for which the algorithm does not produce a covering. The constructions are done in the next section. This leads to the conclusion that the three methods considered in our theorem are the most effective among all the n-th segment methods.

We remark (although we do not prove it here) that for each $n \geq 2$ and $q \geq 2$, the method of the n-th segment has performance guarantee similar to Theorem 1, with some finite constant depending on n and q. For instance for $q = n$ we get the performance $1 + \frac{3}{q}$. Only the method of the first segment may use an unlimited amount of intervals without producing a covering. □

2. Lower-bound examples

In this section, the upper bounds for performance ratio of the method of the n-th segment, which constitute the main result of the paper, are complemented by some constructions which show that these bounds are the tight.

We will first construct *auxiliary examples* of problem instances which cover an interval with density approaching the factor of Lemma 2. Assume that b_i is of the form sq^{-w}, where $s \in \{0, \dots, q^w - n - 1\}$, and that no segment among $\tau_1 S_1, \dots, \tau_{i-1} S_{i-1}$ covers a point from $(b_i, 1]$. After processing the segments of the example, b_{i+t} will have the value $(s+n)q^{-w}$, where t denotes the number of segments in the example. Again, no segment among $\tau_1 S_1, \dots, \tau_{i+t-1} S_{i+t-1}$ will cover a point from $(b_{i+t}, 1]$. By the *density* we mean the ratio $\Delta l / \Delta b$, where $\Delta b = nq^{-w}$ is the newly covered length and Δl denotes the total length of the t intervals. (This notation is not completely in accordance with Lemma 2, since b_i does not jump to b_{i+t} in one step.)

We will construct inductively a sequence of problem instances Ex(1), Ex(2),.... The first example Ex(1) just consists of n segments of length q^{-w}. We have $\Delta l = \Delta b = nq^{-w}$ and hence the density is $\rho_1 = 1$.

Now we describe Ex(k) under the assumption that Ex($k-1$) has been defined. Initially we have $b_i = sq^{-w}$. First, we take many small copies of sequences of segments like in Ex($k-1$), but q^{k+2} times smaller, i. e., we take $w + k + 2$ instead of w. We take as many copies as we can, before b_i reaches $(s+1)q^{-w}$. This means that b_i will end up in the interval $[(s+1)q^{-w} - nq^{-w-k-2}, (s+1)q^{-w}]$, which is contained in $[(s+1)q^{-w} - q^{-w-k}, (s+1)q^{-w}]$. This first part, of length greater than $q^{-w} - q^{-w-k}$, is covered with density ρ_k. Next we generate $n-2$ segments each of the following lengths: $q^{-w-k}, \dots, q^{-w-1}$. None of these segments advance b_i. Finally, we put n segments of length q^{-t}, and b_i moves up to $(s+n)q^{-t}$. The total change Δb is $(q+1)q^{-t}$. Consequently,

$$\Delta l \geq \rho_k (q^{-w} - q^{-w-k}) + (n-2) \sum_{j=w}^{w+k} q^{-j} + 2q^{-w}$$

$$= q^{-w} \left(\rho_k + \frac{(n-2)q}{q-1} + 2 \right) - q^{-w-k} \left(\rho_k + \frac{n-1}{q-1} \right).$$

Since $\rho_k \le 2$ (by Lemma 2), the last expression in parentheses is at most 4, and we get

$$\rho_{k+1} = \frac{\Delta l}{\Delta b} \ge \frac{\rho_k + \frac{(n-2)q}{q-1} + 2}{n} - q^{-k} \cdot \frac{4}{n}.$$

Since ρ_k is increasing, and, by Lemma 2, is bounded above, it must converge to a limit. Taking the limit on both sides of the above inequality and solving for $\rho = \lim_{k\to\infty} \rho_k$ yields $\rho \ge 1 + \frac{1}{q-1} + \frac{1}{n-1} - \frac{1}{(q-1)(n-1)}$. Consequently, the ratio $\Delta l/\Delta b$ differs from the factor of Lemma 2 as little as we wish.

The reader may compare the way how this family of examples was constructed with the proof of Lemma 2. In that proof, Case 2 was the “hard case” which made the bound tight. Thus, in our example, this is the case occurring for most of the intervals covered.

We now use these auxiliary examples to construct a sequence of segments of lengths from $\{q^{-1}, q^{-2}, \ldots\}$ whose total length is arbitrarily close to the bound of Theorem 1 and which does not cover the segment $[0, 1]$ by the method of the n-th segment.

To be specific, we consider only the $(q + 2)$-nd segment method. In the proof of Theorem 1, equation (7) is the one giving the bound for $n = q + 2$. This corresponds to the case where the interval $[0, (q - 1)/q]$ is filled according to Lemma 2, and the rest is filled according to Lemma 3.

Applying many times a scaled-down version of the auxiliary example (with q^{-w} sufficiently small), we move the current bottom up to a point strictly between $1 - 1/q - \epsilon$ and $1 - 1/q$, where $\epsilon > 0$ is an arbitrary small bound; all the time we take sufficiently large values of k in order to ensure that $\Delta l/\Delta b$ exceeds $\rho - \epsilon$. Then, we find an integer $h \ge 3$ such that $q^{-h} < \epsilon$ and we add $n - 2 = q$ segments each of the following lengths: $q^{-h}, \ldots, q^{-3}$. Finally, we add $q-1$ segments of length q^{-2}, and one segment of length q^{-1}. The segment $[0, 1]$ is not totally covered (because of a small piece to the left of $1 - 1/q$), while we have used for the covering a finite sequence of segments of the total length at least

$$\left(1 + \frac{1}{q-1} + \frac{1}{n-1} - \frac{1}{(q-1)(n-1)} - \epsilon\right)\left(1 - \frac{1}{q} - \epsilon\right) + q\sum_{j=3}^{h} q^{-j} + (q-1)q^{-2} + q^{-1}.$$

This total length is arbitrarily close to $1 + \frac{3}{q} - \frac{3q^2-4q-1}{q^4-q^2}$ if ϵ is chosen sufficiently small.

Analogously we proceed for arbitrary n instead of $q + 2$. We omit here the tedious calculation which leads to the conclusion where the point should be to which we move the current bottom applying many times the scaled-down version of the auxiliary example. We present only the conclusions.

If $n \ge 2q$, then we move the current bottom up to a point strictly between $1 - z/q - \epsilon$ and $1 - z/q$, where $\epsilon > 0$ is arbitrarily small; below we explain which integer z should be taken depending on of the value of n. Denote by m the positive integer fulfilling $mq \le n - 1 < (m + 1)q$. If $m \ge q - 1$, then we put $z = q - 1$. If $n = mq + 1$ and $m < q - 1$, we put $z = m$. If $n \ne mq + 1$ and $m < q - 1$, then we put $z = m + 1$. In this case the total length of the used segments is arbitrarily close to a number (we omit the complicated formula for it) which is not smaller than $1 + \frac{3}{q} - \frac{2}{q^2}$.

From now on assume that $n < 2q$.

If $n \le \frac{1}{2}(3+\sqrt{4q-3})$, we move the current bottom up to 1 using only the scaled-down version of the auxiliary example. In this case the total length of the used segments is arbitrarily close to $1 + \frac{1}{n-1} + \frac{1}{q-1}$.

If $\frac{1}{2}(3+\sqrt{4q-3}) < n \le q+2$, we move the current bottom up to a point strictly between $1 - 1/q - \epsilon$ and $1 - 1/q$, where $\epsilon > 0$ is an arbitrarily small bound. In this case the total length of the used segments can arbitrarily close to $1 + \frac{1}{q} + \frac{q-1}{q(n-1)} + \frac{n-1}{q(q-1)}$ for $n \le q$, arbitrarily close to $1 + \frac{3}{q} - \frac{2}{q^2}$ for $n = q+1$, and arbitrarily close to $1 + \frac{3}{q} - \frac{3q^2-4q-1}{q^4-q^2}$ for $n = q+2$.

If $n \ge q+3$, then we move the current bottom up to a point strictly between $1-2/q-\epsilon$ and $1-2/q$, where $\epsilon > 0$ is arbitrarily small. In this case the total length of the used segments can be arbitrarily close to $1 + \frac{3}{q} + \frac{q^2+n^2-2nq-2q-2n+5}{q(q-1)(n-1)}$.

The above values lead to the following conclusions about the total length l of the used segments. If $q = 2$ or $q = 3$, then l may be arbitrarily close to $1 + \frac{3}{q} - \frac{2}{q^2}$ for $n = q+1$, and to a greater number for other n. If $q = 3$ or $q = 4$, then l may be arbitrarily close to $1 + \frac{3}{q} - \frac{3q^2-4q-1}{q^4-q^2}$ for $n = q+2$, and to a greater number for other n. If $q = 6$, then l may be arbitrarily close to $1 + \frac{3}{q} - \frac{4q-8}{q(q-1)(q+2)}$ for $n = q+3$, and to a greater number for other n. If $q \ge 7$, then for every n the total length l may be arbitrarily close to a value greater than $1 + \frac{3}{q} - \frac{4q-8}{q(q-1)(q+2)}$.

3. An application to covering by cubes

Every cube whose side is of the form 2^{-j}, where j is a positive integer, is called *standard*. Every 2^d-adic covering method of the unit segment induces the following method of covering of the unit cube by standard cubes. Let ϕ_k be a bijection between the family of segments of the form $[h2^{-dk}, (h+1)2^{-dk}]$, where $h = 0, \ldots, 2^{dk}-1$, and the family of cubes of the form $\{\, (x_1, \ldots, x_d); h_i 2^{-k} \le x_i \le (h_i+1)2^{-k}$ for $i = 1, \ldots, d \,\}$, where $h_1, \ldots, h_d \in \{0, \ldots, 2^k - 1\}$. Let ϕ_k be given for $k = 1, 2, \ldots$ and let $\phi_k(I) \subset \phi_m(J)$ for every segment I of length 2^{-dk} and for every segment J of length 2^{-dm} such that $I \subset J$ and $k > m$. One possibility of such a bijection between the unit interval and the unit cube is a space-filling curve with its natural parameterization, see Kuperberg [5].

For every standard cube S_i we determine a segment of length equal to the volume of S_i and we use it for the covering of the unit segment according to the given method of covering the unit segment by segments. Simultaneously, we put S_i into the place in the unit cube determined by our bijection.

From Theorem 1 we see that this approach permits an on-line covering the unit cube by every sequence of standard cubes of the total volume at least $1 + \frac{3}{q} - \frac{4q-8}{q(q-1)(q+2)}$ with $q = 2^d$, if $d \ge 3$. (For $d = 2$ we have the constant $1 + 149/240$ instead.) This and the fact that every cube Q of side at most 1 contains a standard cube S such that the volume of the cube Q is smaller than 2^d times the volume of the cube S imply the following theorem.

Theorem 2. *Every sequence of d-dimensional cubes, where $d \geq 3$, of sides at most 1 and of total volume at least*

$$2^d + 3 - \frac{4 \cdot 2^d - 8}{2^{2d} + 2^d - 2}$$

permits an on-line covering of the unit cube of Euclidean d-space.

For $d = 2$ the considerations before Theorem 2 lead to the conclusion that every sequence of squares of sides at most 1 and of total area at least $389/60 \approx 6.483$ permits an on-line covering of the unit square in the Euclidean plane. This is a weaker result than that in [4] which gives the estimate $\frac{7}{4} \cdot 9^{1/3} + \frac{11}{8} \approx 5.265$. For $d \geq 3$ the above theorem improves the estimates given in [3-5].

Note added in Proof. The special case $q = 2$ of Theorem 1, dealing with 2-adic intervals, will be published by the authors in Math. Semesterber. as a solution to [6], with a different proof. This is complemented by a lower-bound construction which shows that, if $\lambda < \frac{4}{3}$, then there is no on-line method which would always work under the assumption that the total length of 2-adic segments is at least λ. Recall that our method of the third segment requires the total length $\lambda = 2$ in order to succeed.

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