Unique Sink Orientations of Cubes

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Suppose we are given (the graph of) an n-dimensional hypercube with its edges oriented so that every face has a unique sink. Such an orientations is called a *unique sink orientation*, and we are interested in finding the unique sink of the whole cube. The basic operations are so-called *vertex evaluations*, where we can access an abritray vertex of the cube, for which we obtain the orientations of the incident edges.

Unique sink orientations occur when the edges of a deformed geometric n-dimensional cube (i.e., a polytope with the combinatorial structure of a cube) are oriented according to some generic linear function. These orientations are easily seen to be acyclic. It is somewhat surprising at first encounter that acyclicity cannot be guaranteed in general. Already for a 3-dimensional cube there is a unique sink orientation with a cycle (of length 6). In fact, the main motivation for studying unique sink orientations are certain linear complementarity problems, which allow this combinatorial abstraction (due to Alan Stickney and Layne Watson), where, indeed, orientations with cycles arise. Similarly, some quadratic optimization problems, like computing the smallest enclosing ball of a finite point set, can be formulated as finding a sink in a unique sink orientation (we do not know whether the resulting orientations are acyclic).

For acyclic unique sink orientations, a randomized procedure due to Bernd Gärtner with an expected number of $O(e^{2\sqrt{n}})$ vertex evaluations has been known; in the cyclic case, a simple randomized $(3/2)^n$ procedure exists (without explicit mention in the literature). We present new algorithms, a deterministic $O(5^{n/3})$ procedure and a randomized $O((43/20)^{n/2})$ procedure for general unique sink orientations (not necessarily acyclic). An interesting aspect of these algorithms is that they do not proceed on a path to the sink (in a simplex like fashion), but they exploit the potential of random access (here in the sense of arbitrary access) to any vertex of the cube.

We believe that unique sink orientations have a rich structure, and there is ample space for improvement on the bounds given above.