Open Problems in Discrete Differential Geometry
COLLECTED BY GÜNTER ROTE

PROBLEM 1 (Sergei Tabachnikov). PAPER MÖBIUS STRIP AND PAPER CYLINDER EVERSION

One can make a smooth Möbius strip from a paper rectangle if its aspect ratio is sufficiently large, but not from a square.

**Question 1.** What is the smallest length \( \lambda \) such that a smooth Möbius band can be made of a \( 1 \times a \) paper rectangle if \( a > \lambda \)?

The known bounds are \( \frac{\pi}{2} \leq \lambda \leq \sqrt{3} \) [1, 2], and it is conjectured that \( \lambda = \sqrt{3} \). For smooth immersions, the answer is \( \lambda = \pi/2 \). See [3, 4, 5] and the references there for developable Möbius bands. A related problem concerns the eversion of a cylinder:

**Question 2.** What is the least perimeter \( \mu \) of a paper cylinder of height \( 1 \) that can be turned inside out in the class of embedded smooth developable surfaces.

The known bounds are \( \pi \leq \mu \leq \pi + 2 \), and for smooth immersions, the answer is \( \mu = \pi \) [2].


PROBLEM 2 (Sergei Tabachnikov). COMMUTING BILLIARD MAPS

Given a smooth convex plane domain, the billiard ball map sends an incoming ray (the trajectory of the billiard ball) that hits the boundary from inside to an outgoing ray according to the law of reflection: the angle of incidence equals the angle of reflection.

Consider two nested convex domains. The two billiard ball maps, \( T_1 \) and \( T_2 \), act on the oriented lines that intersect both domains. If the domains are bounded by confocal ellipses, then the respective billiard ball maps commute; see, e.g., [4].

**Question.** Assume that the two maps commute: \( T_1 \circ T_2 = T_2 \circ T_1 \). Does it follow that the two domains are bounded by confocal ellipses?

For piecewise analytic billiards, this conjecture was proved in [2]. For “outer billiards”, an analogous fact is proved in [3]. Of course, this problem has a multi-dimensional version, open both for inner and outer billiards; see, e.g., [1] on multi-dimensional integrable billiards.

Problems 3 (Mikhail Skopenkov). Inverse problem for alternating-current networks

An alternating-current network [2, Section 2.4] is a (not necessarily planar) graph with a fixed subset of $b$ vertices (boundary vertices) and a complex number $c_{xy}$ with positive real part (conductance) assigned to each edge $xy$. A voltage is a complex-valued function $v_x$ on the set of vertices such that for each nonboundary vertex $y$ we have $\sum_{xy} c_{xy} (v_x - v_y) = 0$, where the sum is over the edges containing the vertex $y$. One can see that the voltage is uniquely determined by its boundary values [2, Section 5.1]. The current flowing into the network through a boundary vertex $y$ is $i(y) := \sum_{xy} c_{xy} (v_x - v_y)$. The network response is the matrix of the linear map taking the vector of voltages at the boundary vertices to the vector of currents flowing into the network through the boundary vertices.

The general electrical-impedance tomography problem is to reconstruct the network from its response. For direct-current planar networks, meaning that all conductances are real and positive, the problem has been solved [1].

Teaser. There is a matrix realizable as the response of the network in the figure to the right, for the boundary vertices $N_1, N_2, N_3$ and some edge conductances $R_1, R_2, R_3$, but not by the network to the left.

Denote by $\Psi_b$ the set of complex $b \times b$ matrices $\Lambda$ having the following 4 properties:

1. $\Lambda$ is symmetric;
2. the sum of the entries of $\Lambda$ in each row is zero;
3. $\text{Re } \Lambda$ is positive semidefinite;
4. if $U = (U_1, \ldots, U_b) \in \mathbb{R}^b$ and $U^T (\text{Re } \Lambda) U = 0$ then $U_1 = \cdots = U_b$.

Question 1. Prove that the set of responses of all possible connected alternating-current networks with $b$ boundary vertices is the set $\Psi_b$.

It is known that Conditions (1–4) are necessary. Sufficiency is known for $b = 2$ and $b = 3$ [2, Theorem 4.7].

Question 2. Provide an algorithm to reconstruct a network and edge conductances for a given response matrix.

Note: Questions 1 and 2 have been solved by Günter Rote.

Question 3. Describe the set of responses of all series-parallel networks.
**Question 4.** Describe the set of responses of all planar networks that have the boundary vertices on the outer face.

**Question 5.** Let the conductance of each edge be either $\omega$ or $1/\omega$, where $\omega$ is a variable. Describe the set of possible responses of such networks as functions in $\omega$.

This is known for $b = 2$ boundary vertices — Foster’s reactance theorem [2, Theorem 2.5].


**PROBLEM 4** (Nina Amenta). **Are face angles determined by dihedral angles?**

Stoker’s conjecture says that, for a convex 3-polytope with given combinatorics, if all dihedral angles are specified (different from 0 and $\pi$), then all face angles are also determined.

**Question.** Is this also true for a non-convex polytope?

One may assume that the polytope is triangulated and that it is homeomorphic to a sphere.

**PROBLEM 5** (Günter Rote). **Existence of offset polytopes**

We are given a non-convex three-dimensional polytope $P$ whose boundary is homeomorphic to a sphere. We want to construct an offset polytope $P_\varepsilon$ in which every face is translated outward by the same small distance $\varepsilon$. $P_\varepsilon$ should not have other faces than the faces coming from $P$, and the boundary of $P_\varepsilon$ should remain homeomorphic to a sphere. If $P$ has a saddle-like vertex of degree 4 or larger, the result is not unique.

**Question.** Does such an offset polytope always exist for sufficiently small $\varepsilon > 0$?

It is enough to solve the problem locally for each vertex $v$ of degree $d \geq 4$. Such a vertex will be blown up into $d - 2$ new vertices, connected by edges that form a tree. The faces of $P_\varepsilon$ should be simply connected when clipped to a neighborhood of $v$.

**PROBLEM 6** (Ulrich Bauer). **Subdivision of discrete conformal structures**

Let $\mathcal{T}$ be a triangulated surface, and let $\lambda, \mu: E_T \to \mathbb{R}_{>0}$ be two discrete conformally equivalent metrics on $\mathcal{T}$, represented as edge lengths of the triangulation. Here, discrete conformal equivalence means that the edge lengths of $\lambda$ and $\mu$ for any edge $ij$ between two vertices $i, j$ are related by $\mu_{ij} = e^{2(u_i + u_j)} \lambda_{ij}$ for some function $u: V_\mathcal{T} \to \mathbb{R}$ on the vertices.

**Question.** Is there a metric subdivision scheme that preserves the conformal equivalence?
Specifically, a subdivision of a simplicial complex $K$ is a complex $K'$ such that $|K| = |K'|$ and each simplex of $K'$ is contained in some simplex of $K$. A metric subdivision scheme is a map sending each simplicial complex $K$ equipped with a metric $\lambda$ to a subdivision $K'$ of $K$ equipped with a metric $\lambda'$. A particular example is the barycentric subdivision. The question is whether there exists a metric subdivision scheme such that the subdivided metrics $\lambda'$ and $\mu'$ are still conformally equivalent.


PROBLEM 7 (Jim Propp, Richard Kenyon). DISK PACKINGS OF MAXIMUM AREA
Consider two disks of radius 1 with centers at $(\pm 1, 1)$. Together with the $x$-axis, they enclose at curved triangular region. Into this region, we want to place infinitely many non-overlapping disks that touch the $x$-axis, such that their total area is maximized.

Question. Is it true that the greedy method of successively placing each new circle into the interstices such that they touch two previously placed circles will give the maximum area?

PROBLEM 8 (Günter Rote). A CURIOUS IDENTITY ON SELF-STRESSES
Take the wheel graph $G$ (the graph of a pyramid) embedded in the plane in general position, with a central vertex $p_0$ that is connected to vertices $p_1, \ldots, p_n$ ($n \geq 3$) forming a cycle. On the $2n$ edges of this graph, we define the following function:

\[
\omega_{i,i+1} := \frac{1}{|p_i p_{i+1} p_0|}, \quad \omega_{0,i} := \frac{1}{|p_{i-1} p_0 p_i|}
\]

for $i = 1, \ldots, n$, where $[q_1 q_2 \ldots q_k]$ denotes signed area of the polygon $q_1 q_2 \ldots q_k$, and $p_{n+1} = p_1$. This function is a self-stress: the equilibrium condition $\sum_i \omega_{ij} (p_j - p_i) = 0$ holds for every vertex $i$, where the summation is over all edges $ij$ incident to $i$. Pick two arbitrary points $a$ and $b$ and define another function $f_{ij}$ on the edges of $G$:

\[
f_{ij} := [ap_i][bp_j],
\]

Then we have the following identity, which was used (and proved) for $n = 3$ in [1].

\[
\sum_{ij \in E(G)} \omega_{ij} f_{ij} = 1
\]

(1)

A different formula for $f_{ij}$ that fulfills (1) is given by a line integral over the segment $p_i p_j$, see [1, Lemma 3.10]:

\[
f'_{ij} := \frac{3}{2} \cdot \|p_i - p_j\| \int_{x=p_i}^{p_j} \|x\|^2 \, ds = \frac{1}{2} \cdot \|p_i - p_j\|^2 \cdot (\|p_i\|^2 + \|p_j\|^2 + \langle p_i,p_j \rangle)
\]

Question 1. Are there other graphs with $n$ vertices and $2n - 2$ edges, for which a self-stress $\omega$ satisfying (1) can be defined? The next candidates with 6 vertices are the graph of a triangular prism with an additional edge, and the complete bipartite graph $K_{3,3}$ with an additional edge.
Question 2. What is the meaning of the identity (1)? Is it an instance of a more general phenomenon? What are the connections to homology and cohomology?


PROBLEM 9 (Hao Chen and Arnau Padrol, reported by Günter M. Ziegler). APPROXIMATELY INScribed POLyTOPES

Steinitz proved in 1928 [3] that not every combinatorial type of 3-polytope can be inscribed, that is, realized with all vertices on a sphere. However, a weak version of this is true: Due to the Koebe–Andreev–Thurston circle packing theorem (see e.g. [2, 4]), every 3-polytope can be realized with all edges tangent to the sphere— and thus it has a representation with

- all vertices outside a sphere
- all facets cutting into the sphere.

The question is whether this extends to higher dimensions:

Question. Does every combinatorial type of $d$-polytope have a realization with

- vertices outside a $(d-1)$-sphere,
- facets cutting into the same $(d-1)$-sphere.

Our conjecture is that this is false for $d > 3$, perhaps already for $d = 4$, but certainly for high dimensions $d$, where we know that there are infinitely many projectively unique polytopes. The examples constructed in [1] are essentially inscribable, but there should be other such polytopes whose “shape” is far off from that of a sphere/quadric. However, we have not been able to construct such a polytope yet.