

Lines pinning lines*

Boris Aronov¹ Otfried Cheong² Xavier Goaoc³ Günter Rote⁴

Abstract

A line ℓ is a *transversal* to a family \mathcal{F} of convex polytopes in \mathbb{R}^3 if it intersects every member of \mathcal{F} . If, in addition, ℓ is an isolated point of the space of line transversals to \mathcal{F} , we say that \mathcal{F} is a *pinning* of ℓ . We show that any minimal pinning of a line by polytopes in \mathbb{R}^3 such that no face of a polytope is coplanar with the line has size at most eight. If in addition the polytopes are pairwise disjoint, then it has size at most six.

1 Introduction

A *line transversal* to a family \mathcal{F} of disjoint compact convex objects is a line that meets every object of the family. Starting with the classic work of Grünbaum, Hadwiger, and Danzer in the 1950's, *geometric transversal theory* studies properties of line transversals and conditions for their existence. There is a sizable body of literature on line transversals in two dimensions. Of particular interest are so-called “Helly-type” theorems, such as the following theorem already conjectured by Grünbaum in 1958, but proven in this form by Tverberg only in 1989: *If \mathcal{F} is a family of at least five disjoint translates of a compact convex figure in the plane such that every subfamily of size five has a line transversal, then \mathcal{F} has a line transversal.* More background on geometric transversal theory can be found in the classic survey of Danzer et al. [4], or in the more recent ones by Goodman et al. [6], Eckhoff [5], Wenger [15], or Holmsen [8].

Much less is known about line transversals in three dimensions. Cheong et al. [3] showed a Helly-type theorem for pairwise disjoint congruent Euclidean balls, Borcea et al. [2] generalized this to families of arbitrary disjoint balls (with an additional ordering condition). On the other hand, Holmsen and Matoušek [9] proved that no such theorem can exist for convex polytopes: For every $n > 2$ they construct a convex polytope K and a family \mathcal{F} of n disjoint translates of K such that \mathcal{F} has no line transversal, but $\mathcal{F} \setminus \{F\}$ has a transversal for every $F \in \mathcal{F}$.

If a line transversal ℓ to a family \mathcal{F} cannot move without missing some $F \in \mathcal{F}$, then we call the line ℓ *pinned* by \mathcal{F} . Here we consider small continuous movements of ℓ in the vicinity of its current position. (We obviously exclude a translation of ℓ parallel to itself, which moves the points on the line but does not change the line as a whole.) In other words, ℓ is pinned if ℓ is an isolated point in the space of line transversals to \mathcal{F} . We will define an appropriate space \mathfrak{L} of lines in Section 2.

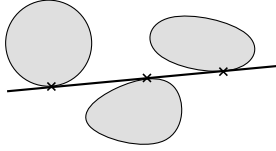
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¹Department of Computer Science and Engineering, Polytechnic Institute of NYU, Brooklyn, New York, USA. aronov@poly.edu

²Department of Computer Science, KAIST, Daejeon, Korea. otfried@kaist.edu

³LORIA - INRIA Nancy Grand Est, Nancy, France. goaoc@loria.fr

⁴Freie Universität Berlin, Institut für Informatik, Berlin, Germany. rote@inf.fu-berlin.de



The line shown on the left is pinned by the three convex figures. Whenever a family \mathcal{F} of convex objects in the plane pins a line ℓ , there are three objects in \mathcal{F} that already pin ℓ ; this fact was already used by Hadwiger [7]. Or, put differently, any *minimal pinning* of a line by convex figures in the plane is of size three. Here, a *minimal pinning* is a family \mathcal{F} pinning a line ℓ such that no proper subset of \mathcal{F} pins ℓ . The Helly-type theorems for balls in three dimensions are based on a similar result: any minimal pinning of a line by pairwise disjoint balls in \mathbb{R}^3 has size at most five [2].

In this paper we prove the analogous result for convex polytopes, with a restriction:

Theorem 1. *Any minimal pinning of a line by possibly intersecting convex polytopes in \mathbb{R}^3 , no facet of which is coplanar with the line, has size at most eight. The number reduces to six if the polytopes are pairwise disjoint.*

In the light of the construction by Holmsen and Matoušek, which shows that there can be no Helly-type theorem for line transversals to convex polytopes, Theorem 1 is perhaps surprising. Any Helly-type theorem can be considered as a guarantee for the existence of a small certificate: Tverberg’s theorem, for instance, says that whenever a family of translates in the plane does not admit a line transversal, we can certify this fact by exhibiting only five objects from the family that already do not admit a line transversal. (This interpretation of Helly-type theorems is the basis for their relation to *LP-type problems* [1].) Holmsen and Matoušek’s construction, on the other hand, shows that there are families \mathcal{F} of translates of a convex polytope that do not admit a line transversal, but such that it is *impossible* to certify this fact by a small subfamily of \mathcal{F} . Now, Theorem 1 can be interpreted as follows: Whenever there is a family \mathcal{F} of convex polytopes and a line ℓ (with the non-coplanarity restriction) such that there is a neighborhood of ℓ that contains no other line transversal to \mathcal{F} , then there is a subfamily $\mathcal{G} \subset \mathcal{F}$ of size at most eight that admits no other line transversal in a neighborhood of ℓ . So, while the absence of a line transversal *globally* cannot be witnessed by a small certificate, *locally* this is the case.

Consider a family \mathcal{F} of convex polytopes pinning a line ℓ_0 . A polytope whose interior intersects ℓ_0 cannot contribute to a minimal pinning, so we can assume that each element of \mathcal{F} is tangent to ℓ_0 . The simplest case is when ℓ_0 intersects a polytope F in a single point interior to an edge e of F (see the figure on the left). In that case, the condition that a line ℓ near ℓ_0 intersects F is characterized by the “side” (a notion we formalize in Section 2) on which ℓ passes the oriented line ℓ_e supporting the edge e . Since pinning is a local property, it follows that we can ignore the polytopes that constitute the family \mathcal{F} , and speak only about the lines supporting their relevant edges. We say that a family \mathcal{F} of *oriented lines* in \mathbb{R}^3 pins a line ℓ_0 if there is a neighborhood of ℓ_0 such that for any oriented line $\ell \neq \ell_0$ in this neighborhood, ℓ passes on the left side of some line in \mathcal{F} . We will prove the following theorem:

Theorem 2. *Any minimal pinning of a line ℓ_0 by lines in \mathbb{R}^3 has size at most eight. If no two of the lines are simultaneously coplanar and concurrent with ℓ_0 , then it has size at most six.*

We will actually give a full characterization of families of lines that minimally pin a line ℓ_0 when either all lines are orthogonal to ℓ_0 , or when they do not contain pairs of lines that are at the same time concurrent and coplanar with ℓ_0 .

When ℓ_0 lies in the plane of a facet of a polytope, the condition that a line close to ℓ_0 intersects the polytope becomes a *disjunction* of two sidedness constraints with respect to lines. Thus it cannot be handled by the methods of Theorem 2. In fact, the non-coplanarity condition

is necessary: We describe a construction that shows that no pinning theorem holds for collections of polytopes when they are allowed both to intersect each other and to touch the line in more than just a single point:

Theorem 3. *There exist arbitrarily large minimal pinnings of a line by convex polytopes in \mathbb{R}^3 .*

We do not know if the condition that the convex polytopes be pairwise disjoint is sufficient to guarantee a bounded minimal pinning number. In fact, we are not aware of any construction of a minimal pinning by pairwise disjoint convex objects in \mathbb{R}^3 of size larger than six.

Pinning is related to the concept of *grasping* used in robotics: an object is *immobilized*, or *grasped*, by a collection of contacts if it cannot move without intersecting (the interior of) one of the contacts. The study of objects (“fingers”) that immobilize a given object has received considerable attention in the robotics community [10, 11], and some Helly-type theorems are known for grasping. For instance, Mishra et al.’s work [12] implies Helly-type theorems for objects in two and three dimensions that are grasped by point fingers. We give two examples of how our pinning theorems naturally translate into Helly-type theorems for grasping a line. First, we can interpret a family \mathcal{F} of lines pinning a line ℓ_0 by considering all lines as solid cylinders of zero radius. Each constraint “cylinder” touches the “cylinder” ℓ_0 on the left. As a result, it is impossible to move ℓ_0 in any way (except to rotate it around or translate it along its own axis), because it would then intersect one of the constraints, so ℓ_0 is grasped by \mathcal{F} . Second, consider a family $\mathcal{F} = \{P_1, \dots, P_n\}$ of polytopes such that P_i is tangent to ℓ_0 in a single point interior to an edge e_i , and let \tilde{P}_i denote the mirror image of P_i with respect to the plane spanned by ℓ_0 and e_i . Since \mathcal{F} pins ℓ_0 if and only if $\tilde{\mathcal{F}}$ grasps it, our pinning theorems directly translate into Helly-type theorems for grasping a line by polytopes.

Outline of the paper. Our main technical contributions are our pinning theorem for lines (Theorem 2), a construction of arbitrarily large minimal pinnings of a line by overlapping polytopes with facets coplanar with the line (Theorem 3), and a classification of minimal pinnings of a line by lines (Theorems 6 and 7).

Sections 2 and 3 are devoted to the proof of the general case of Theorem 2. The idea of the proof is the following. We consider a parameterization of the space of lines by a quadric $\mathfrak{M} \subset \mathbb{R}^5$, where a fixed line ℓ_0 is represented by the origin $0 \in \mathbb{R}^5$, and the set of lines passing on one side of a given line g is recast as the intersection of \mathfrak{M} with a halfspace \bar{U}_g . This associates with a family \mathcal{F} of lines a polyhedral cone C ; C represents all lines not passing on the left side of some element of \mathcal{F} . Then \mathcal{F} pins ℓ_0 if and only if the origin 0 is an isolated point of $C \cap \mathfrak{M}$. We give a characterization of the cones C such that the origin is isolated in $C \cap \mathfrak{M}$ in terms of the trace of C on the hyperplane tangent to \mathfrak{M} in the origin; this is our *Isolation Lemma* (Lemma 3). The Isolation Lemma allows us to analyze the geometry of the intersection $C \cap \mathfrak{M}$ when \mathcal{F} is a pinning of ℓ_0 , and to find a subfamily of size at most eight that suffices to pin.

The subsequent two sections discuss the various configurations of lines that form minimal pinnings, first for constraints that are perpendicular to the pinned line (Section 4) and then for the general case (Section 5). Theorems 1, 2, and 3 are then proven in Section 6.

2 Lines and constraint sets

Sides of lines. Throughout this paper, all lines are oriented unless specified otherwise. Given two non-parallel lines ℓ_1 and ℓ_2 with direction vectors d_1 and d_2 , we say that ℓ_2 *passes to the right of* ℓ_1 if ℓ_2 can be translated by a positive multiple of $d_1 \times d_2$ to meet ℓ_1 , or, equivalently, if

$$\det \begin{pmatrix} p_1 & p'_1 & p_2 & p'_2 \\ 1 & 1 & 1 & 1 \end{pmatrix} < 0, \quad (1)$$

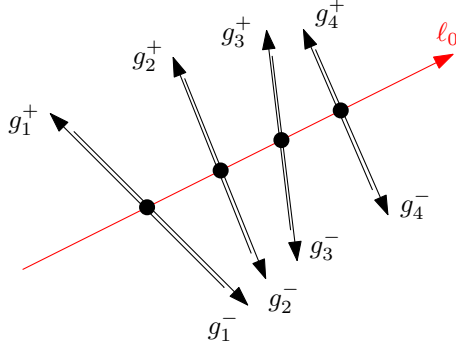


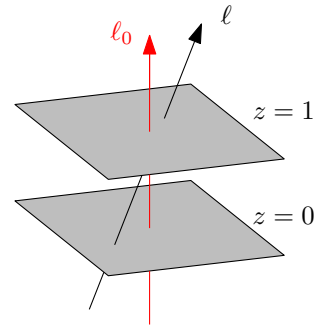
Figure 1: Eight orthogonal constraints that form a minimal pinning of ℓ_0 .

where p_i and p'_i are points on ℓ_i such that $p_i p'_i$ is a positive multiple of d_i .

Constraints. Throughout the paper, ℓ_0 will denote the line to be pinned. A line meeting ℓ_0 in a single point represents a *constraint* on ℓ_0 . A line ℓ *satisfies* a constraint g if and only if ℓ meets g or passes to the right of g . Consequently, the line ℓ_0 is pinned by a family \mathcal{F} of constraints if there is a neighborhood of ℓ_0 such that ℓ_0 is the only line in this neighborhood satisfying all constraints in \mathcal{F} .

We call a constraint *orthogonal* if it is orthogonal to ℓ_0 . We will give an example of eight orthogonal constraints that form a minimal pinning configuration, showing that the constant eight in Theorem 2 is best possible. Four generically chosen non-oriented lines g_1, \dots, g_4 meeting ℓ_0 and perpendicular to it will have at most two common transversals: if no two among g_1, g_2, g_3 are coplanar or concurrent, their transversals define a hyperbolic paraboloid and it suffices to choose g_4 not lying on this surface. Now, let g_i^+ and g_i^- denote the two oriented lines supported by g_i (see Figure 1). Since a line satisfies g_i^+ and g_i^- if and only if it meets g_i , the eight constraints $g_1^+, g_1^-, \dots, g_4^+, g_4^-$ pin ℓ_0 . Suppose we remove one of the eight constraints, say g_1^+ . The common transversals of the three remaining lines g_2, g_3, g_4 form a hyperbolic paraboloid. By our construction, g_1 intersects this quadric surface transversely (since all four lines are orthogonal to ℓ_0 , g_1 cannot be tangent to the quadric, and since the four lines have at most two transversals, g_1 cannot lie in the quadric). Thus, the quadric of transversals of g_2, g_3, g_4 contains lines on both sides of g_1 , and ℓ_0 is no longer pinned. Therefore the eight oriented lines form a minimal pinning, as claimed.

A space of lines. We choose a coordinate system where ℓ_0 is the positive z -axis, and denote by \mathfrak{L} the family of lines whose direction vector makes a positive dot-product with $(0, 0, 1)$. Since pinning is a local property, we can decide whether ℓ_0 is pinned by considering only lines in \mathfrak{L} as alternate positions for ℓ . (The constraint lines are of course not restricted to \mathfrak{L} .) We identify \mathfrak{L} with \mathbb{R}^4 using the intersections of a line with the planes $z = 0$ and $z = 1$: the point $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ represents the line $\ell(u)$ passing through the points $(u_1, u_2, 0)$ and $(u_3, u_4, 1)$. The line ℓ_0 is represented by the origin in \mathbb{R}^4 .



We describe a constraint using the following three parameters: $g(\lambda, \alpha, \delta)$ denotes the constraint that meets ℓ_0 in the point $(0, 0, \lambda)$, makes slope δ with the plane $z = \lambda$, and projects into the xy -plane in a line making an angle α with the positive x -axis. On the line $g(\lambda, \alpha, \delta)$ we can choose two points $(0, 0, \lambda)$ and $(\cos \alpha, \sin \alpha, \lambda + \delta)$. By (1), a line $\ell(u) \in \mathfrak{L}$ satisfies $g(\lambda, \alpha, \delta)$

within the space of lines \mathfrak{L} if and only if

$$\begin{vmatrix} 0 & \cos \alpha & u_1 & u_3 \\ 0 & \sin \alpha & u_2 & u_4 \\ \lambda & \lambda + \delta & 0 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} \leq 0.$$

The set $\mathcal{U}_g \subset \mathbb{R}^4$ of lines in \mathfrak{L} satisfying the constraint $g = g(\lambda, \alpha, \delta)$ is thus

$$\mathcal{U}_g = \{ u \in \mathbb{R}^4 \mid \zeta_g(u) \leq 0 \}, \quad (2)$$

where $\zeta_g: \mathbb{R}^4 \rightarrow \mathbb{R}$ is defined as $\zeta_g(u) = \delta(u_2u_3 - u_1u_4) + \eta_g \cdot u$ and

$$\eta_g = \eta(\lambda, \alpha) = \begin{pmatrix} (1 - \lambda) \sin \alpha \\ -(1 - \lambda) \cos \alpha \\ \lambda \sin \alpha \\ -\lambda \cos \alpha \end{pmatrix}.$$

Pinning by orthogonal constraints. As a warm-up, we consider pinning by *orthogonal* constraints. If g is an orthogonal constraint, then its parameter $\delta = 0$, so the set \mathcal{U}_g is the halfspace $\eta_g \cdot u \leq 0$ in \mathbb{R}^4 . This reduces the problem of pinning by orthogonal constraints to the following question: when is the intersection of halfspaces in \mathbb{R}^4 reduced to a single point? It is well-known that this is equivalent to the fact that the normal vectors surround the origin as an interior point of their convex hull:

Proposition 1. *A finite collection of halfspaces whose boundaries contain the origin intersects in a single point if and only if their outer normals contain the origin in the interior of their convex hull.*

Here, the interior of the convex hull refers to the interior in the ambient space, not the relative interior (in the affine hull).

Proof. If the intersection of the halfspaces contains a point $a \neq 0$, then all normal vectors n must have $n \cdot a \leq 0$. This implies that all normal vectors n , and hence their convex hull, lie in the halfspace $\{ x \mid x \cdot a \leq 0 \}$. Therefore, the convex hull contains no neighborhood of the origin 0.

Conversely, if the origin is on the boundary or outside the convex hull, there must be a (weakly) separating hyperplane $\{ a \cdot x = b \}$ such that $a \cdot n \leq b$ for all normal vectors n , and $a \cdot x \geq b$ for $x = 0$. This implies $a \cdot n \leq 0$ for all normal vectors n , and thus the point $a \neq 0$ lies in all halfspaces. \square

To analyze *minimal* pinning, we make use of the following classic theorem of Steinitz:

Theorem 4 (Steinitz). *If a point y is interior to the convex hull of a set $X \subset \mathbb{R}^d$, then it is interior to the convex hull of some subset Y of at most $2d$ points of X . The size of Y is at most $2d - 1$ unless the convex hull of X has $2d$ vertices that form d pairs (x, x') with y lying on the segment xx' .*

Equivalently, if the intersection of a family H of halfspaces in \mathbb{R}^d is a single point, then there is a subfamily of H of size at most $2d$ whose intersection is already a single point.

For a proof of Steinitz's Theorem that includes the second statement, we refer to Robinson [14, Lemma 2a]. Theorem 4 immediately implies the following lemma.

Lemma 1. *Any minimal pinning of ℓ_0 by orthogonal constraints has size at most eight.*

The example of Figure 1 shows that the bound is tight. By inspection of the formula for η_g we observe that two normals η_{g_1} and η_{g_2} are linearly dependent if and only if the orthogonal constraints g_1 and g_2 are the same line up to orientation. The second statement in Theorem 4 implies therefore that the set of the four pairs of parallel constraints in Figure 1 is in fact the *only* example of a minimal pinning by eight orthogonal constraints.

Linearizing the constraint sets. If the constraint g is not orthogonal, then the boundary $\zeta_g(u) = 0$ of the set \mathcal{U}_g is a quadric through the origin. Since $\text{grad } \zeta_g(0) = \eta_g$, the vector η_g is the outward normal of \mathcal{U}_g at the origin, and we will call it the *normal* of g . Rather than analyzing the geometry of the intersection of the volumes \mathcal{U}_g bounded by quadrics, we linearize these sets by embedding \mathfrak{L} into five-dimensional space. This is based on the observation that the functions $\zeta_g(u)$ have, up to multiplication by a scalar, the same nonlinear term $u_2u_3 - u_1u_4$. The map $\psi: \mathbb{R}^4 \rightarrow \mathbb{R}^5$ defined as

$$\psi: (u_1, u_2, u_3, u_4) \mapsto (u_1, u_2, u_3, u_4, u_2u_3 - u_1u_4)$$

identifies \mathfrak{L} with the quadratic surface

$$\mathfrak{M} = \{ (u_1, u_2, u_3, u_4, u_5) \mid u_5 = u_2u_3 - u_1u_4 \} \subset \mathbb{R}^5.$$

The image of \mathcal{U}_g under this transformation is

$$\{ (u_1, \dots, u_5) \in \mathfrak{M} \mid \delta u_5 + \eta_g \cdot (u_1, \dots, u_4) \leq 0 \},$$

This leads us to define the five-dimensional halfspace

$$\bar{\mathcal{U}}_g = \bar{\mathcal{U}}_{g(\lambda, \alpha, \delta)} = \{ (u_1, \dots, u_5) \in \mathbb{R}^5 \mid \delta u_5 + \eta_g \cdot (u_1, \dots, u_4) \leq 0 \}.$$

We have $\psi(\mathcal{U}_g) = \bar{\mathcal{U}}_g \cap \mathfrak{M}$. If \mathcal{F} is a family of constraints, then the set of lines in \mathfrak{L} satisfying all constraints in \mathcal{F} is identified with

$$\bigcap_{g \in \mathcal{F}} (\bar{\mathcal{U}}_g \cap \mathfrak{M}) = \left(\bigcap_{g \in \mathcal{F}} \bar{\mathcal{U}}_g \right) \cap \mathfrak{M}. \quad (3)$$

Cones and the Isolation Lemma. We restrict our attention to convex polyhedral cones with apex at the origin, that is, intersections of finitely many closed halfspaces whose bounding hyperplanes go through the origin. We will simply refer to them as *cones*. By (3), the set of lines satisfying a family \mathcal{F} of constraints is represented as the intersection of a cone C and \mathfrak{M} . The family \mathcal{F} pins ℓ_0 if and only if the origin is isolated in $C \cap \mathfrak{M}$. Our next step is to characterize cones C such that the origin is isolated in $C \cap \mathfrak{M}$.

Let $T = \{u_5 = 0\}$ denote the hyperplane tangent to \mathfrak{M} at the origin. Let $T^> = \{u_5 > 0\}$ and $T^< = \{u_5 < 0\}$ be the two open halfspaces bounded by T . Similarly, let $\mathfrak{M}^> = \{u_5 > u_2u_3 - u_1u_4\}$ and $\mathfrak{M}^< = \{u_5 < u_2u_3 - u_1u_4\}$ denote the open regions above and below \mathfrak{M} . We define T^\geq , T^\leq , \mathfrak{M}^\geq , and \mathfrak{M}^\leq analogously; refer to Figure 2. The following lemma classifies how a ray $\{tu \mid t > 0\}$ starting in the origin may be positioned with respect to \mathfrak{M} .

Lemma 2. *Let $u \in \mathbb{R}^5$ be a non-zero vector.*

- (i) *If $u \in T \cap \mathfrak{M}$, then the line $\{tu \mid t \in \mathbb{R}\}$ lies in \mathfrak{M} ;*
- (ii) *If $u \in T \cap \mathfrak{M}^>$, then the line $\{tu \mid t \in \mathbb{R}\}$ is contained in $\mathfrak{M}^> \cup \{0\}$;*
- (iii) *If $u \in T \cap \mathfrak{M}^<$, then the line $\{tu \mid t \in \mathbb{R}\}$ is contained in $\mathfrak{M}^< \cup \{0\}$;*
- (iv) *If $u \in T^>$, there is some $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$, $tu \in \mathfrak{M}^>$;*
- (v) *If $u \in T^<$, there is some $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$, $tu \in \mathfrak{M}^<$.*

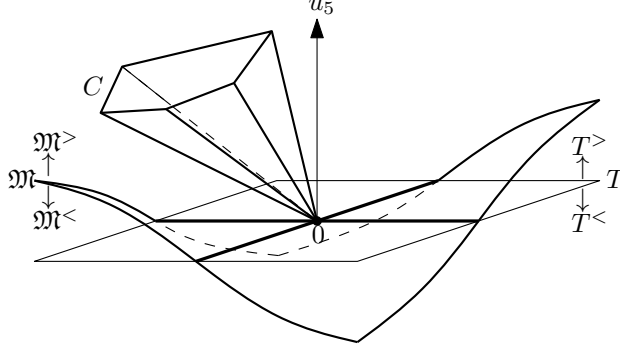


Figure 2: A symbolic drawing of the manifold \mathfrak{M} and its tangent space T at 0 , and a cone C . The hyperplane $T: u_5 = 0$ is shown as a two-dimensional plane, and the intersection $\mathfrak{M} \cap T = \{u_1u_4 = u_2u_3\}$ is represented as two intersecting lines.

Proof. Since \mathfrak{M} is a quadric and T is its tangent hyperplane in the origin, any line in T through the origin either lies in \mathfrak{M} or meets it only in the origin and otherwise stays completely on one side of it. This can be seen directly from the formula for $\mathfrak{M} \cap T$, and it implies statements (i)–(iii). Define now the function $f: x \mapsto x_5 - (x_2x_3 - x_1x_4)$, whose zero set is \mathfrak{M} ; statements (iv)–(v) follow from the observation that $f(tu) = tu_5 - t^2(u_2u_3 - u_1u_4)$, and so the sign of $f(tu)$ is determined, for small t , by the sign of u_5 . \square

We can now characterize the cones whose intersection with \mathfrak{M} contains the origin as an isolated point.

Lemma 3 (Isolation Lemma). *Let C be a cone in \mathbb{R}^5 . The origin is an isolated point of $C \cap \mathfrak{M}$ if and only if either*

- (i) C is a line intersecting T transversely,
- (ii) C is contained in $T^> \cup (T \cap \mathfrak{M}^>) \cup \{0\}$, or
- (iii) C is contained in $T^< \cup (T \cap \mathfrak{M}^<) \cup \{0\}$.

Proof. Assume that the origin is isolated in $C \cap \mathfrak{M}$. First, by Lemma 2 (i), the intersection of C and $T \cap \mathfrak{M}$ is exactly the origin. Thus, $C \subseteq T^> \cup (T \cap \mathfrak{M}^>) \cup T^< \cup (T \cap \mathfrak{M}^<) \cup \{0\}$. Assume, for a contradiction, that C contains both a point $u \in T^> \cup (T \cap \mathfrak{M}^>)$ and a point $v \in T^< \cup (T \cap \mathfrak{M}^<)$, and C is not a line. Since C is not a line, we can ensure that the segment uv does not contain the origin, by perturbing v if necessary. By Lemma 2 (ii)–(v), there is an $\varepsilon > 0$ such that for $t \in (0, \varepsilon)$, $tu \in \mathfrak{M}^>$ and $tv \in \mathfrak{M}^<$. Let $w_t \neq 0$ be the point of \mathfrak{M} on the segment joining tu and tv ; observe that $w_t \in C$, by convexity, and w_t tends to 0 as t goes to 0 , which contradicts the assumption that the origin is isolated in $C \cap \mathfrak{M}$. The condition is therefore necessary.

A line intersecting a quadric transversely meets it in at most two points, so condition (i) is sufficient. Assume condition (ii) holds. If $C = \{0\}$ we are done. Otherwise, the set $A = \{u \in C \mid \|u\| = 1\}$ is compact and nonempty. Let $B = \{u \in \mathfrak{M}^{\leq} \cap T \mid \|u\| \leq 1\}$. Since \mathfrak{M}^{\leq} and T are closed, B is compact, and since $0 \in B$, it is nonempty. Since $C \subset T^> \cup (T \cap \mathfrak{M}^>) \cup \{0\}$, we have $C \cap \mathfrak{M}^{\leq} \cap T = \{0\}$, and thus A and B are disjoint. Let $\tau > 0$ be the distance between A and B . For any $u = (u_1, \dots, u_5) \in C \cap \mathfrak{M} \setminus \{0\}$, we claim that

$$\tau \|u\| \leq u_5 \leq \frac{1}{2} \|u\|^2. \quad (4)$$

The upper bound follows from $u_5 = u_2u_3 - u_1u_4$ and the inequality $xy \leq \frac{x^2+y^2}{2}$ for $x, y \in \mathbb{R}$. For the lower bound, let $v = u/\|u\|$. Since C is a cone, $v \in C$ and so $v \in A$. By assumption, $u \in C \cap \mathfrak{M} \setminus \{0\} \subseteq T^>$, and thus $u_5 > 0$. This implies that the projection $u' = (u_1, \dots, u_4, 0)$

of u on T , is in $\mathfrak{M}^<$. Therefore, by Lemma 2 (iii), $v' = u'/\|u\| \in \mathfrak{M}^<$. Since $\|v'\| \leq \|v\| = 1$, we have $v' \in B$. Thus $\|v - v'\| \geq \tau$, and so $u_5 = \|u - u'\| = \|v - v'\| \cdot \|u\| \geq \tau\|u\|$, completing the proof of (4). Now, (4) implies that any point $u \in C \cap \mathfrak{M}$ other than the origin satisfies $\|u\| \geq 2\tau$. This proves that the origin is isolated in $C \cap \mathfrak{M}$. Condition (ii) is thus sufficient, and the same holds for condition (iii) by symmetry. \square

The tangent hyperplane. The hyperplane T tangent to \mathfrak{M} in the origin plays a special role in the Isolation Lemma. There is a geometric reason for this: $\mathfrak{M} \cap T$ represents exactly those lines of \mathfrak{L} that meet ℓ_0 . Indeed, for $\ell(u)$ to meet ℓ_0 , the two-dimensional points (u_1, u_2) , $(0, 0)$, and (u_3, u_4) have to lie on a line, which is the case if and only if $u_2u_3 - u_1u_4 = 0$. But that is equivalent to $\psi(u) \in T$.

Lemma 4. *If g is a constraint, then $T \cap \mathfrak{M} \cap \partial\bar{\mathcal{U}}_g$ is the union of two two-dimensional linear subspaces.*

Proof. Since $\mathfrak{M} \cap T$ are the lines in \mathfrak{L} that meet ℓ_0 and $\mathfrak{M} \cap \partial\bar{\mathcal{U}}_g$ are the lines in \mathfrak{L} that meet g , $T \cap \mathfrak{M} \cap \partial\bar{\mathcal{U}}_g$ are exactly those lines of \mathfrak{L} that meet both ℓ_0 and g . There are two families of such lines, namely the lines through the point $g \cap \ell_0$, and the lines lying in the plane spanned by ℓ_0 and g . Each family is easily seen to be represented by a two-dimensional linear subspace in \mathbb{R}^5 . \square

Relation to Plücker coordinates. A classic parameterization of lines in space is by means of Plücker coordinates [13]. These coordinates map lines in $\mathbb{P}^3(\mathbb{R})$ to the Plücker quadric $\{x_1x_4 + x_2x_5 + x_3x_6 = 0\} \subset \mathbb{P}^5(\mathbb{R})$. In particular, the oriented line through $(u_1, u_2, 0)$ and $(u_3, u_4, 1)$ has Plücker coordinates

$$(u_3 - u_1, u_4 - u_2, 1, u_2, -u_1, u_1u_4 - u_2u_3),$$

and so the transformation $\mathbb{R}^5 \rightarrow \mathbb{P}^5(\mathbb{R})$ defined as

$$u \mapsto (u_3 - u_1, u_4 - u_2, 1, u_2, -u_1, -u_5)$$

maps $\psi(u)$ to the Plücker coordinates of the line $\ell(u)$. It follows that our manifold \mathfrak{M} is the image of the Plücker quadric under a mapping that sends ℓ_0 to the origin, the hyperplane tangent to the Plücker quadric at ℓ_0 to $\{u_5 = 0\}$, and the lines orthogonal to ℓ_0 to infinity. Our 5-dimensional affine representation has the advantage that \mathfrak{M} admits a parameterization of the form $u_5 = f(u_1, \dots, u_4)$ where f is a homogeneous polynomial of degree two. This is instrumental in our proof of the Isolation Lemma.

We showed that the lines meeting a constraint g are represented by (the intersection of \mathfrak{M} and) a hyperplane in \mathbb{R}^5 . This is no coincidence: the Plücker correspondence implies that this is true for the set of lines meeting *any* fixed line g . Similarly, given any point $u \in \mathfrak{M}$, the hyperplane tangent to \mathfrak{M} at u intersects \mathfrak{M} in exactly those lines that meet the line represented by u —we saw this only for the special hyperplane T . (Of course, both properties can be easily verified in our setting without resorting to Plücker coordinates.)

3 Minimal pinnings by lines have size at most eight

Let $\langle X \rangle$ denote the linear hull of a set $X \subset \mathbb{R}^d$, that is, the smallest linear subspace of \mathbb{R}^d containing X . A j -space is a j -dimensional linear subspace. We start with four lemmas on cones. (Recall that we defined a cone as the intersection of halfspaces whose bounding hyperplanes go through the origin.)

Lemma 5. Let $C = \bigcap_{h \in H} h$ be a cone defined by a family H of halfspaces in \mathbb{R}^d . Then

$$\langle C \rangle = \bigcap_{\substack{h \in H \\ \langle C \rangle \subseteq h}} h = \bigcap_{\substack{h \in H \\ \langle C \rangle \subseteq h}} \partial h.$$

Proof. Clearly $\langle C \rangle \subseteq \bigcap_{h \in H | \langle C \rangle \subseteq h} h$. To show the reverse inclusion, we first pick an arbitrary point x in the relative interior of C . Note that for any halfspace $h \in H$, $x \in \partial h$ implies $\langle C \rangle \subseteq h$.

Let $y \in \bigcap_{h \in H | \langle C \rangle \subseteq h} h$, and consider the segment xy . We show that a neighborhood of x on this segment lies in C , implying $y \in \langle C \rangle$. Indeed, consider $h \in H$. If h contains $\langle C \rangle$ then segment xy is entirely contained in h . If h does not contain $\langle C \rangle$ then x lies in the interior of h , and a neighborhood of x on the segment xy is in h .

The second equality follows from the fact that any linear subspace (in particular, $\langle C \rangle$) that is contained in h must also be contained in ∂h . \square

We also use the following extension of Steinitz's Theorem:

Lemma 6. If the cone defined by a family H of halfspaces in \mathbb{R}^d is a j -space E , then $2d - 2j$ of these halfspaces already define E .

Proof. Every halfspace $h \in H$ contains E , so its bounding hyperplane contains E . Every $h \in H$ can be decomposed as the Cartesian product of E and h 's orthogonal projection on the $(d - j)$ -space F orthogonal to E . A subfamily of H intersects in exactly E if and only if their orthogonal projections intersect in exactly the origin. Since the projection of H on F is a collection of halfspaces that intersect in a single point, by Steinitz's Theorem some $2(d - j)$ of these sets must already intersect in a single point, and the statement follows. \square

In the next two lemmas we consider cones that lie entirely in T_d^\geq , or even in $T_d^> \cup \{0\}$. Since we need these lemmas for arbitrary dimension, we define $T_d = \{x \in \mathbb{R}^d \mid x_d = 0\}$, $T_d^> = \{x \in \mathbb{R}^d \mid x_d > 0\}$, and $T_d^\geq = T_d^> \cup T_d = \{x \in \mathbb{R}^d \mid x_d \geq 0\}$.

Lemma 7. Let $C = \bigcap_{h \in H} h \subseteq T_d^\geq$ be a cone defined by a family H of halfspaces in \mathbb{R}^d . Then

$$\bigcap_{\substack{h \in H \\ \langle C \cap T_d \rangle \subseteq h}} h \subseteq T_d^\geq.$$

Proof. Assume there is a point $y \in T_d^<$ in $\bigcap_{h \in H | \langle C \cap T_d \rangle \subseteq h} h$. Pick a point x in the relative interior of $C \cap T_d$. We consider the segment xy , and show that a neighborhood of x on this segment lies in C , a contradiction to $C \subset T_d^\geq$. Indeed, if $\langle C \cap T_d \rangle \subseteq h$ we have $y \in h$ and so xy lies entirely in h . On the other hand, if $\langle C \cap T_d \rangle \not\subseteq h$, then x lies in the interior of h , and a neighborhood of x lies in h . \square

Lemma 8. Let $C = \bigcap_{h \in H} h$ be a cone defined by a family H of halfspaces in \mathbb{R}^d , with $d \geq 2$, such that no $h \in H$ is bounded by the hyperplane T_d . If C is contained in $T_d^> \cup \{0\}$ then there is a subfamily $H' \subset H$ of size at most $2d - 2$ such that the cone $C' = \bigcap_{h \in H'} h$ defined by H' is contained in $T_d^> \cup \{0\}$.

Proof. The cone C is nonempty, but it does not intersect the hyperplane $F = \{x_d = -1\}$. Helly's theorem thus implies that there is a subset $H_d \subset H$ of size at most d such that $C_d \cap F = \emptyset$, where $C_d = \bigcap_{h \in H_d} h$. Since C_d is a cone, this implies $C_d \subset T_d^\geq$. If $C_d \subset T_d^> \cup \{0\}$, we are done. Otherwise, let $E = \langle C_d \cap T_d \rangle$. Since $C_d \subset T_d^\geq$, $C_d \cap T_d$ is a face of C_d . Since no halfspace in H_d is bounded by T_d , this face cannot be $(d - 1)$ -dimensional, and so the dimension k of E satisfies $1 \leq k \leq d - 2$. For $d = 2$, we already obtain a contradiction, establishing the induction basis for

an inductive proof. So let $d > 2$ and assume that the statement holds for dimensions $2 \leq j < d$. Let $K = \{h \in H_d \mid E \subset h\}$. By Lemma 5 applied to the cone $C_d \cap T_d$, we have $E = \bigcap_{h \in K} h \cap T_d$, and by Lemma 7 we have $\bigcap_{h \in K} h \subset T_d^{\geq}$. For $h \in K$, we define \hat{h} as the projection of h on the $(d - k)$ -space orthogonal to E . Let $\hat{T}_d^>$ be the projection of $T_d^>$. We have $\bigcap_{h \in K} \hat{h} \subset \hat{T}_d^> \cup \{0\}$, and since $2 \leq d - k < d$, the induction hypothesis implies that there is a subset $K' \subset K$ of size at most $2(d - k) - 2$ such that $\bigcap_{h \in K'} \hat{h} \subset \hat{T}_d^> \cup \{0\}$. This implies $\bigcap_{h \in K'} h \subset T_d^> \cup E$. By the original assumption, we have $\bigcap_{h \in H} h \cap E = \{0\}$, and Steinitz's theorem (Theorem 4) inside the subspace E implies that there is a subset $K'' \subset H$ of size at most $2k$ such that $\bigcap_{h \in K''} h \cap E = \{0\}$. Setting $H' = K' \cup K''$ we have $\bigcap_{h \in H'} h \subset T_d^> \cup \{0\}$ with $|H'| \leq 2(d - k) - 2 + 2k = 2d - 2$, completing the inductive step. \square

We are now ready to prove the first half of Theorem 2:

Lemma 9. *Any minimal pinning \mathcal{F} of a line by constraints has size at most eight.*

Proof. Let \mathcal{F} be a minimal pinning of the line ℓ_0 by constraints, let H be the family of halfspaces $\{\bar{U}_g \mid g \in \mathcal{F}\}$, and let C denote the cone $\bigcap_{h \in H} h$. Since \mathcal{F} pins ℓ_0 , the origin is an isolated point of $C \cap \mathfrak{M}$ and we are in one of the cases (i)–(iii) of the Isolation Lemma. In case (i), C is a line. By Lemma 6, C is then equal to the intersection of at most eight halfspaces from H , implying that \mathcal{F} has cardinality at most eight. Without loss of generality, we can now assume that we are in case (ii) of the Isolation Lemma, that is $C \subseteq T^> \cup (T \cap \mathfrak{M}^>) \cup \{0\}$. (Case (iii) is symmetric.)

Let $E = \langle C \cap T \rangle$ and denote the dimension of E by k . As in the proof of Lemma 8, we observe that since T cannot be the boundary of a halfspace in H , we have $E \neq T$ and so $0 \leq k \leq 3$. Let $H' \subset H$ be the set of $h \in H$ with $E \subseteq h$. For $h \in H'$, we define \hat{h} as the projection of h on the $(5 - k)$ -space orthogonal to E . Let \hat{T} , $\hat{T}^>$, and \hat{T}^{\leq} be the projections of T , $T^>$, and T^{\leq} , respectively.

By Lemma 7 we have $\bigcap_{h \in H'} h \subseteq T^{\geq}$. Applying Lemma 5 to the cone $C \cap T$ inside the 4-space T , we have $\bigcap_{h \in H'} h \cap T = E$. Together this implies that $\bigcap_{h \in H'} \hat{h} \subset \hat{T}^> \cup \{0\}$. Applying Lemma 8 in the $(5 - k)$ -dimensional subspace orthogonal to E , we have a subfamily $K \subset H'$ of size at most $2(5 - k) - 2 = 8 - 2k$ such that $\bigcap_{h \in K} \hat{h} \subset \hat{T}^> \cup \{0\}$, implying that $\bigcap_{h \in K} h \subset T^{\geq}$ and $\bigcap_{h \in K} h \cap T = E$.

We have thus found a small subset K of constraints that prevent C from entering $T^<$. By case (ii) of the Isolation Lemma, we still have to ensure that the part of the cone that lies within T does not enter \mathfrak{M}^{\leq} , by an appropriate set of additional constraints. We give a direct geometric argument in the k -dimensional subspace E , arguing separately for each possible value of k :

- If $k = 0$, then $E = \{0\}$ and the at most $8 - 0 = 8$ constraints in K already pin ℓ_0 .
- If $k = 1$, then E is a line contained in T . By Lemma 2 (ii), that line intersects \mathfrak{M} in a single point, which is the origin. Thus the at most $8 - 2 = 6$ constraints in K already pin ℓ_0 .
- If $k = 2$ then $C \cap T$ is a plane, a halfplane, or a convex wedge lying in the 2-space E . We can pick at most two constraints h_1, h_2 in H such that $(\bigcap_{h \in K} h) \cap h_1 \cap h_2 \cap T = C \cap T$. Lemma 3 now implies that the at most $8 - 4 + 2 = 6$ constraints $K \cup \{h_1, h_2\}$ pin ℓ_0 , since their intersection is contained in $T^> \cup (T \cap C) \subseteq T^> \cup (T \cap \mathfrak{M}^>) \cup \{0\}$.
- If $k = 3$, applying Lemma 5 to the cone $C \cap T$, we find that $E = \bigcap_{h \in H'} (h \cap T)$. By Lemma 6, the 3-space E is the intersection of T and two halfspaces in H . This implies that E is of the form $E = T \cap \partial \bar{U}_{g_0}$ with $g_0 \in \mathcal{F}$. Then, by Lemma 4, $E \cap \mathfrak{M}$ is the union of two 2-spaces E_1 and E_2 that intersect along a line f . These two 2-spaces partition the 3-space E

into four quadrants; since $C \cap T$ intersects \mathfrak{M} only in the origin, it must be contained in one of these quadrants. We project $C \cap T$ along f and obtain a two-dimensional wedge. The boundaries of this wedge are projections of edges of the three-dimensional cone $C \cap T$. Each edge is defined by at most two constraints of the three-dimensional cone $C \cap T$, and thus we can find at most four constraints K' of $C \cap T$ that define the same projected wedge, and thus ensure that $\bigcap_{h \in K \cup K'} h \cap T \subseteq (T \cap \mathfrak{M}^>) \cup \{0\}$. Adding these edges to K , we obtain a family of at most $8 - 6 + 4 = 6$ constraints that pin ℓ_0 . \square

All cases in the proof can actually occur—we will see examples in Section 5 when we understand the geometry of pinning configurations better. The example of Figure 1 shows that the constant eight is indeed best possible. However, a look at the proof of Lemma 9 shows that in the case were $C \cap T \neq \{0\}$ (that is, when $k = 1, 2, 3$), a minimal pinning has size at most six. We will make use of this fact later, in the proof of Lemma 19.

The first statement of Theorem 1 (that is, the general case) follows from Lemma 9. The reader interested only in obtaining a finite bound for minimal pinnings by polytopes can skip Sections 4 and 5.

4 Minimal pinning configurations by orthogonal constraints

In this section, we geometrically characterize minimal pinning configurations for a line by orthogonal constraints. We consider our representation of linespace by \mathbb{R}^4 , where the volume of lines satisfying an orthogonal constraint is a halfspace having the origin on its boundary. Recall that, a family of halfspaces through the origin intersects in a single point if and only if the convex hull of their normal vectors contains the origin in its interior. We say that a set of points in \mathbb{R}^d *surrounds the origin* if the origin is in the interior of their convex hull. A set N *minimally surrounds* the origin if N surrounds the origin, but no proper subset of N does.

We first give a description of minimal families of points surrounding the origin in \mathbb{R}^4 as the union of simplices surrounding the origin in a linear subspace (Theorem 5). We then characterize such simplices that can be realized by normals of orthogonal constraints (Lemma 16), and our classification follows (Theorem 6).

4.1 Points minimally surrounding the origin in \mathbb{R}^4

A point set S *surrounds the origin in a linear subspace E* if S spans E and the origin lies in the relative interior of $\text{conv}(S)$. We note that this is true if and only if every point $y \in E$ can be written as $y = \sum_{x \in S} \lambda_x x$ with all $\lambda_x \geq 0$.

A *simplex* of dimension k , or *k -simplex*, is a set of $k + 1$ affinely independent points in \mathbb{R}^d ; we also say *segment* for $k = 1$, *triangle* for $k = 2$, and *tetrahedron* for $k = 3$. We say that a simplex N is *critical* if it surrounds the origin in its linear hull $\langle N \rangle$. Observe that if S and T are two critical simplices then $S \not\subset T$.

Lemma 10. *Let N be a point set with $0 \in \text{conv}(N)$. Then N contains a critical simplex.*

Proof. We simply take a simplex S in N of smallest dimension such that $0 \in \text{conv}(S)$. Such a simplex must exist by Carathéodory's Theorem. \square

Lemma 11. *Let A be a set that minimally surrounds the origin.*

- (i) *For any critical simplex $X \subset A$, no point of $A \setminus X$ lies in the linear hull of X .*
- (ii) *If a critical simplex $X \subset A$ contains k points of a critical k -simplex $Y \subset A$ then $X = Y$.*

Proof. (i) Assume that $p \in A \setminus X$ lies in the linear hull of X . We can then write $p = \sum_{x \in X} \lambda_x x$ with $\lambda_x \geq 0$. Let $A' = A \setminus \{p\}$. Since A surrounds the origin, every point $q \in \mathbb{R}^d$ can be written as $q = \sum_{a \in A} \mu_a a$ with $\mu_a \geq 0$. But this can be rewritten as

$$q = \mu_p p + \sum_{a \in A'} \mu_a a = \sum_{a \in X} \mu_p \lambda_x x + \sum_{a \in A'} \mu_a a,$$

and so q can be expressed as a non-negative sum of the elements of A' . But then A' surrounds the origin, and A is not minimal, a contradiction that proves statement (i).

(ii) If $Y \subset A$ is a critical simplex then any point $p \in Y$ lies in the linear hull of $Y \setminus \{p\}$. Thus, if $X \subset A$ is another critical simplex that contains $Y \setminus \{p\}$, then $p \in \langle X \rangle$, and (i) implies $p \in X$. Thus $Y \subset X$, and hence $Y = X$. \square

Our goal is to describe sets minimally surrounding the origin as unions of (not necessarily disjoint) critical simplices. Our first step is the following ‘‘decomposition’’ lemma.

Lemma 12. *Let A be a critical simplex of dimension at most $d - 1$, let $E = \langle A \rangle$ be its linear hull, and let B be a set of points in \mathbb{R}^d .*

- (i) *$A \cup B$ surrounds the origin in \mathbb{R}^d if and only if the orthogonal projection of B on the orthogonal complement E^\perp of E surrounds the origin in E^\perp .*
- (ii) *If $A \cup B$ minimally surrounds the origin in \mathbb{R}^d then the orthogonal projection of B on the orthogonal complement E^\perp of E minimally surrounds the origin in E^\perp .*
- (iii) *If $\text{conv}(B) \cap E \neq \emptyset$, and $\text{conv}(X) \cap E = \emptyset$ for every $X \subsetneq B$, then B is contained in a critical simplex of $A \cup B$.*

Proof. (i) Let π denote the orthogonal projection on E^\perp . If $A \cup B$ surrounds the origin, then every point $y \in E^\perp \subset \mathbb{R}^d$ can be written as $y = \sum_{x \in A \cup B} \lambda_x x$ with $\lambda_x \geq 0$. Since $\pi(x) = 0$ for $x \in A$, we have

$$y = \pi(y) = \sum_{x \in A \cup B} \lambda_x \pi(x) = \sum_{x \in B} \lambda_x \pi(x),$$

so $\pi(B)$ surrounds the origin in E^\perp .

Assume now that $\pi(B)$ surrounds the origin in E^\perp , and consider a point $y \in \mathbb{R}^d$. Since $\pi(y) \in E^\perp$, we can write $\pi(y) = \sum_{x \in B} \lambda_x \pi(x)$ with $\lambda_x \geq 0$. Let $w = y - \pi(y) + \sum_{x \in B} \lambda_x (\pi(x) - x)$. Since $w \in E$, we can express w as $w = \sum_{x \in A} \mu_x x$ with $\mu_x \geq 0$. Thus

$$y = [y - \pi(y)] + \pi(y) = w - \sum_{x \in B} \lambda_x (\pi(x) - x) + \sum_{x \in B} \lambda_x \pi(x) = \sum_{x \in A} \mu_x x + \sum_{x \in B} \lambda_x x,$$

and $A \cup B$ surrounds the origin.

(ii) If B has a proper subset C such that $\pi(C)$ already surrounds the origin in E^\perp then $A \cup C$ surrounds the origin in \mathbb{R}^d by claim (i). Thus, if $A \cup B$ minimally surrounds the origin in \mathbb{R}^d then $\pi(B)$ minimally surrounds the origin in E^\perp .

(iii) Let \mathcal{A} be the set of facets of A , and let x be an arbitrary point in $\text{conv}(B) \cap E$. Then $\bigcup_{F \in \mathcal{A}} \text{conv}(F \cup \{x\})$ covers $\text{conv}(A)$, and so $0 \in \text{conv}(F \cup \{x\})$ for some facet F of A . This implies $0 \in \text{conv}(F \cup B)$, so by Lemma 10 some simplex $T \subset B \cup F$ is critical. We will conclude the proof by showing that $B \subset T$. Let $X = B \cap T$. Since $0 \in \text{conv}(T) \subset \text{conv}(F \cup X)$, we can write $0 = \alpha y + \beta z$, where $\alpha, \beta \geq 0$, $y \in \text{conv}(F)$ and $z \in \text{conv}(X)$. Since $0 \notin \text{conv}(F)$ we must have $\beta z \neq 0$. From $0 \in E$ and $\alpha y \in E$, we have $\beta z \in E$, which implies $z \in E$, and so $\text{conv}(X) \cap E \neq \emptyset$. By assumption, we now have $X = B$ and therefore $B \subset T$. \square

We will now describe any minimal set surrounding the origin in up to four dimensions as the union of critical simplices. As a warm-up exercise, we state without proof the following lemma about minimal sets surrounding the origin in one or two dimensions, since this will be used in the proof for four dimensions. (The lemma can easily be proven directly, or along the lines of the proof of Theorem 5 below.)

Lemma 13. *A point set that minimally surrounds the origin in \mathbb{R}^1 consists of a positive and a negative point. A point set that minimally surrounds the origin in \mathbb{R}^2 is either a triangle with the origin as an interior point, or a convex quadrilateral with the origin as the intersection point of the diagonals.* \square

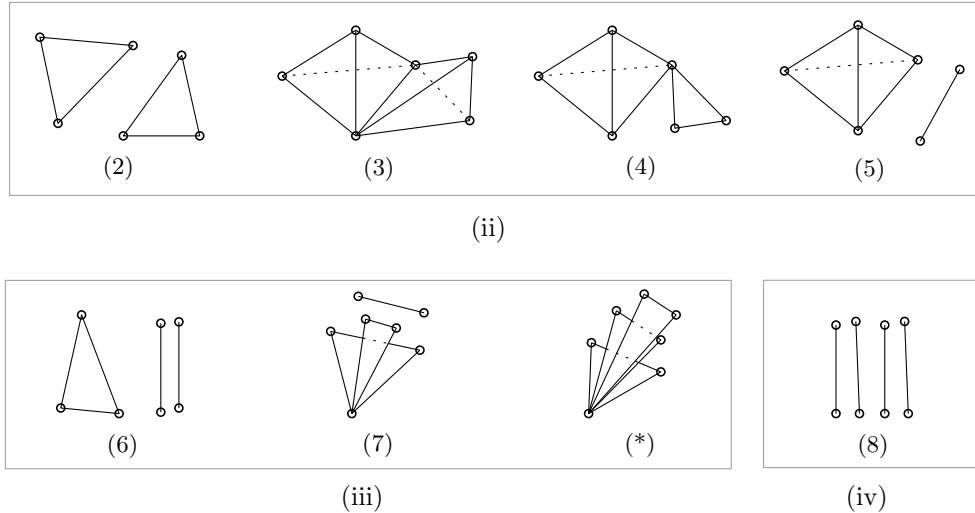


Figure 3: Combinatorial description of nongeneric minimal sets of points surrounding the origin in \mathbb{R}^4 according to Theorem 5.

We now turn to point sets minimally surrounding the origin in \mathbb{R}^4 . The generic case (1) is a 4-simplex surrounding the origin; the remaining cases (2)–(8) are depicted in diagram form in Figure 3. The case (*) has a special label as we will see in Theorem 6 that it cannot be realized by normals of orthogonal constraints. Note that we are not claiming that the critical simplices shown are *all* critical simplices of the point set (although we are not aware of a situation that has additional critical simplices).

Theorem 5. *A set S minimally surrounds the origin in \mathbb{R}^4 if and only if the linear hull of S is \mathbb{R}^4 and one of the following holds: (i) $|S| = 5$ and S is a critical 4-simplex, or (ii) $|S| = 6$ and S is the union of two critical simplices, each of dimension at most three, or (iii) $|S| = 7$ and S is the union of three critical simplices: $k \geq 1$ critical triangles having a single point in common and $3 - k$ disjoint critical segments, or (iv) $|S| = 8$ and S is the disjoint union of four critical segments.*

Proof. We first prove that these cases exhaust all possibilities. Let S be a minimal set of points that surrounds 0. By Lemma 10, some simplex of S is critical. Let A be such a simplex with maximal cardinality, and denote by $B = S \setminus A$ the remaining points of S . Let $E = \langle A \rangle$ denote the linear hull of A and E^\perp the orthogonal complement of E . Let G be the affine hull of B . The linear hull of $E \cup G$ is \mathbb{R}^4 , as otherwise S cannot surround the origin. We consider various cases depending on the cardinality of A .

If $|A| = 5$, then A surrounds the origin and by minimality of S we are in case (i).

If $|A| = 4$, then E is a 3-space and E^\perp is a line. By Lemmas 12 (ii) and 13, B consists of exactly two points, one on each side of E . Since $B \cap E = \emptyset$ and $\text{conv}(B)$ meets E , Lemma 12 (iii) implies that B is contained in some critical simplex $T \subset S$. Since A was chosen with maximal cardinality, we have $|T| \leq 4$, and we are in case (ii) (cases (3)–(5) of Figure 3).

If $|A| = 2$, any critical simplex of S has size exactly two. It is easy to see that there must be exactly four critical segments, and we are in case (iv) (case (8) of Figure 3).

It remains to deal with the case where $|A| = 3$, so that E is a 2-space. Lemmas 12 (ii) and 13 imply that B consists of three or four points.

If $|B| = 3$, then B is a triangle and G is a 2-flat that intersects E in a single point x interior to $\text{conv}(B)$. Since no edge of the triangle B meets E , by Lemma 12 (iii) B is contained in a critical simplex T of S . Since $|T| \leq 3$ we have $T = B$ and S is the disjoint union of two critical triangles. We are thus in case (ii) (case (2) of Figure 3).

If $|B| = 4$, then by Lemma 13 the orthogonal projection of B on E^\perp consists of two critical segments. Thus, B consists of two disjoint pairs, say B_1 and B_2 , whose convex hulls intersect E . By Lemma 12 (iii), each B_i is contained in a critical simplex of S . Since A is of maximal cardinality, we have that B_i either is a critical segment or is contained in a critical triangle $T_i = B_i \cup \{a_i\}$. If at least one B_i is a critical segment, then we are in case (iii) (case (6) or (7) of Figure 3).

Assume finally that both B_i are contained in a critical triangle. If $a_1 \neq a_2$, then $\langle \{a_1, a_2\} \rangle = E$ and $\langle T_1 \cup T_2 \rangle = \mathbb{R}^4$. Since $\langle T_1 \rangle$ and $\langle T_2 \rangle$ are two-dimensional, $\langle T_1 \rangle \cap \langle T_2 \rangle = \{0\}$. It follows that the orthogonal projection of T_2 on $\langle T_1 \rangle^\perp$ is two-dimensional and therefore surrounds the origin in that subspace. Lemma 12 (i) then implies that $T_1 \cup T_2 \subsetneq S$ surrounds the origin, contradicting the minimality of S . Hence, $a_1 = a_2$ and S is the union of three critical triangles with exactly one common point, and we are in case (iii) (case (*) of Figure 3).

We now turn to the converse. We first argue that every point set S according to one of the types (i)–(iv) must surround the origin because $\langle S \rangle = \mathbb{R}^4$ and S is the union of critical simplices: If S does not surround the origin, then there is a closed halfspace h through the origin such that $S \subset h$. Since $\langle S \rangle = \mathbb{R}^4$, there must be a point $p \in S$ in the interior of h . But then the critical simplex $T \subset S$ containing p cannot surround the origin in $\langle T \rangle$, a contradiction.

We proceed to argue the minimality of S : no proper subset of S surrounds the origin. We distinguish the four cases of the theorem.

In case (i), $|S| = 5$ and S is a critical 4-simplex. By definition it surrounds the origin and none of its faces does.

In case (iv), $|S| = 8$ and S is the disjoint union of four critical segments. Since $\langle S \rangle = \mathbb{R}^4$, the directions of these segments are linearly independent, and so S minimally surrounds the origin.

In case (ii), $|S| = 6$. Assume that there is a subset $R \subsetneq S$ that surrounds the origin. Since $|R| \leq 5$, our necessary condition implies that R is a 4-simplex. Let p be the point in S not in R . Since S is of type (ii), it can be written as $S = A \cup B$ where A and B are critical simplices of dimension at most 3. Since neither A nor B can be contained in R , they both contain p and are not disjoint. Without loss of generality, we can assume that A is a tetrahedron and B is a triangle or a tetrahedron. Let $A' = A \setminus \{p\}$ and $B' = B \setminus \{p\}$ and consider the ray r originating in 0 with direction $p\vec{0}$. Since $A = A' \cup \{p\}$ is a critical simplex, r intersects the relative interior of $\text{conv}(A')$ in a point x_A ; similarly, r intersects the relative interior of $\text{conv}(B')$ in a point x_B . Put $E = \langle A \rangle$ and consider the situation in the 3-space E (see Figure 4). There exists a 2-plane Π that strictly separates, in E , $A' \cup \{x_B\}$ from the origin. The affine hull of B' is either a line intersecting E in x_B (if B is a triangle) or a 2-plane intersecting E in the line qx_B (if B is a tetrahedron), where $\{q\} = A' \cap B'$. (Indeed, if the intersection of the affine hull of B' with E has higher dimension, then $B' \subset E$ and $E = \langle R \rangle$, contradicting our assumption that R surrounds

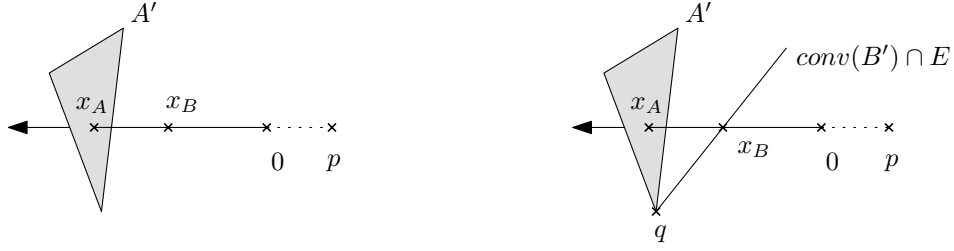


Figure 4: The situation in E in the case where x_B is closer to 0 than x_A (the other case is symmetric): (left) when B is a triangle, (right) when B is a tetrahedron.

the origin.) In either case, there exists a 3-space in \mathbb{R}^4 that strictly separates the origin from $A' \cup B' = R$, so R cannot surround the origin, completing the argument in case (ii).

Finally, in case (iii) $|S| = 7$. Then S can be written as $S = A \cup B \cup C$, where $\{A, B, C\}$ consists of $k \geq 1$ critical triangles sharing exactly one vertex p and $3 - k$ disjoint critical segments. Assume that there exists a proper subset R of S that minimally surrounds the origin. First consider the case where $p \in R$, and note that this implies that any point in $S \setminus R$ belongs to a unique simplex from $\{A, B, C\}$. If R has size 5 it contains one critical simplex from $\{A, B, C\}$, and that contradicts our necessary condition that R be a critical 4-simplex. Thus R must have size 6 and contains two critical simplices from $\{A, B, C\}$, say A and B . Our necessary condition implies that R can be written $R = X \cup Y$ where X and Y are critical simplices of dimension at most 3. Since $|A| \leq 3$, X or Y must contain $|A| - 1$ points from A . Without loss of generality let this be X . Lemma 11 (ii) implies that $X = A$. Then Y contains all points from B except possibly p , and so $Y = B$ by Lemma 11 (ii). Since $A \cup B$ has size at most 5, whereas $X \cup Y = R$ has size 6, we get a contradiction. Now consider the case where $p \notin R$. Let r be the ray originating in 0 with direction vector $p\vec{0}$. Let $Z \in \{A, B, C\}$ and observe that $Z' = Z \cap R$ is either a single point, a critical segment, or a non-critical segment. Moreover, if Z' is a non-critical segment then $Z = Z' \cup \{p\}$ is a triangle and since Z is critical, r intersects the relative interior of $\text{conv}(Z')$. Let G be a 3-space passing through 0, parallel to the segments in $\{A', B', C'\}$ and passing through the points in $\{A', B', C'\}$ (if any). The halfspace in \mathbb{R}^4 bounded by G and containing r has 0 on its boundary and contains R . This implies that R cannot surround 0, a contradiction. \square

Finally, the minimally surrounding sets in \mathbb{R}^3 can easily be obtained from Theorem 5.

Lemma 14. *If a set $S \subset \mathbb{R}^3$ minimally surrounds the origin in \mathbb{R}^3 then either (i) S is a critical tetrahedron, or (ii) S is the union of two critical triangles sharing one point, or (iii) S is the disjoint union of a critical triangle and a critical segment, or (iv) S consists of three disjoint critical segments.*

Proof. S minimally surrounds the origin in \mathbb{R}^3 if and only if S can be extended to a point set minimally surrounding the origin in \mathbb{R}^4 by adding a single critical segment in a transverse direction. We thus obtain all possible configurations from Figure 3, by taking all configurations containing a critical segment, and deleting it. Cases (5), (6), (7), and (8) of Figure 3 are possible, and lead to the four cases of the lemma. \square

4.2 Critical simplices formed by constraints

We now discuss how critical simplices can be formed by the normal vectors of a family of constraints. A necessary condition for a family of normals to form a critical simplex is that they be linearly dependent, but that any proper subset be linearly independent.

We call a family of constraints *dependent* if their normal vectors are linearly dependent. Our first step in rewriting Theorem 5 in terms of constraints is to give a geometric interpretation of dependent families of constraints.

We say that four or more lines ℓ_1, \dots, ℓ_k are in *hyperboloidal position* if they belong to the same family of rulings of a nondegenerate quadric surface. In the case of orthogonal constraints, which are parallel to a common plane, that quadric surface is always a hyperbolic paraboloid.

Lemma 15. *Two orthogonal constraints are dependent if and only if they are identical except possibly for their orientation.*

Three orthogonal constraints are dependent if and only if (i) two of them are, or if (ii) the constraints are coplanar with ℓ_0 , or if (iii) the constraints are concurrent with ℓ_0 .

Four orthogonal constraints are dependent if and only if (i) three of them are, or if (ii) the constraints are in hyperboloidal position, or if (iii) two of the lines are concurrent with ℓ_0 and the other two are coplanar with ℓ_0 .

Proof. Let g_1, \dots, g_k be orthogonal constraints, for $k \in \{2, 3, 4\}$. The normals $\eta_{g_1}, \dots, \eta_{g_k}$ are linearly dependent if and only if they span a linear subspace E of dimension less than k . This is equivalent to saying that the orthogonal complement F of E has dimension larger than $4 - k$. Since $F = \{u \in \mathbb{R}^4 \mid \zeta_{g_i}(u) = 0 \text{ for } 1 \leq i \leq k\}$, F is the space of lines in \mathcal{L} that meet all k constraints g_1, \dots, g_k . If the normals are linearly independent, then F has dimension $4 - k$. If any $k - 1$ normals are linearly independent, but all k are linearly dependent, then any $k - 1$ normals already span the subspace E , and its complement is F . This is equivalent to saying that every constraint must meet all the lines meeting the remaining $k - 1$ constraints.

Consider first two orthogonal constraints g_1 and g_2 . They are dependent if and only if every line meeting g_1 also meets g_2 . This happens if and only if g_1 and g_2 are the same line except possibly for their orientation.

Consider now three orthogonal constraints g_1, g_2, g_3 , and assume that no two are dependent. Then all three are dependent if and only if every line meeting two constraints also meets the third. This is impossible if the constraints are pairwise skew. If two are coplanar with ℓ_0 , say g_1 and g_2 , then their common transversals are exactly the lines in this plane. For g_3 to meet all of them, g_3 has to lie in the same plane as well. Finally, if two constraints, say g_1 and g_2 , meet in a point $p \in \ell_0$ then the common transversals are exactly the lines through p . For g_3 to meet all of them, g_3 has to contain p as well, and the three lines are concurrent with ℓ_0 .

Finally, consider four orthogonal constraints g_1, \dots, g_4 , and assume that no three are dependent. Then every three constraints have a one-dimensional family of common transversals. The four constraints are dependent if and only if every constraint meets every transversal to the other three. If the lines are pairwise skew, then g_4 must lie in the hyperbolic paraboloid formed by the transversals to g_1, g_2, g_3 , and the four lines are in hyperboloidal position. Otherwise, two constraints must be coplanar with ℓ_0 or must meet on ℓ_0 . If two constraints, say g_1 and g_2 , meet in $p \in \ell_0$ then the remaining constraints cannot contain p . The set of transversals to g_1, g_2, g_3 is the set of lines through p and g_3 . For g_4 to meet all these lines, g_4 has to lie in the plane spanned by p and g_3 , and so g_3, g_4, ℓ_0 are coplanar. If no two constraints meet, then two must be coplanar, say g_1 and g_2 , and the constraints g_3 and g_4 intersect the plane spanned by g_1, g_2, ℓ_0 in two distinct points p_1 and p_2 . But there is only one line through p_1 and p_2 , contradicting the assumption that g_1, \dots, g_4 are dependent, and so this case cannot occur. \square

We call a family of constraints a *block* if their normals form a critical simplex. (We will use blocks to build pinning configurations.) The following lemma characterizes blocks geometrically and introduces names for the various types: A k -block consists of k constraints. 3^\times -blocks and 4^\times -blocks contain lines that intersect, while the lines in 3^\parallel -blocks and 4^\parallel -blocks do not intersect. Figure 5 shows all blocks of size at most four.

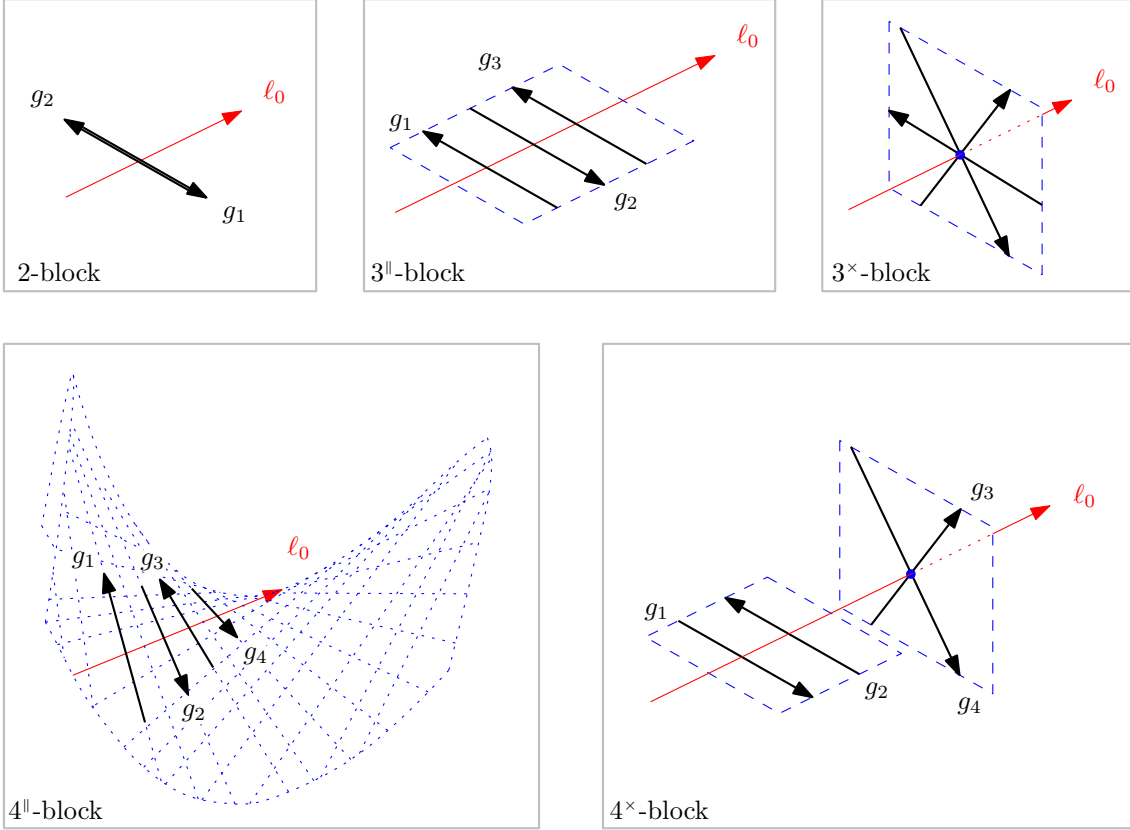


Figure 5: All types of blocks of size at most four.

Lemma 16. *A family of orthogonal constraints forms a block if and only if it contains no proper dependent subfamily and the constraints form one of the following configurations:*

2-block *the two orientations of the same line;*

3-parallel-block *three coplanar constraints with alternating orientations (so the lines are $g(\lambda_1, \alpha, 0)$, $g(\lambda_2, \alpha + \pi, 0)$ and $g(\lambda_3, \alpha, 0)$ with $\lambda_1 < \lambda_2 < \lambda_3$);*

3-times-block *three constraints concurrent with ℓ_0 whose direction vectors positively span ℓ_0^\perp ;*

4-parallel-block *four constraints in hyperboloidal position oriented such that only lines lying in the quadric satisfy all four constraints;*

4-times-block *four constraints $\{g_1, \dots, g_4\}$, where g_1 and g_2 lie in a common plane Π containing ℓ_0 , g_3 and g_4 meet in a point $p \in \ell_0$, and g_1, \dots, g_4 are oriented such that the lines satisfying them are exactly the lines in Π passing through p ;*

5-block *five constraints oriented such that only ℓ_0 satisfies them all.*

Proof. Let \mathcal{F} be a set of $k + 1$ constraints, let $N = \{\eta_g \mid g \in \mathcal{F}\}$ be the set of their normals, and let $E \subset \mathbb{R}^4$ be the set of lines satisfying the constraints in \mathcal{F} . Then N forms a critical k -simplex if and only if E is a linear subspace of dimension $4 - k$.

By construction, this is true for the configurations of constraints described in the lemma. It follows that these configurations are indeed blocks.

It remains to show that if \mathcal{F} is a block, then it falls into one of the six cases. Since every subset of N must be linearly independent, we must have $k \leq 4$. Since N must be linearly dependent, we obtain a first necessary condition from Lemma 15.

If $k = 1$, then we are in the first case of Lemma 15: \mathcal{F} consists of two constraints $\{g_1, g_2\}$ that are either equal or the two orientations of the same line g . Two equal points cannot form a critical segment, so \mathcal{F} is a 2-block.

If $k = 2$, we are in the second case of Lemma 15: \mathcal{F} consists of three constraints $\{g_1, g_2, g_3\}$ that are either coplanar or concurrent with ℓ_0 . If the g_i are coplanar, then E contains the lines in the plane Π spanned by the g_i . This set is already a two-dimensional linear subspace, and so E must be equal to this set. If two constraints met consecutively by ℓ_0 have the same orientation then one of them is redundant, and E contains other lines; and so the orientations must alternate and \mathcal{F} is a 3^{\parallel} -block.

If the g_i are concurrent in some point p on ℓ_0 , then E contains all the lines through p . Again, this set is a two-dimensional linear subspace, and therefore equal to E . Let Π denote the plane containing the three constraints. A line ℓ satisfies a constraint g_i if and only if $\ell \cap \Pi$ lies in the closed halfplane of Π to the right of g_i . The set of points of Π to the right of all three g_i is reduced to $\{p\}$ if and only if the direction vectors of the g_i positively span ℓ_0^\perp . This is a 3^\times -block.

If $k = 3$, we are in the third case of Lemma 15: \mathcal{F} consists either of four constraints in hyperboloidal position, or two lines concurrent with ℓ_0 and two other lines coplanar with ℓ_0 .

If the g_i are in hyperboloidal position, then any line in the other family of rulings of the quadric containing the g_i satisfies all constraints. This set is a line in \mathbb{R}^4 , and so must be identical to E . It follows that \mathcal{F} is a 4^{\parallel} -block. Otherwise, \mathcal{F} consists of two lines g_1 and g_2 coplanar with ℓ_0 , and two lines g_3 and g_4 meeting in a point $p \in \ell_0$. The set E already contains all lines lying in the plane spanned by $g_1 \cup \{p\}$ and passing through p . This set is a line in \mathbb{R}^4 , and therefore identical to E . And so \mathcal{F} is a 4^\times -block.

Finally, if $k = 4$, then N is a 4-simplex surrounding the origin. The five constraints \mathcal{F} are satisfied only by ℓ_0 , and so \mathcal{F} is a 5-block. \square

Remark. Given five orthogonal constraints such that no four are dependent, we can always orient them (that is, reverse some of them) so that they pin ℓ_0 , obtaining a 5-block.

4.3 Characterization of minimal pinning configurations

Combining the descriptions of Theorem 5 with the characterization of Lemma 16, we obtain the following characterization of minimal pinnings of a line by orthogonal constraints. The numbering of cases corresponds to the cases in Figure 3.

Theorem 6. *A family of orthogonal constraints minimally pins ℓ_0 if and only if it forms one of the following configurations:*

- (1) *A single 5-block;*
- (2a) *Two disjoint 3^{\parallel} -blocks defining distinct planes;*
- (2b) *Two disjoint 3^\times -blocks meeting ℓ_0 in distinct points;*
- (3a) *Two 4^{\parallel} -blocks sharing two constraints and defining distinct quadrics;*
- (3b) *Two 4^\times -blocks sharing two constraints, such that their coplanar pairs define distinct planes or their concurrent pairs define distinct points;*
- (3c) *A 4^{\parallel} -block and a 4^\times -block sharing two constraints;*
- (4a) *A 4^{\parallel} -block and a 3^{\parallel} -block sharing one constraint;*
- (4b) *A 4^{\parallel} -block and a 3^\times -block sharing one constraint;*
- (4c) *A 4^\times -block and a 3^{\parallel} -block sharing one constraint such that they define distinct planes;*
- (4d) *A 4^\times -block and a 3^\times -block sharing one constraint such that their concurrent pairs meet ℓ_0 in distinct points;*
- (5a) *A 4^{\parallel} -block and a disjoint 2-block, where the 2-block constraints are not contained in the quadric defined by the 4^{\parallel} -block;*
- (5b) *A 4^\times -block and a disjoint 2-block, where the 2-block constraints are neither coplanar with the coplanar pair nor concurrent with the concurrent pair of the 4^\times -block;*

- (6a) A 3^{\parallel} -block and two 2-blocks, where the four 2-block constraints do not all meet, and where no 2-block constraint is contained in the plane defined by the 3^{\parallel} -block;
- (6b) A 3^{\times} -block and two 2-blocks, where the four 2-block constraints do not all meet, and where no 2-block constraint goes through the common point of the 3^{\times} -block;
- (7) A 3^{\parallel} -block and a 3^{\times} -block sharing one constraint, and a disjoint 2-block that does not lie in the plane of the 3^{\parallel} -block and does not go through the common point of the 3^{\times} -point;
- (8) Four disjoint 2-blocks whose supporting lines are not in hyperboloidal position (that is, all orientations of four lines with finitely many common transversals).

Proof. Let \mathcal{F} be a family of orthogonal constraints and let $N = \{\eta_g \mid g \in \mathcal{F}\}$ be the corresponding family of normals. Recall that \mathcal{F} minimally pins ℓ_0 if and only if N minimally surrounds the origin in \mathbb{R}^4 .

We assume first that \mathcal{F} is of one of the 16 types described in the theorem. For each of the types, we can argue that the line ℓ_0 is pinned, since the sets of lines satisfying each of the blocks have only ℓ_0 as a common element. Since ℓ_0 is pinned, N spans \mathbb{R}^4 . For all the 16 types, the critical simplices of N corresponding to the blocks of \mathcal{F} are of the form described in Theorem 5, so N minimally surrounds the origin, and \mathcal{F} minimally pins ℓ_0 .

It remains to argue the reverse. We assume that \mathcal{F} minimally pins ℓ_0 , that is N minimally surrounds the origin. Then N is of one of the types described in Theorem 5 and shown in Figure 3.

If $|N| = 5$, then N is a critical 4-simplex, and so \mathcal{F} is a 5-block—this is case (1).

If $|N| = 8$, then N is the disjoint union of four critical segments, and so \mathcal{F} is the disjoint union of four 2-blocks. Their supporting lines cannot be in hyperboloidal position (as then \mathcal{F} would not be pinning at all), and so we have case (8).

We now consider $|N| = 6$. If N is the disjoint union of two critical triangles, then we must be in case (2a) or (2b), as the union of a 3^{\parallel} -block and a 3^{\times} -block cannot pin. If N consists of a two critical tetrahedra sharing two constraints, we are in cases (3a)–(3c). If N consists of a critical tetrahedron and a critical triangle sharing one constraint, we are in cases (4a)–(4d). If N consists of critical tetrahedron and a disjoint critical segment, we have cases (5a)–(5b). The additional conditions in cases (2a)–(3b) and (4c)–(4d) hold as otherwise \mathcal{F} would not be pinning at all.

We finally turn to the case $|N| = 7$. Here we need an additional observation: If two 3^{\parallel} -blocks share a constraint, then they both define the same plane. That implies that the critical triangles formed by the normals span the same 2-space, and so these triangles cannot appear together in a point set minimally surrounding the origin by Lemma 11. It follows that \mathcal{F} cannot contain two 3^{\parallel} -blocks sharing a constraint, and the same reasoning applies to two 3^{\times} -blocks. Therefore \mathcal{F} cannot contain three 3-blocks sharing a common constraint, and if it contains two 3-blocks sharing a constraint, then these must be a 3^{\parallel} -block and a 3^{\times} -block. This implies that we must be in cases (6a), (6b) or (7). The additional conditions again hold as otherwise \mathcal{F} would not be pinning. \square

5 Minimal pinning configurations by general constraints

We now return to constraints that are not necessarily orthogonal. As we saw before, the boundary $\zeta_g(u) = 0$ of the set \mathcal{U}_g is a quadric through the origin, with outward normal η_g in the origin. Interestingly, this normal η_g only depends on λ and α , but not on the slope δ . If we consider a constraint $g(\lambda, \alpha, \delta)$ for varying δ , the shape of $\partial\mathcal{U}_g$ changes, but its outward normal in the origin remains the same. In particular, when δ reaches zero, the quadratic term in ζ_g vanishes, and we have $\zeta_g(u) = \eta_g \cdot u$. In other words, linearizing the volume \mathcal{U}_g corresponds to replacing g by its projection g^\perp on the plane perpendicular to ℓ_0 in $\ell_0 \cap g$; \mathcal{U}_{g^\perp} is the halfspace with outer normal

η_g through the origin. We call g^\perp the *orthogonalized constraint* of g , and denote by \mathcal{F}^\perp the family of orthogonalized constraints of \mathcal{F} . (Note that \mathcal{F}^\perp can have smaller cardinality than \mathcal{F} .)

First order pinning. As in the previous section, we call a family of constraints \mathcal{F} *dependent* if their normals are linearly dependent. Since the observation above shows that the normal to a constraint g is identical to the normal of g^\perp , \mathcal{F} is dependent if and only if \mathcal{F}^\perp is dependent. For a characterization of dependent families, we can simply refer to Lemma 15, but that leaves a question: What does it mean geometrically for four constraints if their orthogonalized constraints are in hyperboloidal position? The following lemma clarifies this. Given three pairwise skew lines g_1, g_2, g_3 , we let $\mathcal{B}(g_1, g_2, g_3)$ be the quadric formed by the transversals to the three lines.

Lemma 17. *Two constraints are dependent if and only if they are at the same time coplanar and concurrent with ℓ_0 .*

Three constraints are dependent if and only if (i) two of them are, or if (ii) the constraints are coplanar with ℓ_0 , or if (iii) the constraints are concurrent with ℓ_0 .

Four constraints g_1, g_2, g_3, g_4 are linearly dependent if and only if (i) three of them are, or if (ii) g_4 is tangent to the quadric $\mathcal{B}(g_1, g_2, g_3)$ at $\ell_0 \cap g_4$, or if (iii) two of the lines are concurrent with ℓ_0 and the other two are coplanar with ℓ_0 .

Proof. The normals of constraints are linearly dependent if and only if the normals of their orthogonalized constraints are. This immediately implies the statements for two and three constraints, and nearly implies the statement for four constraints. It remains to show that $g_1^\perp, \dots, g_4^\perp$ are in hyperboloidal position if and only if g_4 is tangent to the quadric $\mathcal{B}(g_1, g_2, g_3)$ at $\ell_0 \cap g_4$.

We can assume that the lines g_1, g_2, g_3 are pairwise skew, so the quadric $\mathcal{B} = \mathcal{B}(g_1, g_2, g_3)$ is defined. Since ℓ_0 is contained in \mathcal{B} , for any point $p \in \ell_0$ there is a unique constraint contained in \mathcal{B} and passing through p . In particular, let $g^* = g(\lambda_4, \alpha^*, \delta^*)$ denote the constraint contained in \mathcal{B} through $p = \ell_0 \cap g_4 = (0, 0, \lambda_4)$. Since the curve $\bigcap_{i=1}^3 \partial \mathcal{U}_{g_i}$ is contained in $\partial \mathcal{U}_{g^*}$, its tangent in the origin, which is orthogonal to the normals $\eta_{g_1}, \dots, \eta_{g_3}$, is contained in the tangent plane to \mathcal{U}_{g^*} at the origin. This implies that η_{g^*} lies in the linear subspace E spanned by $\eta_{g_1}, \dots, \eta_{g_3}$. Let $w \in \mathbb{R}^4$ be such that $E = \{w \cdot u = 0 \mid u \in \mathbb{R}^4\}$. Consider another constraint $g = g(\lambda_4, \alpha, \delta)$ through p . The normal η_g lies in E if and only if

$$0 = w \cdot \eta_g = w \cdot \eta(\lambda_4, \alpha) = \begin{pmatrix} w_1(1 - \lambda_4) + w_3\lambda_4 \\ -w_2(1 - \lambda_4) - w_4\lambda_4 \end{pmatrix} \cdot \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix},$$

that is when $(\sin \alpha, \cos \alpha)$ is orthogonal to a fixed vector. Modulo π , that is, up to reversal of the line, there is only one solution for α , namely when $\alpha = \alpha^*$.

It follows that the normals of g_1, g_2, g_3 and g are linearly dependent if and only if g is coplanar and concurrent with g^* and ℓ_0 . This is true if and only if g is tangent to \mathcal{B} in p . \square

By approximating the volumes \mathcal{U}_g to first order, we get:

Lemma 18. *Let \mathcal{F} be a family of constraints. If \mathcal{F}^\perp pins ℓ_0 then \mathcal{F} pins ℓ_0 . If \mathcal{F} pins ℓ_0 and no four constraints in \mathcal{F} are dependent, then \mathcal{F}^\perp pins ℓ_0 .*

Proof. Since the sets \mathcal{U}_g are bounded by algebraic surfaces of constant degree, the origin 0 is isolated in the intersection of such volumes if and only if there exists no smooth path moving away from 0 inside that intersection. If the tangent vector at 0 to a smooth path γ makes a positive dot product with η_g , then γ locally exits \mathcal{U}_g . If \mathcal{F}^\perp pins ℓ_0 , then the normals to \mathcal{F}^\perp surround the origin and any vector must make a positive dot product with the normal to at least one of the constraints in \mathcal{F}^\perp . Since these are also the normals of the constraints in \mathcal{F} , no

smooth path can move away from 0 inside $\bigcap_{g \in \mathcal{F}} \mathcal{U}_g$, and \mathcal{F} also pins ℓ_0 . The same argument shows that if \mathcal{F} pins ℓ_0 , then $\bigcap_{g^\perp \in \mathcal{F}^\perp} \mathcal{U}_{g^\perp}$ must have empty interior. In that case, if it is not a single point then four of the normals to the constraints in \mathcal{F}^\perp are linearly dependent, and the statement follows. \square

Lemma 18 implies that the minimal pinning examples we gave in the previous section are surprisingly robust: We can start with a family of orthogonal constraints pinning ℓ_0 , rotate each constraint by changing its δ -parameter arbitrarily, and the resulting family will still pin ℓ_0 . However, the lemma does not exclude the possibility that there are pinnings that are not robust in this sense, and indeed this is the case. We saw in the previous section that minimal pinnings by orthogonal constraints consist of between five and eight lines. Surprisingly, it is possible to pin using only *four* non-orthogonal constraints, as we will see below.

Our Isolation Lemma analyzes the intersection of the cone $C = \bigcap_{g \in \mathcal{F}} \bar{\mathcal{U}}_g$ with \mathfrak{M} near 0 in terms of the trace of C on the hyperplane T . Perhaps surprisingly, this trace has a simple geometric interpretation. Consider the map $\phi: \mathbb{R}^4 \rightarrow \mathbb{R}^5$ with $\phi(u_1, u_2, u_3, u_4) = (u_1, u_2, u_3, u_4, 0)$. Clearly, ϕ defines a bijection between \mathfrak{L} and the hyperplane $T = \{u_5 = 0\}$. Moreover, given a constraint g , a line $\ell(u)$ satisfies g^\perp if and only if $\phi(u) \in \bar{\mathcal{U}}_g$. Thus, $\phi^{-1}(C \cap T)$ represents, in our \mathbb{R}^4 parameterization of \mathfrak{L} , those lines that satisfy \mathcal{F}^\perp . In particular, \mathcal{F}^\perp pins ℓ_0 if and only if $C \cap T = \{0\}$. In the proof of Lemma 9, we observed that if \mathcal{F} minimally pins ℓ_0 and $C \cap T \neq \{0\}$ then \mathcal{F} has size at most six. This implies:

Lemma 19. *If \mathcal{F} is a minimal set of constraints pinning ℓ_0 such that \mathcal{F}^\perp does not pin ℓ_0 then \mathcal{F} has size at most six.* \square

Examples of higher-order pinnings. In the proof of Lemma 9, we had considered two cases, namely where $C \cap T = \{0\}$ (this includes the case where C is a line intersecting \mathfrak{M} transversely), and where $E = \langle C \cap T \rangle$ is a k -space with $1 \leq k \leq 3$. In the former case only ℓ_0 satisfies \mathcal{F}^\perp , so \mathcal{F}^\perp pins ℓ_0 . In the latter case, however, all lines in $\phi^{-1}(C \cap T)$ satisfy \mathcal{F}^\perp , and so \mathcal{F}^\perp does not pin. We give a few such examples, also showing that dimensions 1, 2, and 3 are all possible for $E = \langle C \cap T \rangle$.

First, consider the four constraints

$$\begin{aligned} g_0 &= \{(-t, 0, 0) \mid t \in \mathbb{R}\}, \\ g_1 &= \{(t, t, 1) \mid t \in \mathbb{R}\}, \\ g_2 &= \{(-t, -2t, 2) \mid t \in \mathbb{R}\}, \text{ and} \\ g_3 &= \{(t, 3t, 3) \mid t \in \mathbb{R}\}, \end{aligned}$$

oriented in the direction of increasing t . The four constraints lie in the quadric \mathcal{B} defined by the equation $y = xz$ and have alternate orientations, so they form a 4^{||}-block. A line ℓ satisfies the four constraints if and only if it lies in the other family of rulings of \mathcal{B} . These lines are of the form $\{(a, at, t) \mid t \in \mathbb{R}\}$, for $a \in \mathbb{R}$. We now replace g_0 by g'_0 , by rotating the constraint slightly around the origin inside the plane $y = 0$ (the tangent plane to \mathcal{B} in the origin). For concreteness, we choose g'_0 to be the line $\{(-t, 0, -t/100)\}$. This means that for points on g'_0 , we have $y - xz = 0 - t^2/100 < 0$ for $t \neq 0$. This implies that g'_0 touches \mathcal{B} in the origin, and otherwise lies entirely in the volume $y < xz$ bounded by \mathcal{B} . In order to satisfy the four constraints g'_0, g_1, g_2, g_3 , a line near ℓ_0 would have to intersect \mathcal{B} at least three times (points of tangency counted twice), and thus lies in \mathcal{B} as \mathcal{B} is a quadric. A line close to ℓ_0 , but distinct from it, that is contained in \mathcal{B} must violate g'_0 . It follows that the family $\mathcal{F} = \{g'_0, g_1, g_2, g_3\}$ pins ℓ_0 . Since $\mathcal{F}^\perp = \{g_0, g_1, g_2, g_3\}$ does not pin ℓ_0 , this example already shows that the condition of

independence in the second statement of Lemma 18 is necessary. Note that in this example the space $E = \langle C \cap T \rangle$ is one-dimensional (it is the set of transversals to \mathcal{F}^\perp).

Second, consider the family \mathcal{F} consisting of the six lines $g_1 = g(0, -\pi/2, -1)$, $g_2 = g(0, \pi/2, -1)$, $g_3 = g(1, 0, 1)$, $g_4 = g(1, \pi, 1)$, $g_5 = g(2, 3\pi/4, 0)$, and $g_6 = g(2, -\pi/4, 0)$. The family \mathcal{F}^\perp consists of three 2-blocks, that is all orientations of three lines. The set of lines satisfying \mathcal{F}^\perp is the set of transversals to these three lines, again a one-dimensional family. We can verify that \mathcal{F} pins ℓ_0 using the Isolation Lemma. The halfspaces corresponding to the lines are:

$$\begin{aligned}\bar{U}_{g_1} &: -u_1 - u_5 \leq 0, \\ \bar{U}_{g_2} &: u_1 - u_5 \leq 0, \\ \bar{U}_{g_3} &: -u_4 + u_5 \leq 0, \\ \bar{U}_{g_4} &: u_4 + u_5 \leq 0, \\ \bar{U}_{g_5} &: -u_1 - u_2 + 2u_3 + 2u_4 \leq 0, \text{ and} \\ \bar{U}_{g_6} &: u_1 + u_2 - 2u_3 - 2u_4 \leq 0.\end{aligned}$$

The constraints g_1 and g_2 imply $u_5 \geq 0$, and constraints g_3 and g_4 imply $u_5 \leq 0$. Together they enforce $u_5 = 0$ and then $u_1 = u_4 = 0$. Plugging this into the last two constraints we obtain $u_2 = 2u_3$. Since the 1-space $\{(0, 2t, t, 0, 0) \mid t \in \mathbb{R}\}$ intersects \mathfrak{M} in the origin only, \mathcal{F} pins ℓ_0 . None of the constraints is redundant, as can be checked by showing that for each g_i there is a point in $T \cap \mathfrak{M}$ satisfying all but this constraint.

Third, replace the lines g_5 and g_6 in the previous family by $g'_5 = g(2, -3\pi/4, 0)$ and $g'_6 = g(3, \pi/4, 0)$. This produces two different halfspaces

$$\begin{aligned}\bar{U}_{g'_5} &: u_1 - u_2 - 2u_3 + 2u_4 \leq 0, \text{ and} \\ \bar{U}_{g'_6} &: -2u_1 + 2u_2 + 3u_3 - 3u_4 \leq 0.\end{aligned}$$

Plugging $u_1 = u_4 = u_5 = 0$ into these constraints we obtain $-u_2/2 \leq u_3 \leq -2u_2/3$. This is a two-dimensional wedge in the (u_2, u_3) -plane. It follows that $E = \langle C \cap T \rangle$ is the 2-space $u_1 = u_4 = u_5 = 0$. The intersection $E \cap \mathfrak{M}$ consists of the two 1-spaces $u_2 = 0$ and $u_3 = 0$, which intersect the wedge $-u_2/2 \leq u_3 \leq -2u_2/3$ in the origin only. It follows that $\mathcal{F} = \{g_1, g_2, g_3, g_4, g'_5, g'_6\}$ pins ℓ_0 . Again, we can check minimality by verifying that no constraint is redundant.

Finally, we consider the family $\mathcal{F} = \{g_1, g_2, g''_3, g''_4, g''_5, g''_6\}$, where $g''_3 = g(-1, \pi - \tau_3, 0)$, $g''_4 = g(-1/2, -\tau_2, 0)$, $g''_5 = g(1/4, \tau_2, 0)$, and $g''_6 = g(1/3, \pi + \tau_3, 0)$, with $\tau_2 = \arctan 2 \approx 63.4^\circ$ and $\tau_3 = \arctan 3 \approx 71.6^\circ$. The corresponding halfspaces are

$$\begin{aligned}\bar{U}_{g_1} &: -u_1 - u_5 \leq 0, \\ \bar{U}_{g_2} &: u_1 - u_5 \leq 0, \\ \bar{U}_{g''_3} &: 6u_1 + 2u_2 - 3u_3 - u_4 \leq 0, \\ \bar{U}_{g''_4} &: -6u_1 - 3u_2 + 2u_3 + u_4 \leq 0, \\ \bar{U}_{g''_5} &: 6u_1 - 3u_2 + 2u_3 - u_4 \leq 0, \text{ and} \\ \bar{U}_{g''_6} &: -6u_1 + 2u_2 - 3u_3 + u_4 \leq 0\end{aligned}$$

The first two constraints again ensure $u_5 \geq 0$. To construct $C \cap T$, we note that $u_5 = 0$ implies $u_1 = 0$. Plugging $u_1 = u_5 = 0$ into the remaining four constraints, we obtain a three-dimensional cone with apex at the origin that lies in the octant $u_2, u_3, u_4 > 0$. It follows that $E = \langle C \cap T \rangle$ is the 3-space $u_1 = u_5 = 0$. The intersection $E \cap \mathfrak{M}$ consists of the two 2-spaces $u_2 = 0$ and $u_3 = 0$, both of which intersect C only in the origin. It follows that \mathcal{F} pins ℓ_0 , and minimality is verified by checking that no constraint is redundant.

The unique nondegenerate minimal pinning of higher order. Call a pair of constraints $\{g_1, g_2\}$ *degenerate* if they are at the same time coplanar and concurrent with ℓ_0 . This is equivalent to $g_1^\perp = g_2^\perp$ or $\{g_1^\perp, g_2^\perp\}$ forming a 2-block. We do not attempt to give a full characterization of all possible pinning configurations \mathcal{F} where \mathcal{F}^\perp does not pin. However, we observe that all but one of the examples given above contained degenerate pairs of constraints. We can show that the example we gave is in fact unique in this respect.

Theorem 7. *Let \mathcal{F} be a minimal set of constraints pinning ℓ_0 not containing degenerate pairs and such that \mathcal{F}^\perp does not pin ℓ_0 . Then \mathcal{F} consists of exactly four constraints and \mathcal{F}^\perp is a 4^{||}-block.*

Proof. Let $C = \bigcap_{g \in \mathcal{F}} \bar{U}_g$. Since \mathcal{F}^\perp does not pin, we have $C \cap T \neq \{0\}$. Let $E = \langle C \cap T \rangle$, and define $\mathcal{G} = \{g \in \mathcal{F} \mid E \subset \bar{U}_g\}$. By Lemma 5 applied to the cone $C \cap T$, we have $\bigcap_{g \in \mathcal{G}} \bar{U}_g \cap T = E$. Since $\phi^{-1}(\bar{U}_g \cap T) = \mathcal{U}_{g^\perp}$, the set $D = \phi^{-1}(E)$ represents, in our parameterization of \mathfrak{L} by \mathbb{R}^4 , the set of lines satisfying \mathcal{G}^\perp . In other words, \mathcal{G}^\perp is a family of orthogonal constraints such that the set D of lines satisfying \mathcal{G}^\perp is a k -dimensional subspace, where $k \in \{1, 2, 3\}$.

We now argue that $k = 1$ and that D is the set of lines satisfying a 4^{||}-block. Let N denote the set of normals to the constraints in \mathcal{G}^\perp , and observe that N surrounds the origin in the $(4 - k)$ -dimensional subspace orthogonal to D . Since \mathcal{F} contains no degenerate pair, \mathcal{G}^\perp contains no 2-block, and so N contains no critical segment. This immediately implies $k \neq 3$. If $k = 2$, then N surrounds the origin in a 2-space. Since N contains no critical segment, it must contain a critical triangle, and by Lemma 16 D is the set of lines satisfying a 3-block. D is thus either the set of lines lying in fixed plane $\Pi \supset \ell_0$, or the set of lines through a fixed point $p \in \ell_0$. In both cases, all such lines meet ℓ_0 . However, if $\ell(u)$ meets ℓ_0 then $\psi(u) \in T$ and $\psi(u) = \phi(u)$. It follows that $C \cap T \subset E \subset \mathfrak{M}$, and \mathcal{F} cannot pin ℓ_0 , a contradiction. It follows that $k = 1$, so D is one-dimensional and N surrounds the origin in the 3-space orthogonal to D . Since N contains no critical segment, there are two cases by Lemma 14: First, D could be the set of lines satisfying two 3-blocks sharing one constraint, which is the set of lines lying in a fixed plane and going through a fixed point. All such lines meet ℓ_0 , a contradiction. Second, D could be the set of lines satisfying a 4-block. If this is a 4[×]-block, then again all lines in D meet ℓ_0 , a contradiction. We have thus established that D is the set of lines satisfying a 4^{||}-block, that is one family of rulings of a quadric \mathcal{B} , while \mathcal{G}^\perp is a subset of the other family of rulings of \mathcal{B} .

To conclude, we will now show that \mathcal{F} has size four. Since \mathcal{F} pins ℓ_0 and $C \cap T \neq \{0\}$, we are in case (ii) or (iii) of the Isolation Lemma. Without loss of generality, we assume case (ii), and have $C \subset T^\triangleright \cup \{\mathfrak{M}^\triangleright \cap T\} \cup \{0\}$. This implies $C \cap T \subset \mathfrak{M}^\triangleright \cup \{0\}$, and by Lemma 2 (ii) we have $E \subset \mathfrak{M}^\triangleright \cup \{0\}$. By Lemma 7 we have $\bigcap_{g \in \mathcal{G}} \bar{U}_g \subset T^\triangleright$, and since $\bigcap_{g \in \mathcal{G}} \bar{U}_g \cap T = E \subset \mathfrak{M}^\triangleright \cup \{0\}$, this implies $\bigcap_{g \in \mathcal{G}} \bar{U}_g \subset T^\triangleright \cup \{\mathfrak{M}^\triangleright \cap T\} \cup \{0\}$. By Lemma 3, the family \mathcal{G} pins ℓ_0 , and by minimality of \mathcal{F} we have $\mathcal{F} = \mathcal{G}$, and so the constraints in \mathcal{F}^\perp are in hyperboloidal position.

Let $Y = \{u_5 \leq -1\}$. Since $\bigcap_{g \in \mathcal{F}} \bar{U}_g \subset T^\triangleright$, we have $(\bigcap_{g \in \mathcal{F}} \bar{U}_g) \cap Y = \emptyset$. Since $E \subset \partial \bar{U}_g$ for $g \in \mathcal{F}$ and E is parallel to ∂Y , we can project all halfspaces on the 4-space orthogonal to E and apply Helly's theorem there to obtain a five-element subset of $\{\bar{U}_g \mid g \in \mathcal{F}\} \cup \{Y\}$ with empty intersection. Since Y must be one of the five elements, there is a four-element subset $\mathcal{F}_1 \subset \mathcal{F}$ with $\bigcap_{g \in \mathcal{F}_1} \bar{U}_g \subset T^\triangleright$. Consider the cone $C_1 = \bigcap_{g \in \mathcal{F}_1} \bar{U}_g$ and let $E_1 = \langle C_1 \cap T \rangle$. If E_1 is one-dimensional, then $E = E_1$, and \mathcal{F}_1 pins ℓ_0 , implying that $\mathcal{F} = \mathcal{F}_1$. Since \mathcal{F}^\perp is in hyperboloidal position, it is a 4^{||}-block. It remains to consider the case that E_1 is at least two-dimensional. Let $\mathcal{F}_2 = \{g \in \mathcal{F}_1 \mid E_1 \subset \bar{U}_g\}$. By Lemma 5, we have $\bigcap_{g \in \mathcal{F}_2} \bar{U}_g \cap T = E_1$. The normals of the constraints in \mathcal{F}_2^\perp would have to surround the origin in a 1-space or 2-space, but this is impossible for constraints in hyperboloidal position without a 2-block. \square

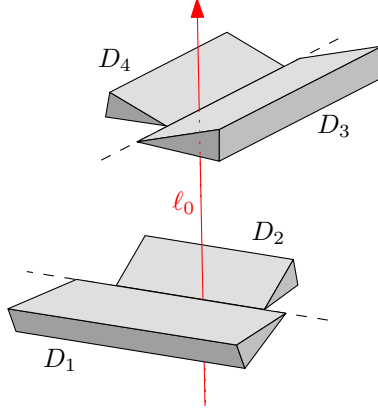


Figure 6: Four polytopes that restrict transversals to a two-dimensional set.

6 Proofs of Theorems 1, 2, and 3

We can now finish the proof of Theorem 2, which states that any minimal pinning of a line by constraints has size at most eight. The bound reduces to six if no two constraints are simultaneously concurrent and coplanar with ℓ_0 , that is form a degenerate pair.

Proof of Theorem 2. The first statement was proven in Lemma 9. For the second statement, consider be a minimal pinning \mathcal{F} of ℓ_0 by constraints, no two forming a degenerate pair. If \mathcal{F}^\perp does not pin ℓ_0 then by Lemma 9 we have that \mathcal{F} has size at most six. If \mathcal{F}^\perp pins ℓ_0 but is not minimal, then some subfamily $\mathcal{G} \subsetneq \mathcal{F}$ is such that \mathcal{G}^\perp pins ℓ_0 ; Lemma 18 yields that \mathcal{G} pins ℓ_0 , a contradiction. If \mathcal{F}^\perp is a minimal pinning of ℓ_0 , since it cannot contain a 2-block (otherwise \mathcal{F} would contain a degenerate pair) it must have size at most six by Theorem 6. \square

We now return to families of convex polytopes that pin a line, and prove Theorem 1, which asserts that any minimal pinning of a line by polytopes in \mathbb{R}^3 has size at most eight if no facet of a polytope is coplanar with the line. The number reduces to six if, in addition, the polytopes are pairwise disjoint.

Proof of Theorem 1. Consider a family \mathcal{F} of convex polytopes pinning the line ℓ_0 , such that no facet of a polytope is coplanar with ℓ_0 . Then, for each polytope $F \in \mathcal{F}$, ℓ_0 intersects F either in the interior of an edge e^F or in a vertex v . In the former case, a line ℓ in a neighborhood of ℓ_0 intersects F if and only if it satisfies a constraint supporting e^F . In the latter case, there are exactly two silhouette edges e_1^F and e_2^F incident to v in the direction of ℓ_0 , and a line ℓ in a neighborhood of ℓ_0 intersects F if and only if it satisfies the two constraints supporting e_1^F and e_2^F . It follows that the family of these constraints pins ℓ_0 , and so Theorem 2 implies that ℓ_0 is already pinned by eight of the constraints. The corresponding at most eight polytopes pin ℓ_0 . If we now make the additional assumption that the polytopes are pairwise disjoint, then no two constraints can be coplanar and concurrent with ℓ_0 . It then follows from the second statement in Theorem 2 that six constraints suffice to pin the line, and the statement for pairwise disjoint polytopes follows. \square

Last, we give a construction of arbitrarily large minimal pinnings of a line by polytopes in \mathbb{R}^3 , proving Theorem 3.

Proof of Theorem 3. We again identify ℓ_0 with the z -axis. We first pick two polytopes D_1 and D_2 such that their common transversals in the vicinity of ℓ_0 are precisely the lines intersecting the y -axis. Similarly, we pick two polytopes D_3 and D_4 that restrict the transversals to pass through the line $\{(t, 0, 1) \mid t \in \mathbb{R}\}$, as in Figure 6. A line $\ell(u)$ meets all four polytopes if and

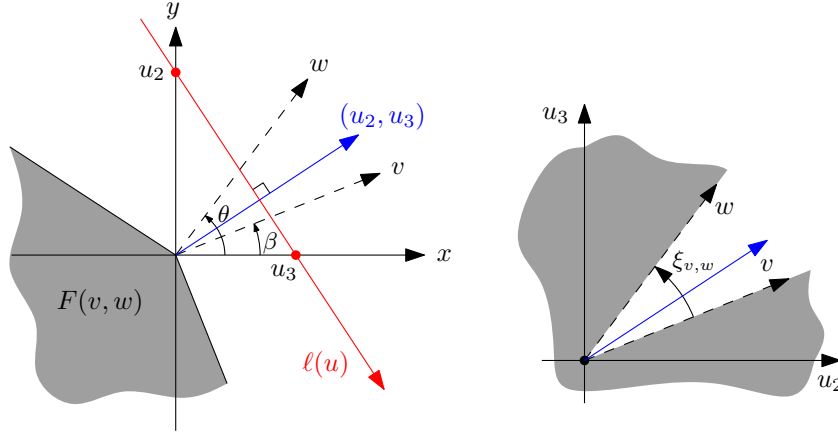


Figure 7: The polyhedron $F(v, w)$ and the set of lines intersecting it.

only if $u_1 = u_4 = 0$. We can therefore analyze the situation in the u_2u_3 -plane. We add two other polytopes D_5 and D_6 (not pictured) to enforce $2u_2 \geq u_3$ and $2u_3 \geq u_2$; these polytopes are bounded by the oriented lines $\{(t, -t, -1) \mid t \in \mathbb{R}\}$ and $\{(t, -t, 2) \mid t \in \mathbb{R}\}$. In the u_2u_3 -plane, the set of lines meeting D_1, \dots, D_6 is the closed wedge $W = \{(u_2, u_3) \mid u_2/2 \leq u_3 \leq 2u_2\}$.

Consider two angles, β and θ , with $0 < \beta < \theta < \pi/2$. Let $v = (v_x, v_y) = (\cos \beta, \sin \beta)$, $w = (w_x, w_y) = (\cos \theta, \sin \theta)$ be two unit vectors, and define the unbounded polyhedral wedge $F(v, w) = \{(x, y, z) \mid v_x x + v_y y \leq 0 \text{ and } w_x x + w_y y \leq 0\}$. The left-hand side of Figure 7 shows a projection along ℓ_0 . A line $\ell(u)$ with $u_2, u_3 > 0$ and $u_1 = u_4 = 0$ misses $F(v, w)$ if and only if the vector $(u_2, u_3) \in \mathbb{R}^2$ falls in the (closed, counterclockwise) acute angular interval $\xi_{v,w} = [\beta, \theta]$ between v and w . In other words, the set of lines intersecting $F(v, w)$ looks like the gray shape in the u_2u_3 -plane depicted on the right hand side of Figure 7, namely the plane with the closed wedge corresponding to $\xi_{v,w}$ removed.

We pick n pairs of vectors $(v^1, w^1), (v^2, w^2), \dots, (v^n, w^n)$, with the property that together the wedges cover W , and such that the middle vector $v^i + w^i$ of $\xi_{v^i w^i}$ lies in W but does not lie in $\xi_{v^j w^j}$ for $j \neq i$. The family $\{F_1, F_2, \dots, F_n\}$ of shapes $F_i = F(v^i, w^i)$ has the property that ℓ_0 is the only line in W intersecting the entire family, but for any $1 \leq i \leq n$ there is an entire sector of such lines that intersect all F_j with $j \neq i$. It follows that the family $\mathcal{F} = \{D_1, \dots, D_6, F_1, \dots, F_n\}$ pins ℓ_0 , but has no pinning subfamily of size smaller than n (some D_i could be redundant, but none of the F_i is).

In the final step, we crop the polyhedral wedges to create a family of bounded polytopes with the same property. Since pinning is determined by lines in a neighborhood of ℓ_0 only, we can clearly make D_1, \dots, D_6 bounded. For each F_i , select the following “linear” path $\gamma_i(t)$ in \mathfrak{L} starting at the origin: $\gamma_i(t) = (u_1(t), u_2(t), u_3(t), u_4(t)) = (0, v_y^i + w_y^i, v_x^i + w_x^i, 0) \cdot t$, for $0 \leq t \leq 1$. In the projection on the xy -plane, the line $\gamma_i(t)$ is, for any $t > 0$, perpendicular to the vector $v^i + w^i$. Therefore $\gamma_i(t)$ does not intersect F_i , but by construction it intersects each F_j with $j \neq i$. Let $P_{ij}(t)$ be the point where $\gamma_i(t)$ enters or exits F_j . A straightforward calculation shows that $P_{ij}(t)$ moves linearly away from ℓ_0 along a line perpendicular to ℓ_0 .

We can therefore crop each F_j to a bounded polytope, ensuring that it still contains all points $P_{ij}(t)$ for $i \neq j$ and $0 < t \leq 1$, and hence intersects all lines on the paths $\gamma_i(t)$ for $i \neq j$. As a result, the family \mathcal{F} still pins ℓ_0 , but for any F_i the family $\mathcal{F} \setminus \{F_i\}$ is not a pinning as witnessed by the path γ_i . \square

7 Concluding remarks

We have shown that minimal pinning by convex polytopes have bounded size if the line ℓ_0 is not coplanar with a facet of a polytope. It seems that this condition can be slightly relaxed: if ℓ_0 intersects a polytope in a vertex v and lies in the plane of a facet incident to v , then one can argue separately in T^{\geq} and T^{\leq} . However, we do not know if the result generalizes any further.

If ℓ_0 meets the relative interiors of two different edges of a polytope facet (and is thus coplanar with this facet), then a line ℓ near ℓ_0 intersects the polytope if and only if it satisfies one of the two line constraints that support the front edge and the back edge of the facet. It follows that the set of lines intersecting the polytope is the union of two “halfspaces.” Since this is not a convex shape, our techniques do not seem to apply.

We have shown that intersecting convex polytopes can have minimal pinning of arbitrary size. We conjecture that *pairwise disjoint* convex polytopes have bounded pinning number. In fact, as mentioned in the introduction, we are not aware of any construction of a minimal pinning by *arbitrary* pairwise disjoint compact convex objects of size larger than six.

Lemma 9 is a key lemma. The most difficult case is when E is three-dimensional, and in fact the proof only works because $E \cap \mathfrak{M}$ must consist of two intersecting 2-spaces. The proof would not go through if $E \cap \mathfrak{M}$ were allowed to be, for example, a circular cone. Perhaps this is an indication that there might be no corresponding result in higher dimensions.

We saw in the introduction that a family of lines pinning a line can be considered as a grasp of that line. In grasping, one often considers *form closure*, which means that the object is immobilized even with respect to infinitesimally small movements. For instance, an equilateral triangle with a point finger at the midpoint of every edge is immobilized, as it cannot be moved in any way, but it is not in form closure because an infinitesimal rotation around its center is possible. It is easy to see that all grasps listed in Theorem 6 are form closure grasps in this sense. The grasp caused by a 4-pinning, however, is not a form-closure grasp, as ℓ_0 can be moved infinitesimally in the quadric defined by three of the lines.

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References

- [1] N. Amenta. Helly-type theorems and generalized linear programming. *Discrete & Computational Geometry*, 12:241–261, 1994.
- [2] C. Borcea, X. Goaoc, and S. Petitjean. Line transversals to disjoint balls. *Discrete & Computational Geometry*, 39:158–173, 2008.
- [3] O. Cheong, X. Goaoc, A. Holmsen, and S. Petitjean. Hadwiger and Helly-type theorems for disjoint unit spheres. *Discrete & Computational Geometry*, 39:194–212, 2008.
- [4] L. Danzer, B. Grünbaum, and V. Klee. Helly’s theorem and its relatives. In V. Klee, editor, *Convexity*, Proc. of Symposia in Pure Math., pages 101–180. Amer. Math. Soc., 1963.

- [5] J. Eckhoff. Helly, Radon and Carathéodory type theorems. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, pages 389–448. North-Holland, 1993.
- [6] J. E. Goodman, R. Pollack, and R. Wenger. Geometric transversal theory. In J. Pach, editor, *New Trends in Discrete and Computational Geometry*, volume 10 of *Algorithms and Combinatorics*, pages 163–198. Springer-Verlag, Heidelberg, Germany, 1993.
- [7] H. Hadwiger. Über Eibereiche mit gemeinsamer Treffgeraden. *Portugaliae mathematica*, 6:23–29, 1957.
- [8] A. Holmsen. Recent progress on line transversals to families of translated ovals. In J. E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry: Twenty Years Later*, volume 453 of *Contemporary Mathematics*, pages 283–298. American Mathematical Society, 2008.
- [9] A. Holmsen and J. Matoušek. No Helly theorem for stabbing translates by lines in \mathbb{R}^d . *Discrete & Computational Geometry*, 31:405–410, 2004.
- [10] X. Markenscoff, L. Ni, and C. H. Papadimitriou. The geometry of grasping. *Internat. J. Robot. Res.*, 9:61–74, 1990.
- [11] M. T. Mason. *Mechanics of Robotic Manipulation*. MIT Press, Cambridge, USA, 2001.
- [12] B. Mishra, J. T. Schwartz, and M. Sharir. On the existence and synthesis of multifinger positive grips. *Algorithmica*, 2:541–558, 1987.
- [13] H. Pottmann and J. Wallner. *Computational Line Geometry*. Springer-Verlag, 2001.
- [14] C. V. Robinson. Spherical theorems of Helly type and congruence indices of spherical caps. *American Journal of Mathematics*, 64:260–272, 1942.
- [15] R. Wenger. Helly-type theorems and geometric transversals. In Jacob E. Goodman and Joseph O’Rourke, editors, *Handbook of Discrete & Computational Geometry*, chapter 4, pages 73–96. CRC Press LLC, Boca Raton, FL, 2nd edition, 2004.