

Approximation and Hardness for Token Swapping

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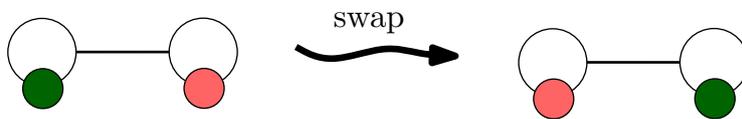
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Two tokens on adjacent vertices swap

Abstract

Given a graph $G = (V, E)$ with $V = \{1, \dots, n\}$, we place on every vertex a token T_1, \dots, T_n . A swap is an exchange of tokens on adjacent vertices. We consider the algorithmic question of finding a shortest sequence of swaps such that token T_i is on vertex i . We are able to achieve essentially matching upper and lower bounds, for exact algorithms and approximation algorithms. For exact algorithms, we rule out any $2^{o(n)}$ algorithm under the ETH. This is matched with a simple $2^{O(n \log n)}$ algorithm based on a breadth-first search in an auxiliary graph. We show one general 4-approximation and show APX-hardness. Thus, there is a small constant $\delta > 1$ such that every polynomial time approximation algorithm has approximation factor at least δ .

Our results also hold for a generalized version, where tokens and vertices are colored. In this generalized version each token must go to a vertex with the same color.

1 Introduction

In the theory of computation, we regularly encounter the following type of problem: Given two configurations, we wish to transform one to the other. In these problems we also need to fix a family of operations that we are allowed to perform. Then, we need to solve two problems: (1) Determine if one can be transformed to the other; (2) If so, find a shortest sequence of such operations. Motivations come from the better understanding of solution spaces, which is beneficial for design of local-search algorithms, enumeration, and probabilistic analysis. See [10] for examples. The study of *combinatorial reconfigurations* is a young growing field [3].

Among problem variants in combinatorial reconfiguration, we study the *token swapping problem* on a graph. The problem is defined as follows. We are given an undirected connected graph with n vertices v_1, \dots, v_n . Each vertex v_i holds exactly one token $T_{\pi(i)}$, where π is a permutation of $\{1, \dots, n\}$. In one step, we are allowed to swap tokens on a pair of adjacent

A condensed version of this report will be presented at the 24th Annual European Symposium on Algorithms in Aarhus, Denmark, in August 2016 [18].

vertices, that is, if v_i and v_j are adjacent, v_i holds the token T , and v_j holds the token T' , then the swap between v_i and v_j results in the configuration where v_i holds T' , v_j holds T , and all the other tokens stay in place. Our objective is to determine the minimum number of swaps so that every vertex v_i holds the token T_i . It is known (and easy to observe) that such a sequence of swaps always exists [23].

We also study a generalized problem called the *colored token swapping problem*. Here every token and every vertex is given a color and we want to find a minimum sequence of swaps such that every token arrives on a vertex with the same color. In case that every token and every vertex is given a unique color, this is equivalent to the uncolored version. This problem was introduced in its full generality by Yamanaka, E. Demaine, Ito, Kawahara, Kiyomi, Okamoto, Saitoh, Suzuki, Uchizawa, and Uno [23], but special cases had been studied before. When the graph is a path, the problem is equivalent to counting the number of adjacent swaps in bubble sort, and it is folklore that this is exactly the number of inversions of the permutation π (see Knuth [17, Section 5.2.2]). When the graph is complete, it was already known by Cayley [5] that the minimum number is equal to n minus the number of cycles in π (see also [15]). Note that the number of inversions and the number of cycles can be computed in $O(n \log n)$ time. Thus, the minimum number of swaps can be computed in $O(n \log n)$ time for paths and complete graphs. Jerrum [15] gave an $O(n^2)$ -time algorithm to solve the problem for cycles. When the graph is a star, a result of Pak [20] implies an $O(n \log n)$ -time algorithm. Exact polynomial-time algorithms are also known for complete bipartite graphs [23] and complete split graphs [25]. Polynomial-time approximation algorithms are known for trees with approximation factor two [23] and for squares of paths with factor two [13]. Since Yamanaka et al. [23], it has remained open whether the problem is polynomial-time solvable or NP-complete, even for general graphs, and whether there exists a constant-factor polynomial-time approximation algorithm for general graphs.

Bonnet, Miltzow, and Rzażewski [2] have recently strengthened our results: Deciding whether a token swapping problem on a graph with n vertices has a solution with k swaps is W[1]-hard with respect to the parameter k , and it cannot be solved in $n^{o(k/\log k)}$ time unless the ETH fails. In addition, they show NP-hardness on graphs with treewidth at most 2. Finally, they complement their lower bounds by showing that token swapping (and some variants of it) are FPT on nowhere dense graphs. This includes planar graphs and graphs of bounded treewidth (still with the number k of swaps as the parameter).

Our results. In this paper, we present upper and lower bounds on the algorithmic complexity of the token swapping problem. For exact computation, we derive a straightforward super-exponential time algorithm in Section 2. We also analyze the algorithm under the parameter k and Δ , where k is the number of required swaps and Δ is the maximum degree of the underlying graph.

Theorem 1 (Simple Exact Algorithm). *Consider the token swapping problem on a graph G with n vertices and m edges. Denote by k the optimal number of required swaps and by Δ the maximum degree of G . An optimal sequence can be found within the following running times:*

1. $O(n! mn \log n) = 2^{O(n \log n)}$
2. $O(m^k n \log n) = O(n^{2k+1} \log n)$
3. $(2k\Delta)^k \cdot O(n \log n)$

The space requirement of these algorithms is equal to the runtime limit divided by $n \log n$.

With more problem specific insights, we could derive polynomial time constant factor approximation algorithms, see Section 3. This resolves the first open problem by Yamanaka et al. [23]. We describe the algorithm for the uncolored version in Section 3. In Section 4, we show a general technique to get also approximation algorithms for the more general colored

version. The approximation algorithm we suggest makes deep use of the structure of the problem and is nice to present.

Theorem 2 (Approximation Algorithm). *There is a 4-approximation of the colored token swapping problem on general graphs and a 2-approximation on trees.*

Our main aim was to complement these algorithmic findings with corresponding conditional lower bounds. For this purpose, we design a gadget called an *even permutation network*¹, in Section 5. This can be regarded as a special class of instances of the token swapping problem. It allows us to achieve *any* even permutation of the tokens between dedicated input and output vertices. Using even permutation networks allows us to reduce further from the colored token swapping problem to the more specific uncolored version. Even permutation networks together with the NP-hardness proof for the *colored* token swapping problem implies NP-hardness of the uncolored token swapping problem. Yamanaka et al. [24] already proved that the colored token swapping problem is NP-complete. This gives a comparably simple way to answer the open question by Yamanaka et al. [24]. However, this reduction has some polynomial blow up and thus does not give tight lower bounds. We believe that even permutation networks are of general interest and we hope that similar gadgets will find applications in other situations.

Our reductions are quite technical and, so, we try to modularize them as much as possible. In the process, we show hardness for a number of intermediate problems. In Section 6, we show a *linear* reduction from 3SAT to a problem of finding disjoint paths in a structured graph. (A precise definition can be found in that section.) Section 7 shows how to reduce further to the colored token swapping problem. Section 8 is committed to the reduction from the colored to the uncolored token swapping problem. For this purpose, we attach an even permutation network to each color class. The blow up remains linear as each color class has *constant* size. Section 9 finally puts the three previous reductions together in order to attain the main results. For our lower bounds we need to use the exponential time hypothesis (ETH), which essentially states that there is no $2^{o(N)}$ algorithm for 3SAT, where N denotes the number of variables.

Theorem 3 (Lower Bounds). *The token swapping problem has the following properties:*

1. *It is NP-complete.*
2. *It cannot be solved in time $2^{o(n)}$ unless the Exponential Time Hypothesis fails, where n is the number of vertices.*
3. *It is APX-hard.*

These properties also hold, when we restrict ourselves to instances of bounded degree.

Therefore, our algorithmic results are almost *tight*. Even though our exact algorithm is completely straightforward our reductions show that we cannot hope for much better results. The algorithm has complexity $2^{O(n \log n)}$ and we can rule out $2^{o(n)}$ algorithms. Thus there remains only a $\log n$ -factor in the exponent remaining as a gap.

In addition, we provide a 4-approximation algorithm and show that there is a small constant c such that there exists no c -approximation algorithm. This determines the approximability status up to the constant c . We want to point out that a crude estimate for the constant c in Theorem 3 is $c \approx 1 + \frac{1}{1000}$. We do not believe that it is worth to compute c exactly. Instead, we hope that future research might find reductions with better constants. Note that all our lower bounds for the uncolored token swapping problem carry over immediately to the colored version.

¹Not to be confused with a sorting network.

Related concepts. The famous 15-puzzle is similar to the token swapping problem, and indeed a graph-theoretic generalization of the 15-puzzle was studied by Wilson [22]. The 15-puzzle can be modeled as a token-swapping problem by regarding the empty square of the 4×4 grid as a distinguished token, but there remains an important difference from our problem: in the 15-puzzle, two adjacent tokens can be swapped only if one of them is the distinguished token. For the 15-puzzle and its generalization, the reachability question is not trivial, but by the result of Wilson [22], it can be decided in polynomial time. However, it is NP-hard to find the minimum number of swaps even for grids [21].

The token swapping problem can be seen as an instance of the minimum generator sequence problem of permutation groups. There, a permutation group is given by a set of generators π_1, \dots, π_k , and we want to find a shortest sequence of generators whose composition is equal to a given target permutation τ . This is the problem that Jerrum [15] studied. He gave an $O(n^2)$ -time algorithm for the token swapping problem on cycles. He proved that the minimum generator sequence problem is PSPACE-complete [15]. The token swapping problem is the special case when the set of generators consists only of transpositions, namely those transpositions that correspond to the edges of G .

In the literature, we also find the problem of *token sliding*, but this is different from the token swapping problem. In the token sliding problem on a graph G , we are given two independent sets I_1 and I_2 of G of the same size. We place one token on each vertex of I_1 , and we perform a sequence of the following sliding operations: We may move a token on a vertex v to another vertex u if v and u are adjacent, u has no token, and after the movement the set of vertices with tokens forms an independent set of G . The goal is to determine if a sequence of sliding operations can move the tokens on I_1 to I_2 . The problem was introduced by Hearn and Demaine [12], and they proved that the problem is PSPACE-complete even for planar graphs of maximum degree three. Subsequent research showed that the token sliding problem is PSPACE-complete for perfect graphs [16] and graphs of bounded treewidth [19]. Polynomial-time algorithms are known for cographs [16], claw-free graphs [4], trees [7] and bipartite permutation graphs [9].

Several other models of swapping have been studied in the literature [11, 8, 6].

2 Simple Exact Algorithms

We start presenting our results with the simplest one. There is an exact, exponential time algorithm which, after the hardness results that we obtain in Section 9, will prove to be almost tight to the lower bounds.

Theorem 1 (Simple Exact Algorithm). *Consider the token swapping problem on a graph G with n vertices and m edges. Denote by k the optimal number of required swaps and by Δ the maximum degree of G . An optimal sequence can be found within the following running times:*

1. $O(n! mn \log n) = 2^{O(n \log n)}$
2. $O(m^k n \log n) = O(n^{2k+1} \log n)$
3. $(2k\Delta)^k \cdot O(n \log n)$

The space requirement of these algorithms is equal to the runtime limit divided by $n \log n$.

Proof. The algorithm is breadth-first search in the configuration graph. The nodes \mathcal{V} of the configuration graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consist of all $n! = 2^{O(n \log n)}$ possible configurations of tokens on the vertices of G . Two configurations A and B are adjacent if there is a single swap that transforms A to B . Each configuration has $m \leq \binom{n}{2} = O(n^2)$ adjacent configurations. Thus the total number of edges is $n!m/2$. A shortest path in \mathcal{G} from the start to the target configuration gives a shortest sequence of swaps. Breadth-first search finds this shortest path.

We assume that the graph fits into main memory and its nodes can be addressed in constant time. We can maintain the set of visited nodes (permutations) in a compressed binary trie, where searches, insertions, and deletions can be done in $O(\log n!) = O(n \log n)$ time. The running time of breadth-first search is thus $O(n \log n) \cdot (|\mathcal{V}| + |\mathcal{E}|) = O(n!mn \log n) = 2^{O(n \log n)}$. This establishes the first claim.

In case that the number of required swaps is k , breadth-first search explores the configuration graph for only k steps. Each configuration has at most $m = O(n^2)$ neighbors. Thus the total number of explored configurations is bounded by $m^k \leq n^{2k}$. This implies the second claim.

For the the third claim, we need the observation that there is an optimal sequence of swaps that swaps only misplaced tokens. Assume that $S = (s_1, \dots, s_k)$ is an optimal sequence of swaps and s_i is a swap, where two tokens are swapped that were at the correct position before the swap. We construct an optimal sequence $S' = (s'_1, \dots, s'_k)$, with one less swap of correctly placed tokens. The new sequence S' skips the swap s_i and adds it to the very end. It is easy to check that S' is indeed a valid sequence of swaps and has one less swap of already correctly placed tokens. Note that if k swaps are sufficient to bring the tokens into the target configuration, at most $2k$ tokens are misplaced.

If the maximum degree of the graph is bounded by Δ , then, by the aforementioned observation, the breadth-first search needs to branch in at most $2k\Delta$ new configurations from each configuration. This implies the third claim. \square

3 A 4-Approximation Algorithm

We describe an approximation algorithm for the token swapping problem, which has approximation factor 4 on general graphs and 2 on trees. First, we state the following simple lemma.

Lemma 4. *Let $d(T_i)$ be the distance of token T_i to the target vertex i . Let L be the sum of distances of all tokens to their target vertices:*

$$L := \sum_{i=1}^n d(T_i).$$

Then any solution needs at least $L/2$ swaps.

Proof. Every swap reduces L by at most 2. \square

We are now ready to describe our algorithm. It has two atomic operations. The first one is called an *unhappy swap*: This is an edge swap where one of the tokens swapped is already on its target and the other token reduces its distance to its target vertex (by one).

The second operation is called a *happy swap chain*. Consider a path of $\ell + 1$ distinct vertices $v_1, \dots, v_{\ell+1}$. We swap the tokens over edge (v_1, v_2) , then (v_2, v_3) , etc., performing ℓ swaps in total. The result is that the token that was on vertex v_1 is now on vertex $v_{\ell+1}$ and all other tokens have moved from v_j to v_{j-1} . If every swapped token reduces its distance by at least 1, we call this a happy swap chain of length ℓ . Note that a happy swap chain could consist of a single swap on a single edge. A single swap that is part of a happy swap chain is called a *happy swap*. When our algorithm applies a happy swap chain, there will be an edge between $v_{\ell+1}$ and v_1 , closing a cycle. In this case, the happy swap chain performs a cyclic shift. Figure 1 illustrates this definition with an example. Note that a happy swap chain might consist of a single swap. We leave it to the reader to find such an example.

Lemma 5. *Let $G = (V, E)$ be an undirected graph with a token placement where not every token is at its target vertex. Then there is a happy swap chain or an unhappy swap.*

Proof. Given a token placement on $G = (V, E)$, we define the directed graph F on V as follows. For each undirected edge $e = \{v, w\}$ of G , we include the directed edge (v, w) in F if the token on v reduces its distance to its target vertex by swapping along e . Note that

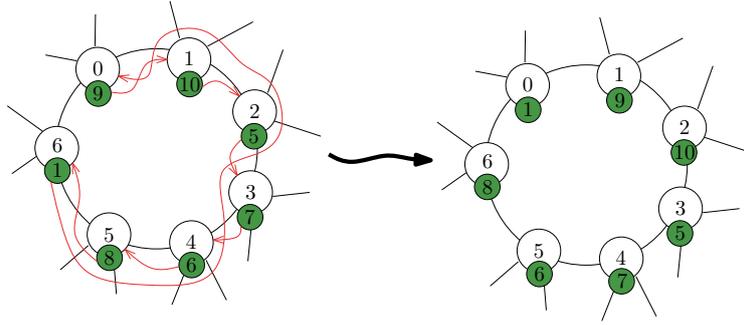


Figure 1: Before and after a happy swap chain. The swap sequence is, in this order, 6 – 5, 5–4, 4–3, 3–2, 2–1, 1–0. Token 1 swaps with every other token, moving counter-clockwise; every other token moves one step clockwise.

for a pair of vertices, both directed edges might be part of F . We can perform a happy swap chain whenever we find a directed cycle in F . The outdegree of a vertex v in F is 0 if and only if the token on v has target vertex v . Assume that not every token is in its target position. Choose any vertex v that does not hold the right token and construct a directed path from v by following the directed edges of F . This procedure will either revisiting a vertex, and we get a directed cycle, or we encounter a vertex with outdegree 0, and we get an unhappy swap. \square

The lemma gives rise to our algorithm: Search for a happy swap chain or unhappy swap; when one is found it is performed, until none remains. If there is no such swap, the final placement of every token is reached. This algorithm is polynomial time (follows from the proof of Lemma 5). Moreover, it correctly swaps the tokens to their target position with at most $2L$ swaps.

Lemma 6. *Let T_i be a token on vertex i . If T_i participates in an unhappy swap, then the next swap involving T_i will be a happy swap.*

Proof. Refer to Figure 2. Let the vertices i, j be the ones participating in the unhappy swap and let e be the edge that connects them. On vertex j is token T_i that got unhappily removed from its target vertex and on vertex j is token S' whose target is neither i nor j . Based on Lemma 5, our algorithm performs either unhappy swaps or happy swap chains. Note that, currently, edge e cannot participate in an unhappy swap. This is because none of its endpoints holds the right token. Moreover, token T_i cannot participate in an unhappy swap that does not involve edge e , as that would not decrease its distance. Therefore, there has to be a happy swap chain that involves token T_i . \square

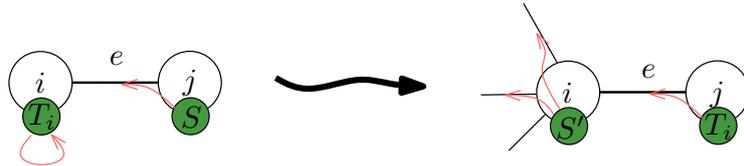


Figure 2: After a Token T_i makes an unhappy swap along edge e , T_i wants to go back to vertex i and is not willing to go to any other vertex. Also whatever token S' will be on vertex i , it has no desire to stay there. This implies that the next swap involving T_i will be part of a happy swap chain.

Theorem 7. *Any sequence of happy swap chains and unhappy swaps is at most 4 times as long as an optimal sequence of swaps on general graphs and 2 times as long on trees.*

Proof. Let L be the sum of all distances of tokens to its target vertex. We know that the optimal solution needs at least $L/2$ swaps as every swap reduces L by at most 2 (Lemma 4). We will show that our algorithm needs at most $2L$ swaps, and this implies the claim.

A happy swap chain of length l reduces the total sum of distances by $l + 1$. Thus

$$\#(\text{happy swaps}) < L.$$

By Lemma 6,

$$\#(\text{unhappy swaps}) \leq \#(\text{happy swaps}),$$

and this implies

$$\#(\text{swaps}) = \#(\text{unhappy swaps}) + \#(\text{happy swaps}) \leq 2 \cdot \#(\text{happy swaps}) < 2L.$$

On trees, this algorithm is a 2-approximation algorithm, as the longest possible cycle in F (as in Lemma 5) has length 2, and thus every happy swap reduces L by *two*. This implies

$$\#(\text{happy swaps}) = L/2. \quad \square$$

4 The Colored Version of the Problem

In this section, we consider the version of the problem when some tokens are indistinguishable. More precisely, each token T_i has a color C_i , each vertex j has a color D_j , and the goal is to let each token arrive at a vertex of its own color. We call this the *colored token swapping problem*. Of course we have to assume that the number of tokens of each color is the same as the number of vertices of that color.

We will see that all approximation bounds from the previous sections carry over to this problem. Our approach to the problem is easy:

1. We first decide which token goes to which target vertex.
2. We then apply one of the algorithms from the previous sections for n distinct tokens.

For Step 1, we solve a bipartite minimum-cost matching problem based on distances in the graph: For each color k separately, we set up a complete bipartite graph between the tokens T_i of color k and the vertices j of color k . The cost d_{ij} of each arc is the graph distance from the starting position of token T_i to vertex j . We solve the assignment problem with these costs and obtain a perfect matching between tokens and vertices. We put the optimal matchings for the different colors together and get a bijection $\pi: \{T_1, \dots, T_n\} \rightarrow V$ between all tokens and all vertices, of total cost

$$L^* = \sum_{i=1}^n d_{i\pi(i)}.$$

Finally, for Step 2, we renumber each vertex $\pi(T_i)$ to i and apply an algorithm for distinct tokens, moving token T_i to the vertex whose original number is $\pi(T_i)$.

The reason why the approximation bounds from the previous sections carry over to this setting is that they are based on the lower bound in Lemma 4 from the sum of the graph distances.

Lemma 8. *Consider any (not necessarily optimal) swap sequence S that solves the colored version of the problem, and let L be the sum of the distances between each token's initial and final position. Then $L \geq L^*$.*

Proof. Since S solves the problem, it must move each token T_i to some vertex $\sigma(T_i)$ of the same color, defined by an assignment $\sigma: \{T_1, \dots, T_n\} \rightarrow V$ which might be different than π . The sum of the distances then is

$$L = \sum_{i=1}^n d_{i\sigma(i)}, \tag{1}$$

must satisfy $L \geq L^*$, since the assignment π was constructed as the assignment minimizing the expression (1) among all assignments σ that respect the colors. \square

The assignment problem on a graph with $N + N$ nodes and arbitrary costs can be solved in $O(N^3)$ time. This must be summed over all colors, giving a total time bound of $O(n^3)$ for Step 1. If the color classes are small, the running time is of course better. Depending on the graph class, the running time may also be reduced. For example, on a path, the optimal matching can be determined in linear time, assuming that the colors are consecutive integers.

Theorem 2 (Approximation Algorithm). *There is a 4-approximation of the colored token swapping problem on general graphs and a 2-approximation on trees.*

5 Even Permutation Networks

Before we dive in the details of the reductions for token swapping (Sections 6-8), we first introduce the even permutation network. This is the gadget we have been advertising in the introduction. We consider it stand-alone and this is why we present it here, independently of the other parts of the reduction.

Consider a family of token swapping instance $I(\pi)$, which depends on a permutation π from some specific set of input vertices V_{in} to some set of output vertices V_{out} . The permutation π specifies, for each token initially on V_{in} the target vertex in V_{out} . For every other token, the target vertex is independent of π . If the optimal number of swaps to solve $I(\pi)$ is the same for every *even* permutation π , we call this family an *even permutation network*, see Figure 3. We will refer to the permutation also as *assignment*. Recall that a permutation is even, if the number of inversions of the permutation is even. Realizing all permutations at the same cost is impossible, since every swap changes the parity, and hence all permutations reachable by a given number of swaps have the same parity.

We will use the even permutation networks later, to reduce from the *colored* token swapping problem to the token swapping problem, see Lemma 13 and Figure 10. The idea is to identify one color class with the input vertices of a permutation network. We use the simple trick of doubling the whole input graph before attaching the permutation networks. We thereby ensure that realizing even permutations is sufficient: the product of two permutations of the same parity is always even.

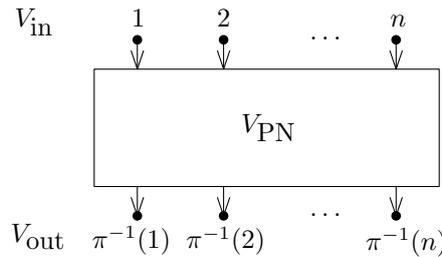


Figure 3: The interface of a permutation network PN

The constructed network has n input vertices V_{in} and n output vertices V_{out} , plus $O(n^3)$ inner vertices V_{PN} . For each token on $V_{\text{PN}} \cup V_{\text{out}}$, a fixed target vertex in $V_{\text{in}} \cup V_{\text{PN}}$ is defined. The target vertices of V_{in} lie in V_{out} , but are still unspecified. In this section, and only in this section, we refer to all tokens initially placed on $V_{\text{PN}} \cup V_{\text{out}}$ as *filler tokens*.

Lemma 9. *For every n , there is a permutation network PN with n input vertices V_{in} , n output vertices V_{out} , and $O(n^3)$ additional vertices V_{PN} , which has the following properties, for some value T :*

- *For every target assignment π between the inputs V_{in} and the outputs V_{out} that is an even permutation, the shortest swapping sequence has length T .*

- For any other target assignment, the shortest realizing sequence has length at least $T + 1$.
- The same statement holds for any extension of the network PN , which is the union of PN with another graph H that shares only the vertices V_{in} with PN , and in which the starting positions of the tokens assigned to V_{out} may be anywhere in H .

The last clause concerns not only all assignments between V_{in} and V_{out} that are odd permutations, but also all other conceivable situations where the tokens destined for V_{out} do not end up in V_{in} , but somewhere else in the graph H . The lemma confirms that such non-optimal solutions for H cannot be combined with solutions for PN to yield better swapping sequences than the ones for which the network was designed.

Proof. The even permutation network will be built up hierarchically from small gadgets. Each gadget is built in a layered manner, subject to the following rules.

1. There is a strict layer structure: The vertices are partitioned into layers U_1, \dots, U_t of the same size.
2. Each gadget has its own input layer $V = U_1$ at the top and its output layer $V' = U_t$ at the bottom, just as the overall network PN .
3. Edges may run between two vertices of the same layer (*horizontal edges*), or between adjacent layers (*downward edges*).
4. Every vertex has at most one neighbor in the successor layer and at most one neighbor in the predecessor layer.

The goal is to bring the input tokens from the input layer V_{in} to the output layer V_{out} . By Rule 3, the cheapest conceivable way to achieve this is by using only the downward edges, and then every such edge is used precisely once.

The gadget in Figure 4 has an additional input vertex a with a fixed destination, indicated by the label a . This assignment forces us to use also horizontal edges, incurring some extra cost as compared to letting the input tokens travel only downward.

It can be checked easily that we only need to consider candidate swapping sequences in which every downward edge is used precisely once. This has the following consequences.

1. We can compare the cost of swapping sequences by counting the horizontal edges used.
2. Each of the filler tokens, which fill the vertices of the graph, moves one layer up to a fixed target location. Thus we can predict the final position of all the auxiliary filler tokens that fill the network.

Let us now look at the gadget of Figure 4. It has three inputs 1,2,3 that connect the gadget with other gadgets, plus an additional input a with a specified target. The initial vertex of a and the target vertex of a are not connected to other parts of the graph. The possible permutations of 1,2,3 that arise on the output are in the table below the network, together with the number $\#h$ of used horizontal edges (i.e., the relative cost).

Let us check this table. To get the token a into the correct target position, we must get it from the fourth track to the third track, using (at least) two horizontal edges (AB or DE or AE). With this minimum cost of 2, we reach the identity permutation 1, 2, 3. If we use edge C , we can also reach the permutations 3, 2, 1 (together with A, B) and 2, 1, 3 (together with D, E), at a cost of 3. All other possibilities that bring the tokens from V_{in} to V_{out} and bring a to the correct target cost more, including “crazy” swapping sequences where the tokens 1, 2, 3, a make intermediate upward moves.

In summary, the gadget can either realize the identity permutation, or it can swap 1 with one of the other tokens at an extra cost of 1. We call this the *swapping gadget*, and we will symbolize it like in the right part of Figure 4, omitting the auxiliary token a .

The next gadget is the *shift gadget*, which is composed of two swapping gadgets as shown in Figure 5. When putting together gadgets, the tracks must be filled up with vertices on

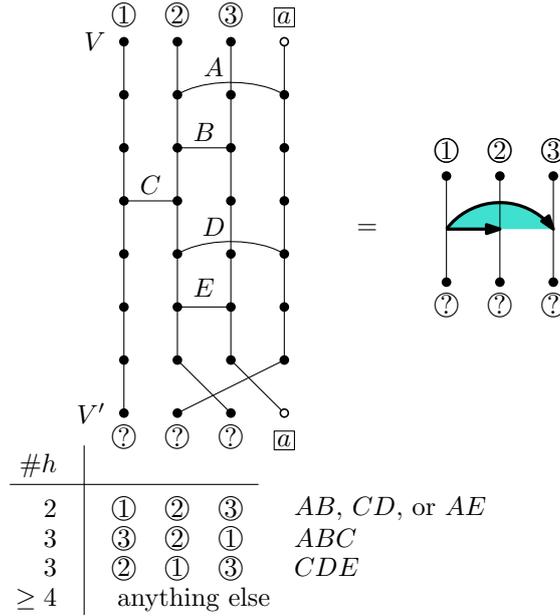


Figure 4: The gadget for swapping 1 with 2 or 3. The number of horizontal edges used is denoted by #h.

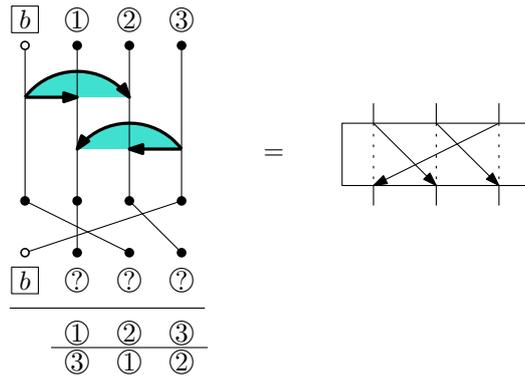


Figure 5: The gadget for an optional cyclic shift of 1,2,3 by one position

straight segments to maintain the strict layer structure, but we don't show these trivial paths in the pictures.

We have a token b with a fixed target, which is not connected to the rest of the graph. To bring b from the first track to the fourth track and hence to the target, we cannot use the identity permutation in the swap gadgets; in both swapping gadgets, we must use one of the more expensive swapping choices. There are two choices that put b on the correct target: The choice “swap $b \leftrightarrow 2$ and then swap $b \leftrightarrow 3$ ” leads to the identity permutation 1, 2, 3. The choice “swap $b \leftrightarrow 1$ and then swap $b \leftrightarrow 3$ ” leads to the cyclic shift 3, 1, 2. We thus conclude that this gadget has two minimum-cost solutions: The identity, or a cyclic shift to the right. The shift gadget is symbolically drawn as a rectangular box as shown in the right part of Figure 5, indicating the optional cyclic shift.

From the shift gadget, we can now build the whole permutation network. The idea is similar to the realization of a permutation as a product of adjacent transpositions, or to the bubble-sort sorting network. Figure 6 shows how a cascade of $n - 1$ shift gadgets can bring any input token to the last position. We add a cascade of $n - 2$ shift gadgets that can bring any of the remaining tokens to the $(n - 1)$ -st position, and so on. In this way, we can bring any desired token into the n -th, $(n - 1)$ -st, \dots , 3rd position. The first two positions are

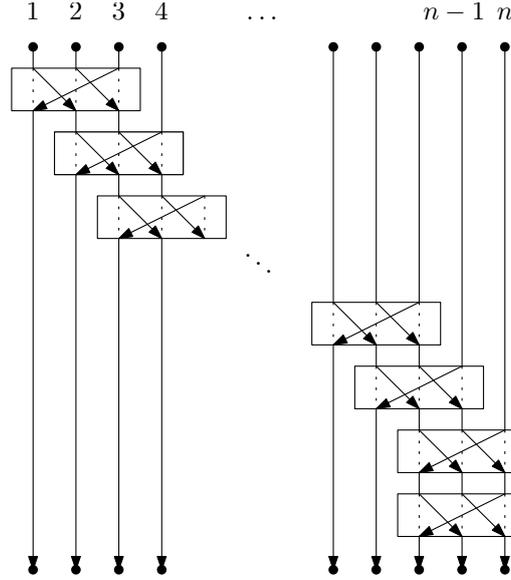


Figure 6: The gadget for bringing any desired token to the last position

then fixed, and thus we can realize half of all permutations, namely the even ones. □

6 Reduction to a Disjoint Paths Problem

For the lower bounds of the Token Swapping problem, we study some auxiliary problems. The first problem is a special multi-commodity flow problem.

Disjoint Paths on a Directed Acyclic Graph (DP)

Input: A directed acyclic graph $G = (V, E)$ and a bijection $\varphi: V^- \rightarrow V^+$ between the sources V^- (vertices without incoming arcs) and the sinks V^+ (vertices without outgoing arcs), with the following properties:

1. The vertices can be partitioned into layers V_1, V_2, \dots, V_t such that, for every vertex in some layer V_j , all incoming arcs (if any) come from the same layer V_i , with $i < j$. Note that i need not be the same for every vertex in V_j .
2. Every layer contains at most 10 vertices.
3. For every $v \in V^-$, there is a path from v to $\varphi(v)$ in G . Let $n(v)$ denote the number of vertices on the shortest path from v to $\varphi(v)$.
4. The total number of vertices is $|V| = \sum_{v \in V^-} n(v)$.

Question: Is there a set of vertex-disjoint directed paths P_1, \dots, P_k with $k = |V^-| = |V^+|$, such that P_i starts at some vertex $v \in V^-$ and ends at $\varphi(v)$?

By Property 4, the k paths must completely cover the vertices of the graph. The graphs that we will construct in our reduction have in fact the stronger property that *any* directed path from $v \in V^-$ to $\varphi(v)$ contains the same number $n(v)$ of vertices.

In our construction, we will label the source and the sink that should be connected by a path by the same symbol X , and “the path X ” refers to this path. In the drawings, the arcs will be directed from top to bottom.

Lemma 10. *There is a linear-size reduction from 3SAT to the Disjoint Paths Problem on a Directed Acyclic Graph (DP).*

Proof. Let x_1, \dots, x_n be the variables and C_1, \dots, C_m the clauses of the 3SAT formula. Each variable x_i is modeled by a variable path, which has a choice between two tracks. The track is determined by the choice of the first vertex on the path after x_i : either x_i^T or x_i^F . This choice models the truth assignment. There is also a path for each clause. In addition, there will be supplementary paths that fill the unused variable tracks. Figure 7 shows an example of a variable x_i that appears in three clauses C_u, C_v , and C_w . Consequently, the two tracks x_i^T and x_i^F , which run in parallel, pass through three clause gadgets, which are shown schematically as gray boxes in Figure 7 and which are drawn in greater detail in Figure 8. The bold path in Figure 7 corresponds to assigning the value *false* to x_i : the path follows the track x_i^F . For a variable that appears in ℓ clauses, there are $\ell + 1$ supplementary paths. In Figure 7, they are labeled s_1, \dots, s_4 . The path s_j covers the unused track (x_i^T in the example) between the $(j - 1)$ -st and the j -th clause in which the variable x_i is involved.

Initially, the path x_i can choose between the tracks x_i^T and x_i^F ; the other track will be covered by a path starting at s_1 . This choice is made possible by a *crossing gadget*. Each variable has two crossing gadgets attached, one at the beginning of the variable path and one at the end. Those gadgets consist of 6 vertices: x_i, s_1, x_i^T, x_i^F and two auxiliary vertices that allow the variable to change tracks. In Figure 7, the crossing gadgets appear at the top and the bottom. Note, that a variable can only change tracks in the crossing gadgets; the last supplementary path $s_{\ell+1}$ allows the path x_i to reach its target sink.

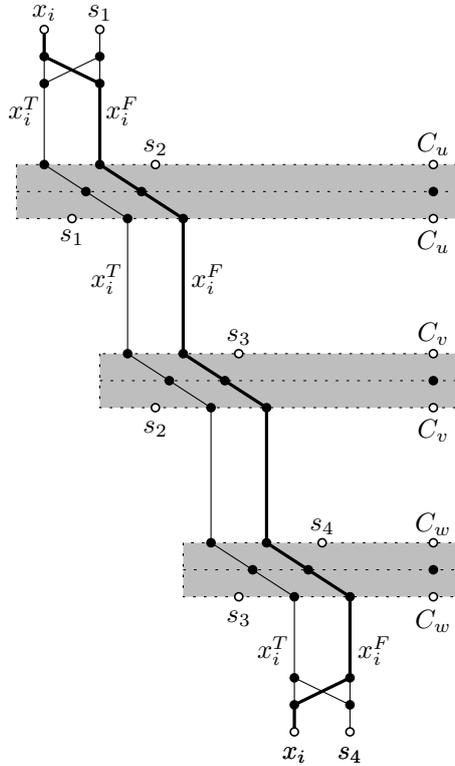


Figure 7: Schematic representation of a variable x_i . Sources and sinks are marked by white vertices, and their labels indicated the one-to-one correspondence φ between sources and sinks. Arcs are directed from top to bottom.

Figure 8 shows a clause gadget C_z in greater detail. It consists of three successive layers and connects the three variables that occur in the clause. The clause itself is represented by a clause path that spans only these three layers. A supplementary path starts at the first layer and one ends at the third layer of each clause gadget. Each layer of the clause gadget has at most 10 vertices: three for each variable, two that are on the tracks x_i^T, x_i^F , one for the supplementary path of the variable. There is at most three variables per clause. Finally, there is a vertex that corresponds to the clause itself.

Let C_z be the current clause which is the j th clause in which x_i appears, and it does so as a positive literal (as in Figure 8a). Each track, say x_i^T , connects to the corresponding vertex in the middle and bottom layer of the clause gadget and to s_j (the end of the supplementary path). The track of the literal that does not appear in this clause (x_i^F in this case) is also connected to the vertex of the clause on the middle layer (z in Figure 8). Moreover, the middle layer vertex of this track is connected to the top and bottom layer vertices that correspond to the clause. The supplementary path s_{j+1} starts in this clause gadget and goes through the middle layer and then connects to the vertices of both tracks on the bottom layer. Figure 8 depicts all these connections.

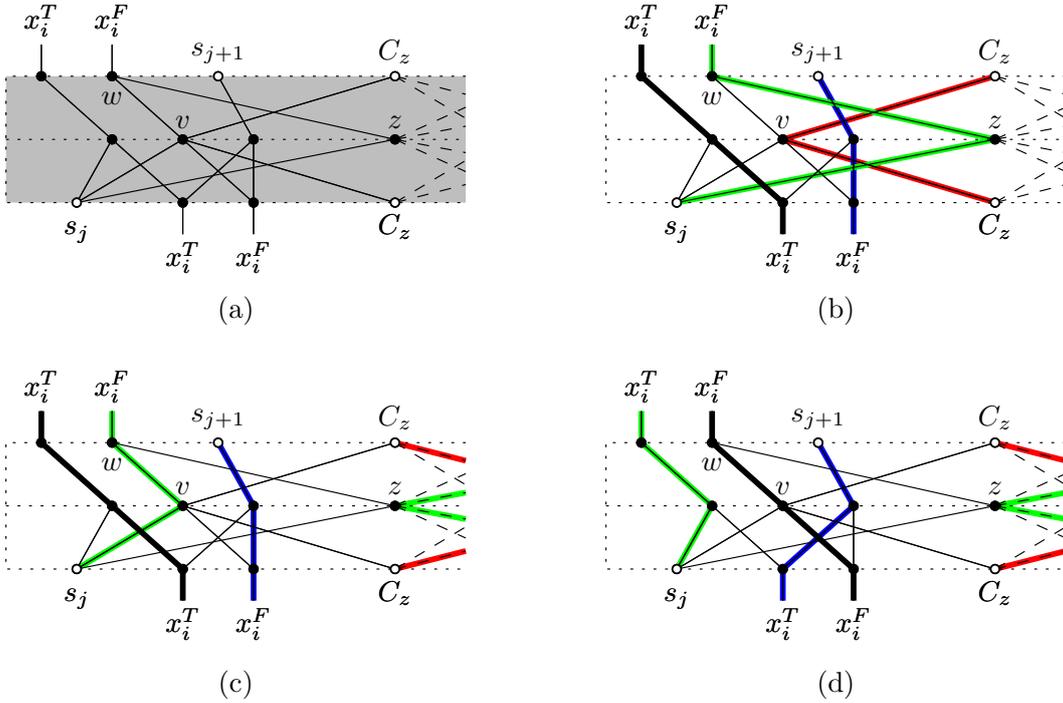


Figure 8: (a) Gadget for a clause C_z containing a variable x_i as a positive literal. The clause involves two other variables, whose connections are indicated by dashed lines. (b–d) The possibilities of paths passing through the gadget: (b) $x_i = \text{true}$, the clause is fulfilled, and the clause path makes its detour via the vertex v . (c) $x_i = \text{true}$, the clause is fulfilled, and the clause path makes its detour via another vertex. (d) $x_i = \text{false}$, this variable does not contribute to fulfilling the clause, and the clause path *has to* make its detour via another vertex. For a negative literal, the detour vertex v and the upper neighbor w of z would be placed on the other track, x_i^T .

We can make the following observations about the interaction between the variable x_i and the clause C_z . We, further, illustrate those in Figure 8.

1. A variable path (shown in black) that enters on the track x_i^T or x_i^F must leave the gadget on the same track. Equivalently, a path cannot change track except in the crossing gadgets.
2. The clause path (in red) can make a detour through vertex v (w.r.t. Figure 8) only if the variable x_i makes the clause true, according to the track chosen by the variable path. In this case, the supplementary path s_j covers the intermediate vertex z of the clause.
3. If variable x_i makes the clause true, the clause path may also choose a different detour, in case more variables make the clause true.

4. The supplementary path s_j (shown in green) can reach its sink vertex.
5. The supplementary path s_{j+1} (in blue) can reach the track the track x_i^T or x_i^F which is not used by the variable path.

Since each clause path *must* make a detour into one of the variables, it follows from Property 2 that a set of disjoint source-sink paths exists if and only if all clauses are satisfiable.

The special properties of the graph that are required for the Disjoint Paths Problem on a Directed Acyclic Graph (DP) can be checked easily. Whenever a vertex has one or more incoming arcs, they come from the previous layer of the same clause gadget. We can assign three distinct layers to each clause gadget and to each crossing gadget. Thereby, we ensure that every layer contains at most 10 vertices. It follows from the construction that the graph has just enough vertices that the shortest source-sink paths can be disjointly packed, but we can also check this explicitly. If there are n variables and m clauses, there are $30m + 12n$ vertices in G : 30 per clause gadget, plus 12 for the two crossing gadgets of each variable. For a variable x_i that appears in ℓ_i clauses, we have $n(x_i) = 3(\ell_i + 2)$. Each clause path C_z contains $n(C_z) = 3$ vertices, and each of the $\sum(\ell_i + 1) = 3m + n$ supplementary paths s_j has length $n(s_j) = 6$. This gives in total $\sum n(v) = 3\sum \ell_i + 6n + 3m + 6(3m + n) = 9m + 6n + 3m + 18m + 6n = 30m + 12n = |V|$, as required. \square

7 Reduction to Colored Token Swapping

We have shown in Lemma 10 how to reduce 3SAT to the Disjoint Paths Problem on a Directed Acyclic Graph (DP) in linear space. Now, we show how to reduce from this problem to the colored token swapping problem.

Lemma 11. *There exists a linear reduction from DP to the colored token swapping problem.*

Proof. Let G be a directed graph and φ be a bijection, as in the definition of DP. We place k tokens t_1, \dots, t_k of distinct colors on the vertices in V^- , see Figure 9 for an illustration. Their target positions are in V^+ as determined by the assignment φ . We define a color for each layer V_1, \dots, V_{t-1} . Each vertex in layer V_i which is not a sink is colored by the corresponding color. Recall that for each vertex v all ingoing edges come from the same layer, which we denote by L_v . On each vertex v which is not a source, we place a token with the color of layer L_v . We call these tokens *filler tokens*. We set the threshold T to $|V(G)| - k$. This equals the number of filler tokens.

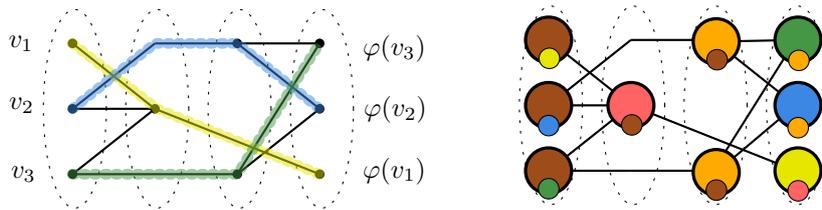


Figure 9: Left: an instance of the disjoint paths problem with $t = 4$ layers and $k = 3$ source-sink pairs, together with a solution; arcs are directed from left to right. Right: an equivalent instance of the colored token swapping problem.

We have to show that there are k paths with the properties above if and only if there is a sequence of at most T swaps that brings every token to its target position.

\Rightarrow Given those paths, swap tokens t_1, \dots, t_k along their respective paths. In this way every token gets to its target vertex. (Recall that the paths P_1, \dots, P_k partition the set of vertices.)

\Leftarrow Assume we can swap every token to its target position in T swaps. Since each of the T filler tokens must be swapped at least once, we conclude that every filler token swaps

exactly once. In particular, every swap must swap a filler token and a non-filler token, which is a token that starts on a vertex in V_1 , and the filler token moves to a lower layer and the non-filler token to a higher layer.

We denote by P_1, \dots, P_k the paths that the non-filler tokens t_1, \dots, t_k follow. By the above argument, P_1, \dots, P_k must partition the vertex set. The paths start and end at the correct position, because the tokens t_1, \dots, t_k do. \square

We conclude that the Colored Token Swapping Problem is NP-hard. This has already been proved by Yamanaka, Horiyama, Kirkpatrick, Otachi, Saitoh, Uehara, and Uno [24], even when there are only three colors.

The reduction of Lemma 11 produces instances of the colored token swapping problem with additional properties, which are directly derived from the properties of DP. For later usage, we summarize them in the following definition.

Structured Colored Token Swapping Problem

Input: A number $k \in \mathbb{N}$ and a graph $G = (V, E)$, where each vertex has a not necessarily unique color. Further on each vertex sits a colored token.

1. There exists a partition of the vertices V into layers V_1, \dots, V_t such that each layer has at most 10 vertices.
2. Each edge can be oriented so that all outgoing edges of a vertex go to layers with larger index.
3. There exists a bijection φ between the sources $V^- \subseteq V$ and the sinks $V^+ \subseteq V$, such that for each vertex $v \in V^-$ the token on v has target vertex $\varphi(v) \in V^+$.
4. All non-sink vertices of layer $V_i \setminus V^+$ have the same color, for all $i = 1, \dots, t$.
5. All non-source vertices $v \in V_i \setminus V^-$ have all ingoing edges to the same layer V_j with $j < i$.

Question: Does there exist a sequence of k swaps such that each token is on a vertex with the same color?

Corollary 12. *There exists a linear reduction from DP to the structured colored token swapping problem.*

8 Reduction to the Token Swapping Problem

In this section we describe the final reduction which results in an instance of the token swapping problem. To achieve this we make use of the even permutation network gadget from Section 5.

Lemma 13. *There exists a linear reduction from the structured colored token swapping problem to the token swapping problem.*

Proof. Let I be an instance of the structured colored token swapping instance. We denote the graph by G , the layers by V_1, \dots, V_t , the sources by V^- , the sinks by V^+ and the threshold for the number of swaps by k .

We construct an instance J of the token swapping problem. The graph \overline{G} consists of two copies of G . For each set $V_j \setminus V^+$, we add *one* permutation network to the union of both copies. In other words, the two copies of $V_j \setminus V^+$ serve as inputs of the permutation network. We denote the output vertices of the permutation network attached to the copies of $V_j \setminus V^+$ by V'_j . The filler tokens that were destined for $V_j \setminus V^+$ have V'_j as their new final destination, see Figure 10. It is not important how we assign each token to a target vertex, as long as this assignment corresponds to an *even* permutation. Further each token gets a unique label. The threshold \overline{k} for the number of swaps is defined by $2k$ plus the number of swaps needed for the permutation networks.

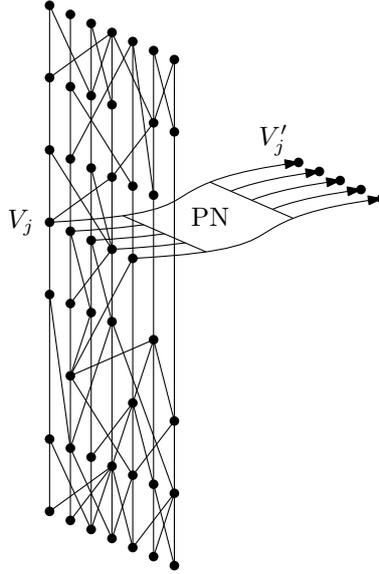


Figure 10: Attaching a permutation network to let the filler tokens destined for V_j arrive at their final destinations V'_j . The small “vertical” network in this figure represents the network constructed in Lemma 11. Note that we did not duplicate G in the figure.

We show at first that this reduction is linear. For this it is sufficient to observe that the size of each layer is constant. And thus also the permutation network attached to these layers have constant size each.

Now we show correctness. Let I be an instance of the structured colored token swapping problem and J the constructed instance as described above.

[\Rightarrow] Let S be a valid sequence of swaps that brings every token on G to a correct target position within k swaps. We need to show that there is a sequence of \bar{k} swaps that brings each token in \bar{G} to its unique target position. We perform S on each copy of G . Thereafter every token is swapped through the permutation network to its target position. Note that the permutation between the input and output is an *even* permutation. This is because we doubled the graph G . Therefore, the number of swaps in the permutation network is constant regardless of the permutation of the filler tokens in layer V_i , by Lemma 9. This also implies that the total number of swaps is \bar{k} .

[\Leftarrow] Assume that there is a sequence S' of \bar{k} swaps that brings each token of J to its target position. This implies that each filler token went through the permutation network to its correct target vertex. In order to do that each filler token must have gone to some input vertex of its corresponding permutation network. As the number of swaps inside each permutation network is independent of the permutation of the tokens on the input vertices, there remain exactly $2k$ swaps to put all tokens in each copy of G at its right place. This implies that each token in G can be swapped to its correct position in I in k swaps as claimed. \square

9 Hardness of the Token Swapping Problem

In this section we put together all the reductions. They imply the following theorem.

Theorem 3 (Lower Bounds). *The token swapping problem has the following properties:*

1. *It is NP-complete.*
2. *It cannot be solved in time $2^{o(n)}$ unless the Exponential Time Hypothesis fails, where n is the number of vertices.*

3. It is APX-hard.

These properties also hold, when we restrict ourselves to instances of bounded degree.

Proof. For the NP-completeness, we reduce from 3SAT. For the lower bound under ETH, we need to use the Sparsification Lemma, see [14], and reduce from 3SAT instances where the number of clauses is linear in the number of variables. This prevents a potential quadratic blow up of the construction. For the inapproximability result, we reduce from 5-OCCURRENCE-MAX-3-SAT. (In this variant of 3SAT each variable is allowed to have at most 5 *occurrences*.) This gives us some additional structure that we use for the argument later on. In all three cases the reduction is exactly the same.

Let f be a 3SAT instance. We denote by $K(f)$ the instance of the token swapping problem after applying the reduction of Lemma 10, Lemma 11 and Lemma 13 in this order. (It is easy to see that the graph of $K(f)$ has bounded degree.)

As all three reductions are correct, we can immediately conclude that the problem is NP-hard. NP-membership follows easily from the fact that a valid sequence of swaps is at most quadratic in the size of the input and can be checked in polynomial time.

The Exponential Time Hypothesis (ETH) asserts that 3SAT cannot be solved in $2^{o(n')}$, where n' is the number of variables. The Sparsification Lemma implies that 3SAT cannot be solved in $2^{o(m')}$, where m' is the number of clauses. As the reductions are linear the number of vertices in $K(f)$ is linear in the number of clauses of f . Thus a subexponential-time algorithm for the token swapping problem implies a subexponential-time algorithm for 3SAT and contradicts ETH.

To show APX-hardness, we do the same reductions as before, but we reduce from 5-OCCURRENCE-MAX-3-SAT. Thus we can assume that each variable in f appears in at most 5 clauses. This variant of 3SAT is also APX-hard, see [1]. Assume a constant fraction of the clauses of f are not satisfiable. We have to show that we need an additional constant fraction on the total number of swaps. For this, we assume that the reader is familiar with the proofs of Lemma 10, 11 and 13. It follows from these proofs that there is a constant sized gadget in $K(f)$ for each clause of f . Also there are certain tokens that represent variables and the paths they take correspond to a variable assignment. We denote with x, y, z the variables of some clause C , and we denote with T_x, T_y, T_z the tokens corresponding to these variables and G_C the gadget corresponding to C . We need two crucial observations. In case that the paths that the tokens T_x, T_y and T_z take do not correspond to an assignment that makes C true, at least one more swap is needed that can be attributed to this clause gadget G_C . In case that a token T_x changes its track, which corresponds to another assignment of the variable, then at least one more swap needs to be performed that can be attributed to all its clauses with value $1/5$. These two observations follow from the proofs of the above lemmas, as otherwise it would be possible to “cheat” at each clause gadget and the above lemmas would be incorrect. The observations also imply the claim. Let f be a 3SAT formula with a constant fraction of the clauses not satisfiable. Assume at first that the swaps are ‘honest’ in the sense that the variable tokens T_x does not change its track and corresponds consistently with the same assignment. In this case, by the first observation, we need at least one extra swaps per clause. And thus a constant fraction of extra swaps, compared to the total number of swaps. In a dishonest sequence of swaps, changing the track of some variable token T_x fixes at most five clauses. This implies at least one extra swap for every five unsatisfied clauses, which is a constant fraction of all the swaps as the total number of swaps is linear in the number of clauses of f . This finishes the APX-hardness proof. \square

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References

- [1] Sanjeev Arora and Carsten Lund. Hardness of approximations. In Dorit Hochbaum, editor, *Approximation Algorithms for NP-Hard Problems*, pages 399–446. PWS Publishing, 1996.
- [2] Édouard Bonnet, Tillmann Miltzow, and Paweł Rzażewski. Complexity of token swapping and its variants. Preprint, July 2016. [arXiv:1607.07676](https://arxiv.org/abs/1607.07676).
- [3] Paul Bonsma, Takehiro Ito, Marcin Kamiński, and Naomi Nishimura. Invitation to combinatorial reconfiguration. Presentation at the First International Workshop on Combinatorial Reconfiguration (CoRe 2015), <http://www.ecei.tohoku.ac.jp/alg/core2015/page/template.pdf>, February 2015.
- [4] Paul Bonsma, Marcin Kamiński, and Marcin Wrochna. Reconfiguring independent sets in claw-free graphs. In R. Ravi and Inge Li Gørtz, editors, *Algorithm Theory — SWAT 2014*, volume 8503 of *Lecture Notes in Computer Science*, pages 86–97. Springer, 2014. doi:10.1007/978-3-319-08404-6_8.
- [5] Arthur Cayley. Note on the theory of permutations. *Philosophical Magazine Series 3*, 34:527–529, 1849. doi:10.1080/14786444908646287.
- [6] Gruia Călinescu, Adrian Dumitrescu, and János Pach. Reconfigurations in graphs and grids. *SIAM Journal on Discrete Mathematics*, 22(1):124–138, 2008. doi:10.1137/060652063.
- [7] Erik D. Demaine, Martin L. Demaine, Eli Fox-Epstein, Duc A. Hoang, Takehiro Ito, Hirotaka Ono, Yota Otachi, Ryuhei Uehara, and Takeshi Yamada. Linear-time algorithm for sliding tokens on trees. *Theoretical Computer Science*, 600:132–142, 2015. doi:10.1016/j.tcs.2015.07.037.
- [8] Ruy Fabila-Monroy, David Flores-Peñaloza, Clemens Huemer, Ferran Hurtado, Jorge Urrutia, and David R. Wood. Token graphs. *Graphs and Combinatorics*, 28:365–380, 2012. doi:10.1007/s00373-011-1055-9.
- [9] Eli Fox-Epstein, Duc A. Hoang, Yota Otachi, and Ryuhei Uehara. Sliding token on bipartite permutation graphs. In Khaled Elbassioni and Kazuhisa Makino, editors, *Algorithms and Computation*, volume 9472 of *Lecture Notes in Computer Science*, pages 237–247. Springer, 2015. doi:10.1007/978-3-662-48971-0_21.
- [10] Parikshit Gopalan, Phokion G. Kolaitis, Elitza N. Maneva, and Christos H. Papadimitriou. The connectivity of Boolean satisfiability: Computational and structural dichotomies. *SIAM J. Comput.*, 38(6):2330–2355, 2009. doi:10.1137/07070440X.
- [11] Daniel Graf. How to sort by walking on a tree. In Nikhil Bansal and Irene Finocchi, editors, *Algorithms — ESA 2015*, volume 9294 of *Lecture Notes in Computer Science*, pages 643–655. Springer, 2015. doi:10.1007/978-3-662-48350-3_54.
- [12] Robert A. Hearn and Erik D. Demaine. PSPACE-completeness of sliding-block puzzles and other problems through the nondeterministic constraint logic model of computation. *Theoretical Computer Science*, 343(1-2):72–96, 2005. doi:10.1016/j.tcs.2005.05.008.
- [13] Lenwood S. Heath and John Paul C. Vergara. Sorting by short swaps. *Journal of Computational Biology*, 10(5):775–789, 2003. doi:10.1089/106652703322539097.
- [14] Russell Impagliazzo and Ramamohan Paturi. On the complexity of k -SAT. *J. Comput. Syst. Sci.*, 62(2):367–375, 2001. doi:10.1006/jcss.2000.1727.
- [15] Mark R. Jerrum. The complexity of finding minimum-length generator sequences. *Theoretical Computer Science*, 36:265–289, 1985. doi:10.1016/0304-3975(85)90047-7.

- [16] Marcin Kamiński, Paul Medvedev, and Martin Milanič. Complexity of independent set reconfigurability problems. *Theoretical Computer Science*, 439:9–15, 2012. doi:10.1016/j.tcs.2012.03.004.
- [17] Donald E. Knuth. *Sorting and Searching*, volume III of *The Art of Computer Programming*. Addison-Wesley, 1973.
- [18] Tillmann Miltzow, Lothar Narins, Yoshio Okamoto, Günter Rote, Antonis Thomas, and Takeaki Uno. Approximation and hardness for token swapping. In Piotr Sankowski and Christos Zaroliagis, editors, *Algorithms — ESA 2016. Proc. 24th Annual European Symposium on Algorithms, Aarhus*, Leibniz International Proceedings in Informatics (LIPIcs), pages 185:1–185:15. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPIcs.ESA.2016.185.
- [19] Amer E. Mouawad, Naomi Nishimura, Venkatesh Raman, and Marcin Wrochna. Reconfiguration over tree decompositions. In Marek Cygan and Pinar Heggernes, editors, *Parameterized and Exact Computation*, volume 8894 of *Lecture Notes in Computer Science*, pages 246–257. Springer, 2014. doi:10.1007/978-3-319-13524-3_21.
- [20] Igor Pak. Reduced decompositions of permutations in terms of star transpositions, generalized Catalan numbers and k -ARY trees. *Discrete Mathematics*, 204(1-3):329–335, 1999. Selected papers in honor of Henry W. Gould. doi:10.1016/S0012-365X(98)00377-X.
- [21] Daniel Ratner and Manfred Warmuth. The $(n^2 - 1)$ -puzzle and related relocation problems. *Journal of Symbolic Computation*, 10(2):111–137, 1990. doi:10.1016/S0747-7171(08)80001-6.
- [22] Richard M. Wilson. Graph puzzles, homotopy, and the alternating group. *Journal of Combinatorial Theory, Series B*, 16(1):86–96, 1974. doi:10.1016/0095-8956(74)90098-7.
- [23] Katsuhisa Yamanaka, Erik D. Demaine, Takehiro Ito, Jun Kawahara, Masashi Kiyomi, Yoshio Okamoto, Toshiki Saitoh, Akira Suzuki, Kei Uchizawa, and Takeaki Uno. Swapping labeled tokens on graphs. *Theoretical Computer Science*, 586:81–94, 2015. Special issue for the conference *Fun with Algorithms 2014*. doi:10.1016/j.tcs.2015.01.052.
- [24] Katsuhisa Yamanaka, Takashi Horiyama, David Kirkpatrick, Yota Otachi, Toshiki Saitoh, Ryuhei Uehara, and Yushi Uno. Swapping colored tokens on graphs. In Frank Dehne, Jörg-Rüdiger Sack, and Ulrike Stege, editors, *Algorithms and Data Structures*, volume 9214 of *Lecture Notes in Computer Science*, pages 619–628. Springer, 2015. doi:10.1007/978-3-319-21840-3_51.
- [25] Gaku Yasui, Kouta Abe, Katsuhisa Yamanaka, and Takashi Hirayama. Swapping labeled tokens on complete split graphs. Technical Report 2015-AL-153-14, IPSJ SIG, Jozankei, Sapporo, Hokkaido, June 2015.