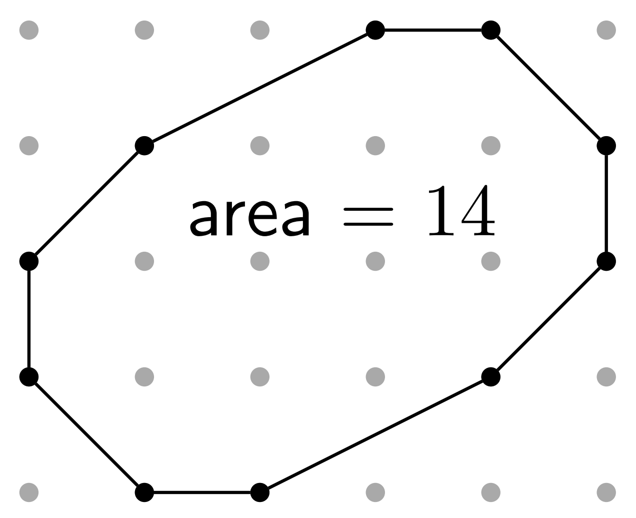
Lattice Polygons: Optimization and Counting

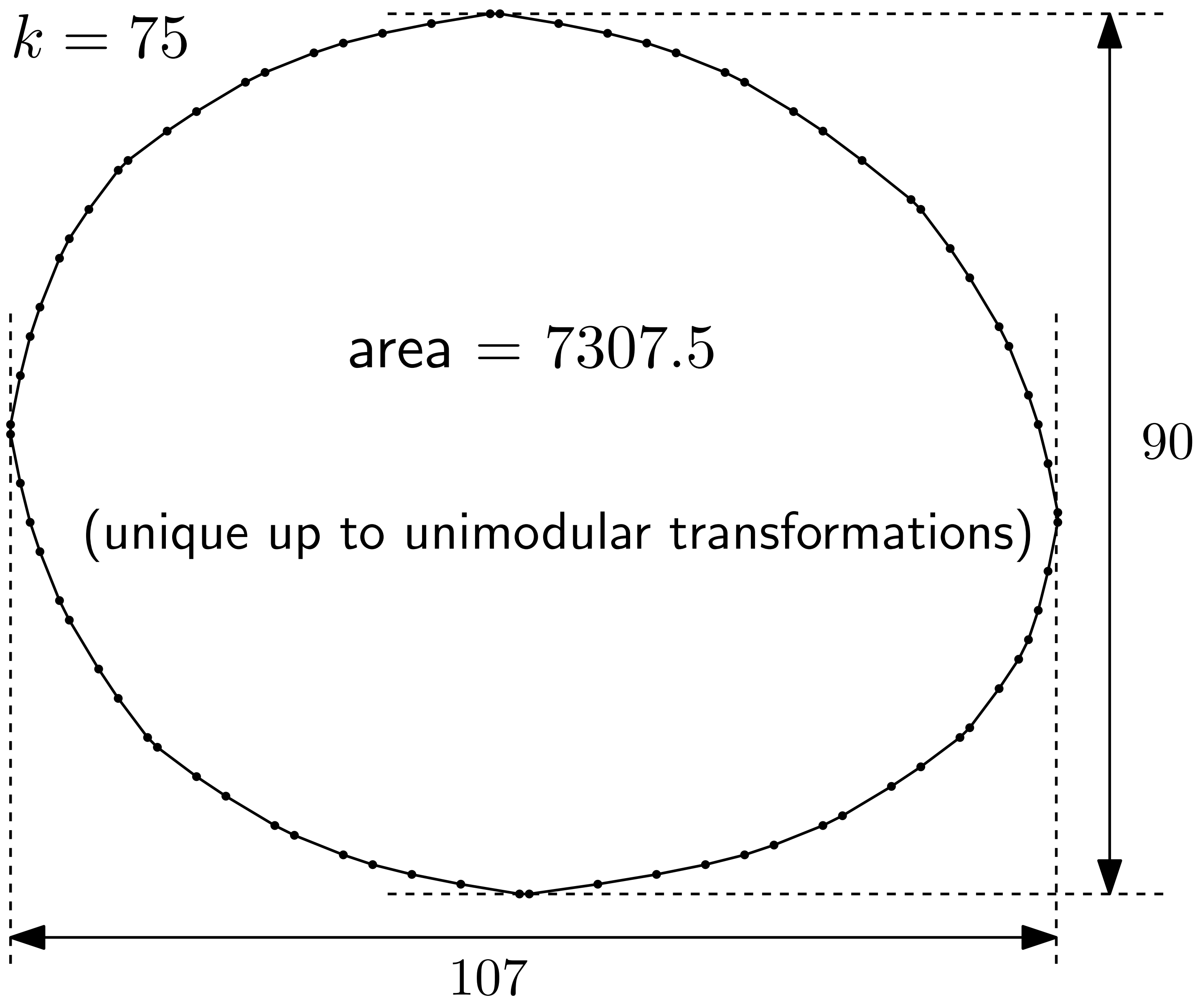
Günter Rote

Freie Universität Berlin

Minimum-area lattice k-gon [OEIS, A070911]

$k = 10$





Bárány and Tokushige (2003):

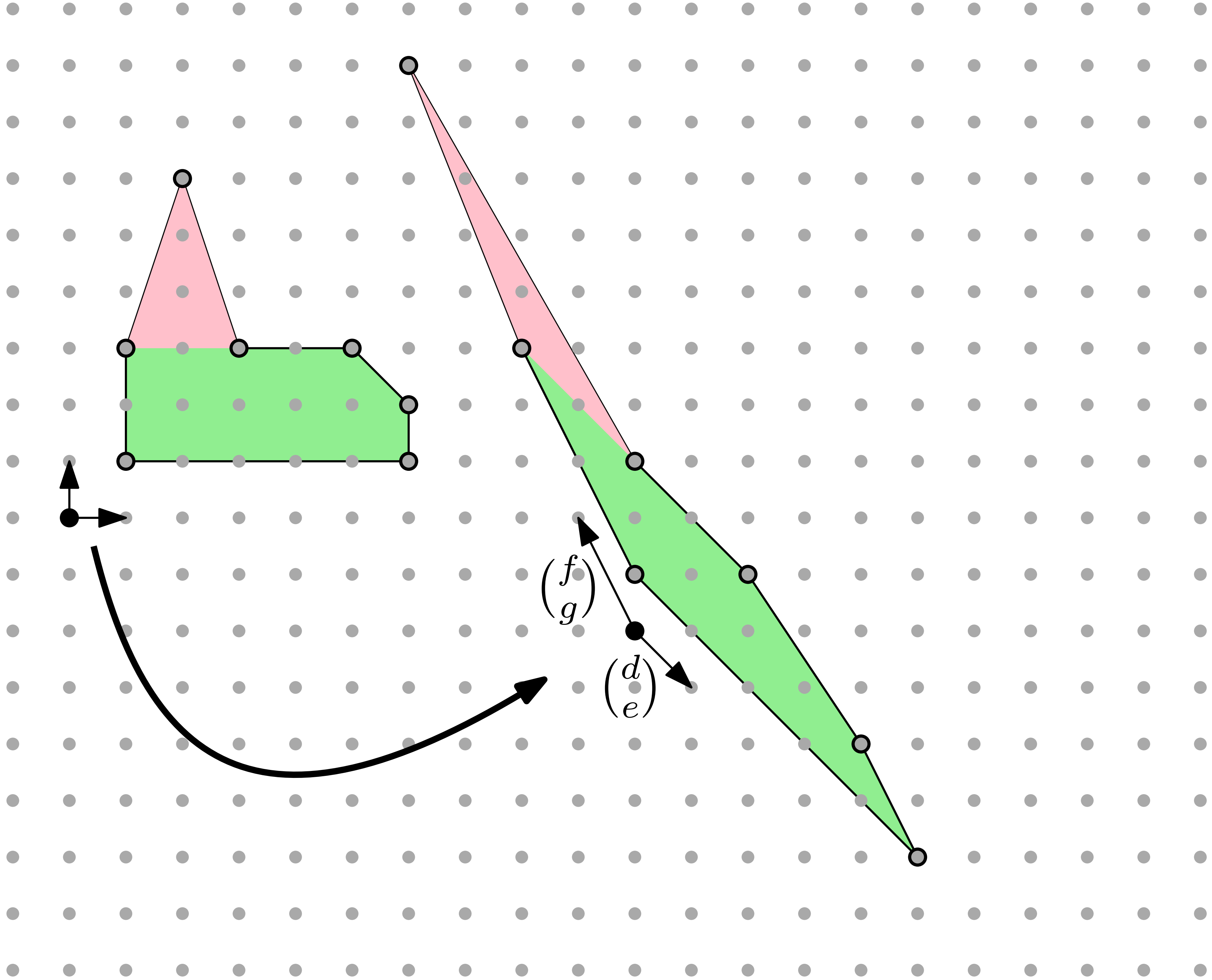
area $\sim Ck^3$ as $k \to \infty$, C algebraic.

$C = 0.0185067\ldots$ (conjectured)

Unimodular transformations

$x \mapsto Mx + t, \quad t \in \mathbb{Z}^d,\ M \in \mathbb{Z}^{d\times d},\ \det M = \pm 1. \implies M^{-1} \in \mathbb{Z}^{d\times d}$

Lattice-preserving affine transformation, bijection on $\mathbb{Z}^{d\times d}$



$$M = \begin{pmatrix} d & f \\ e & g \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Unimodular transformations

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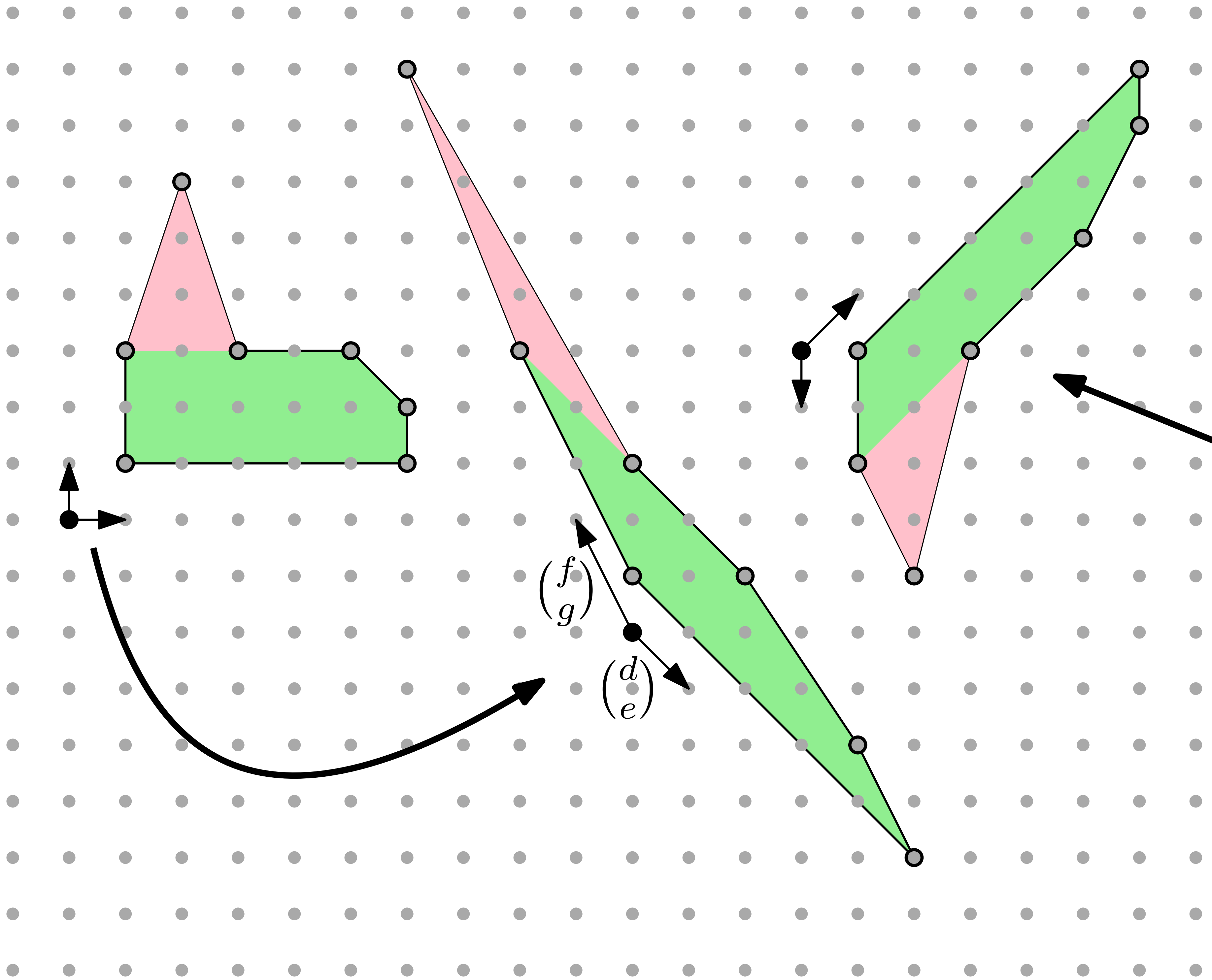


$M = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, $\det M = -1$

$M = \begin{pmatrix} d & f \\ e & g \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

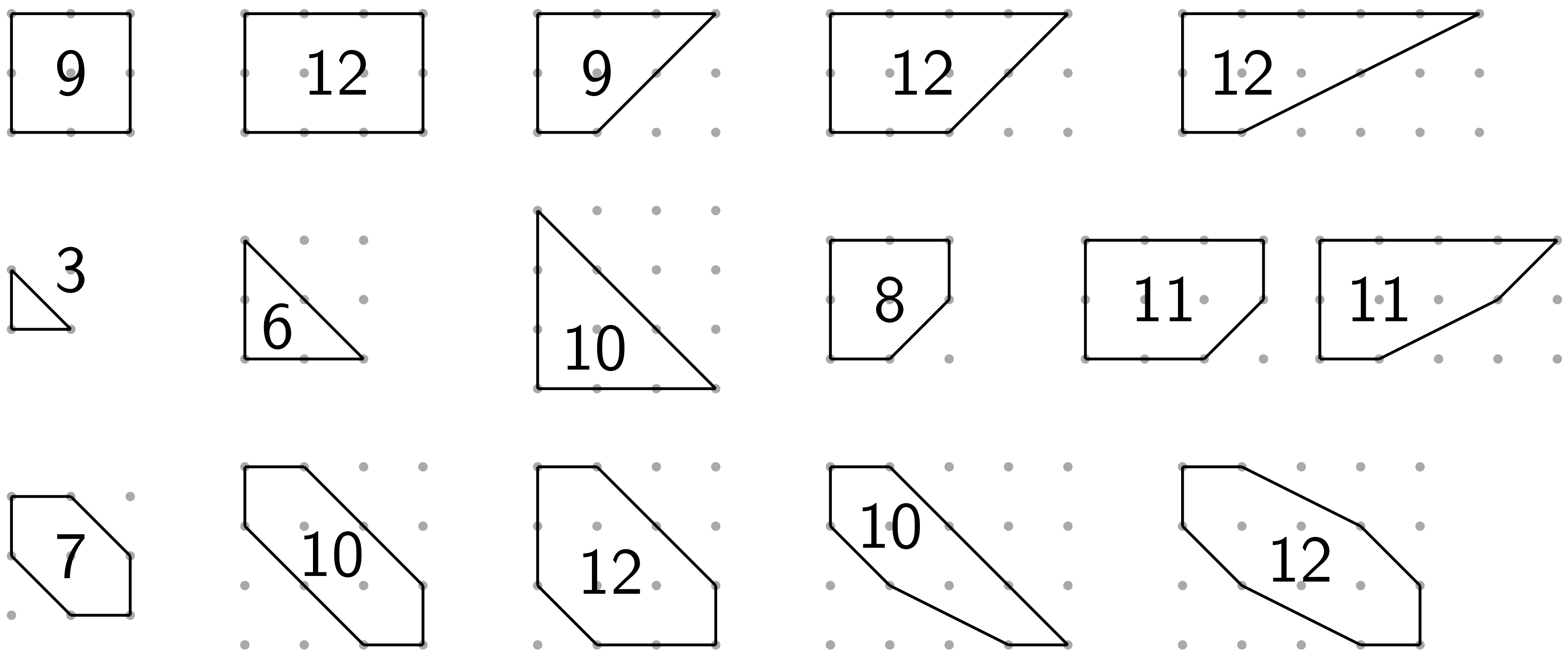
Smooth polygons

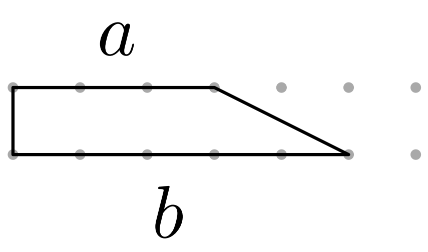
“Finitely many smooth d-polytopes with n lattice points” (2015)

There are 41 equivalence classes of smooth lattice polygons with at most 12 lattice points.

k = #vertices	3	4	5	6	7	8
polygons	3	30	3	4	0	1

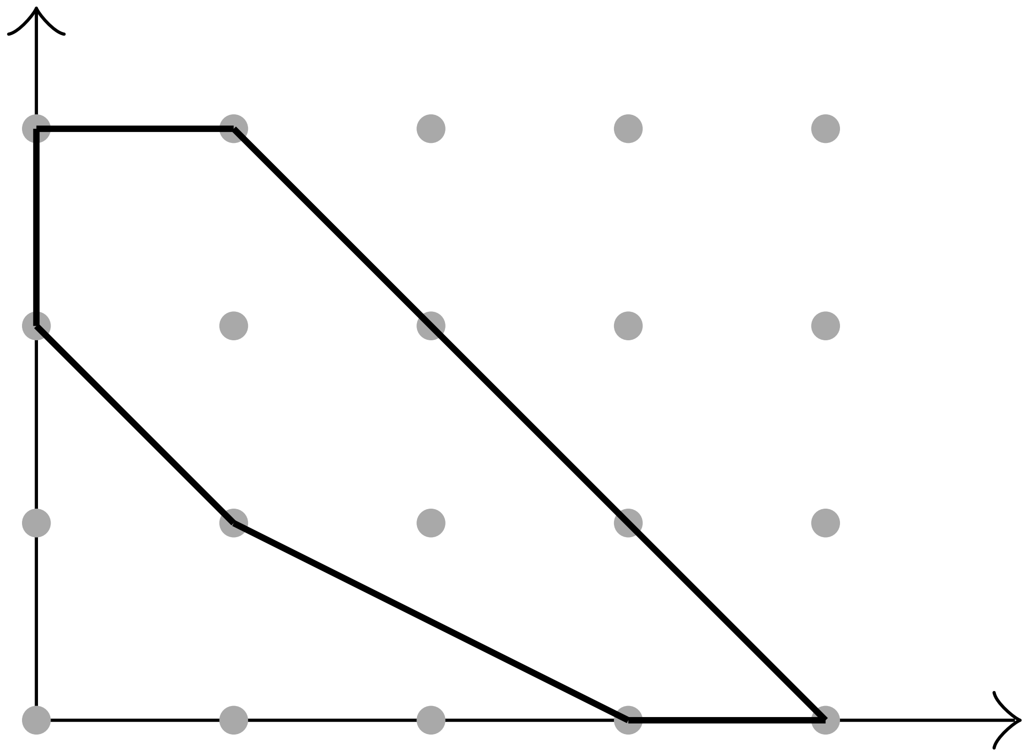
[Tristram Bogart, Christian Haase, Milena Hering, Benjamin Lorenz, Benjamin Nill, Andreas Paffenholz, Günter Rote, Francisco Santos, Hal Schenck 2015]

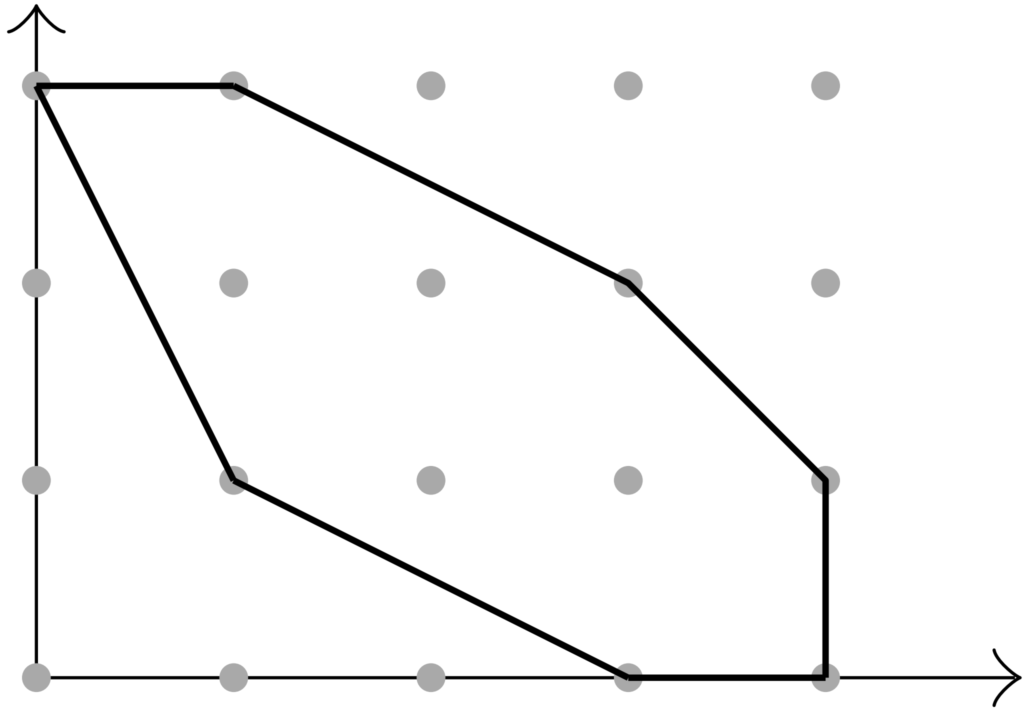




$$(a,b) = \begin{cases} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9) \\ (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8) \\ (3,3), (3,4), (3,5), (3,6), (3,7) \\ (4,4), (4,5), (4,6) \\ (5,5) \end{cases}$$

Smooth

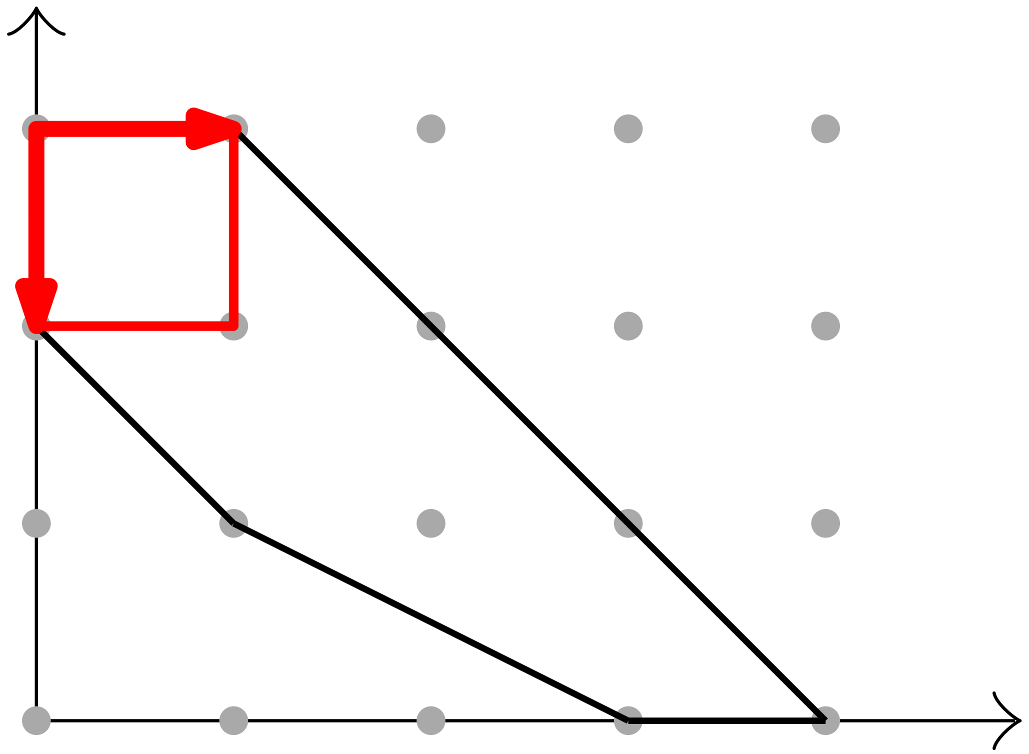


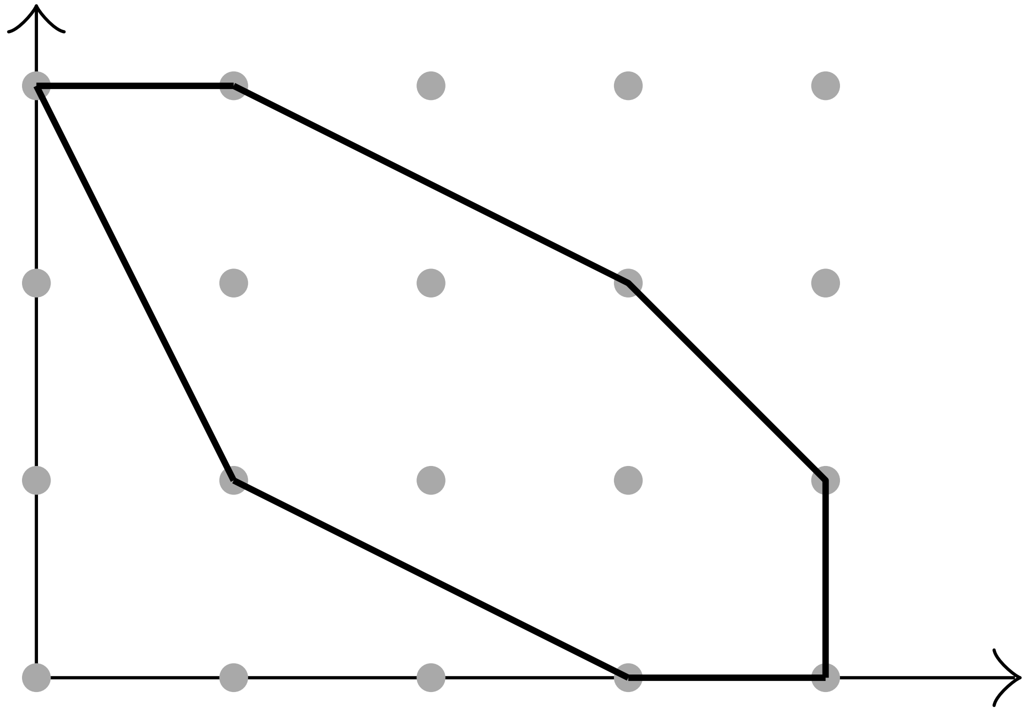


smooth polygons: Consecutive edge directions span a parallelogram of unit area.

[*smooth* d-polytopes: All normal cones are unimodular: They are spanned (using nonnegative combinations) by d integer vectors (extreme rays) that generate (through integer combinations) all integer vectors.]

Smooth





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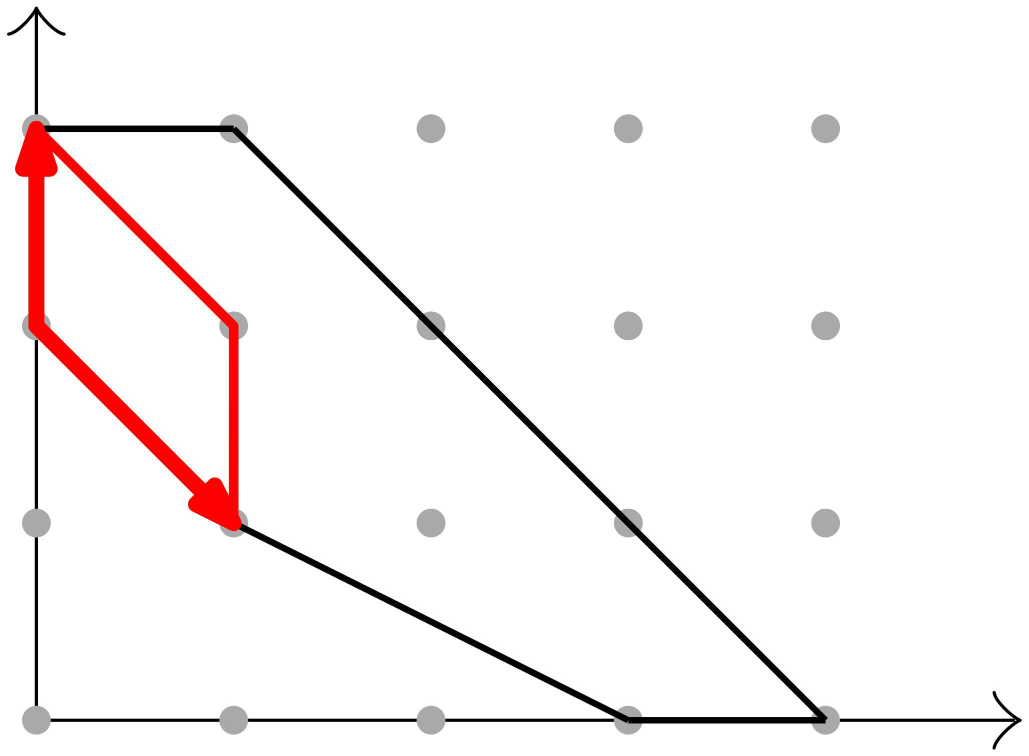

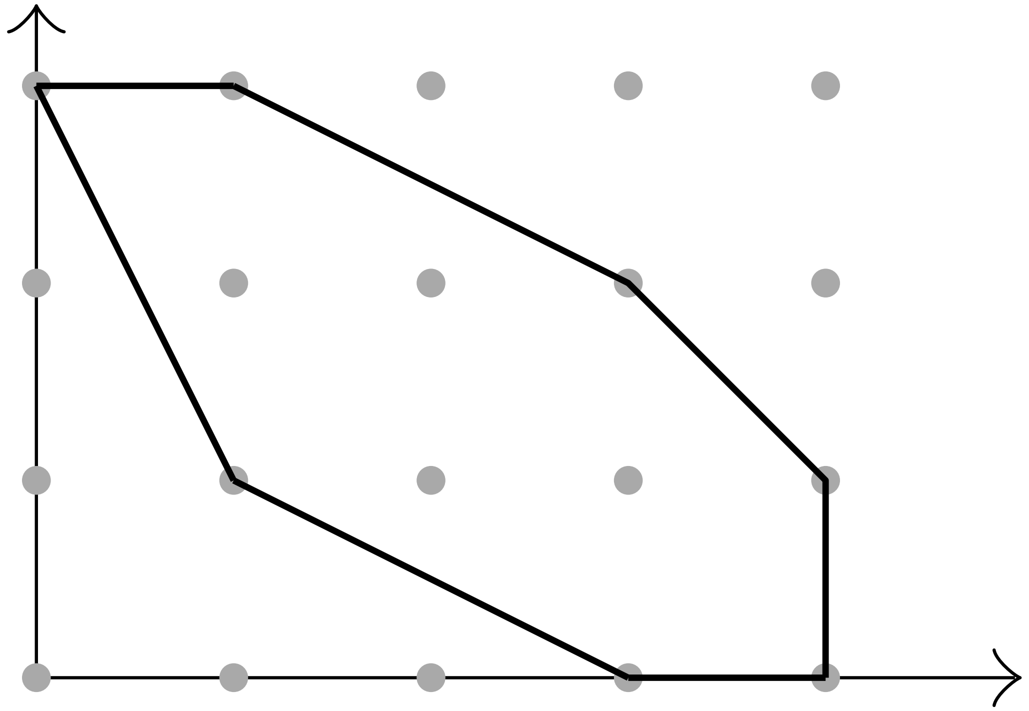

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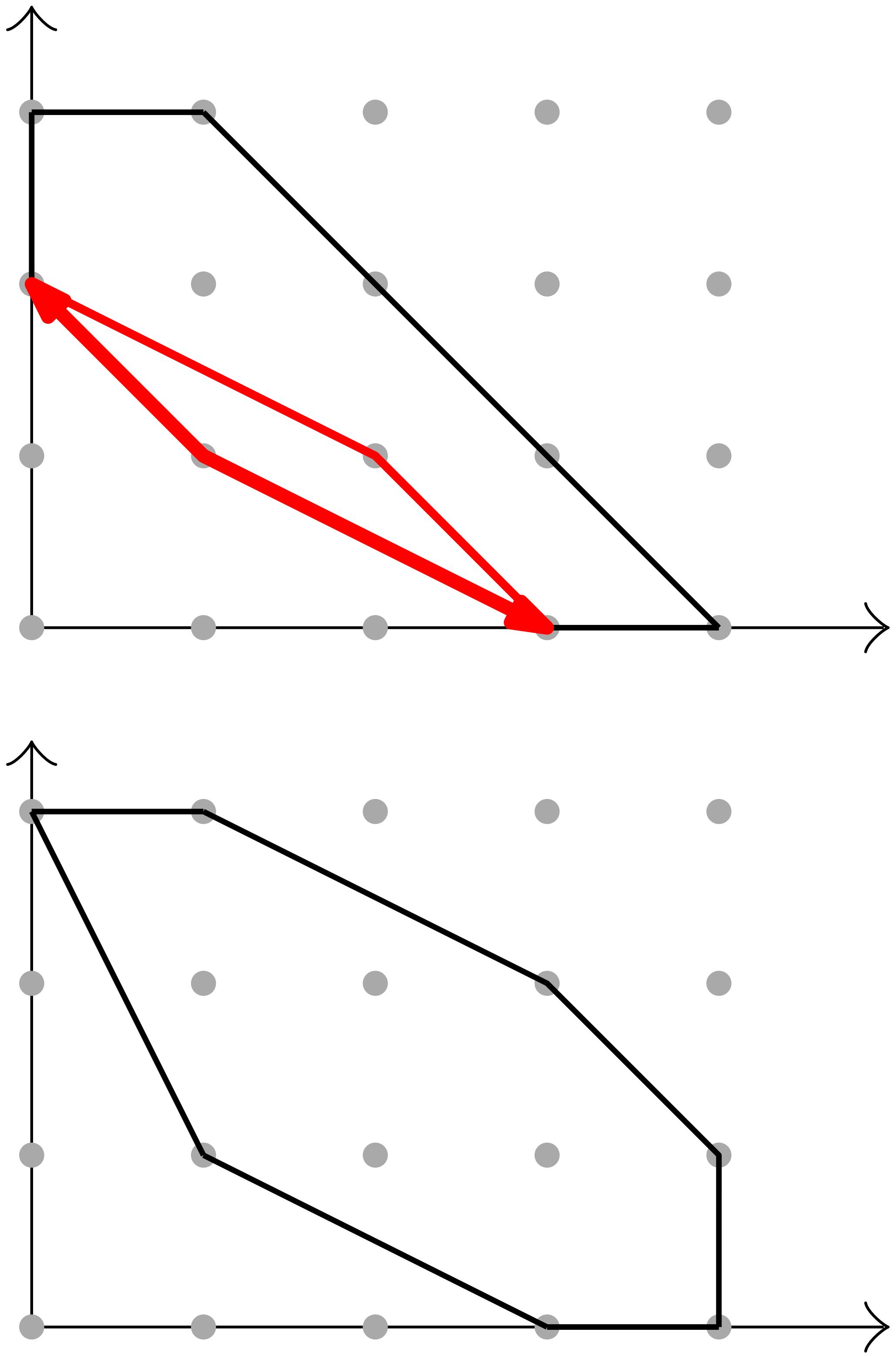

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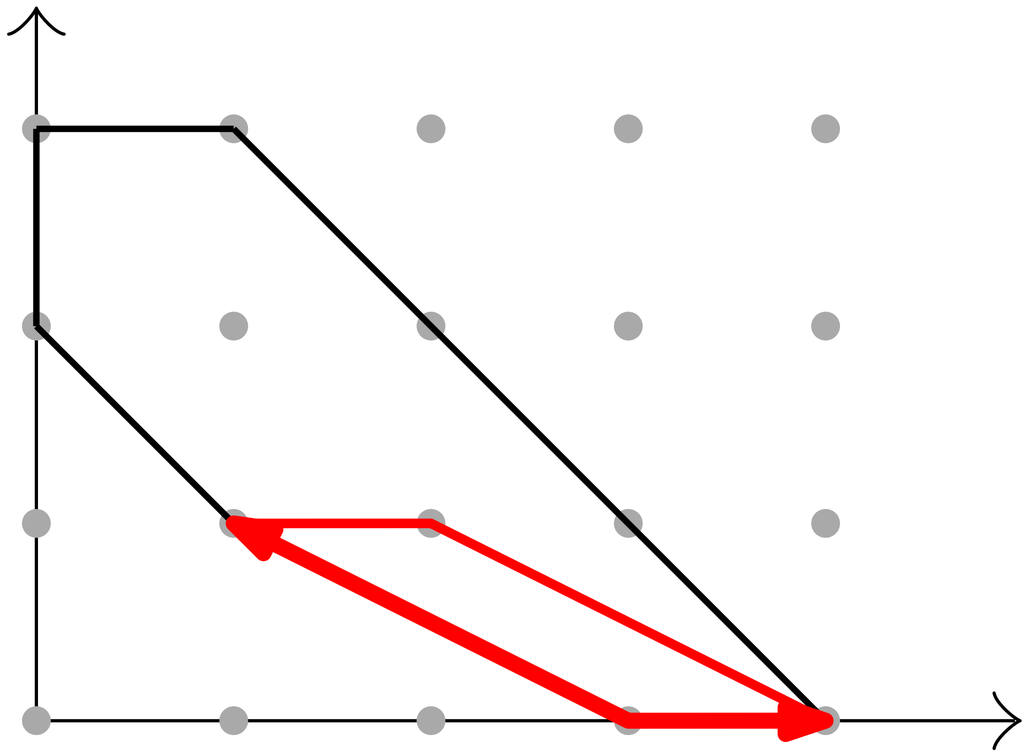

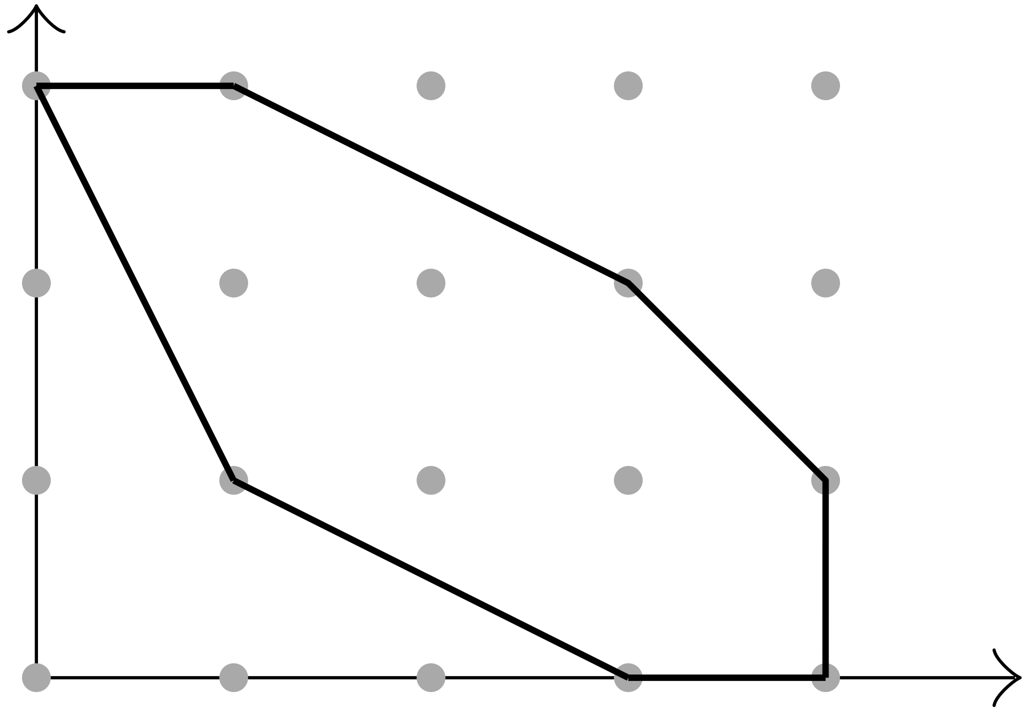

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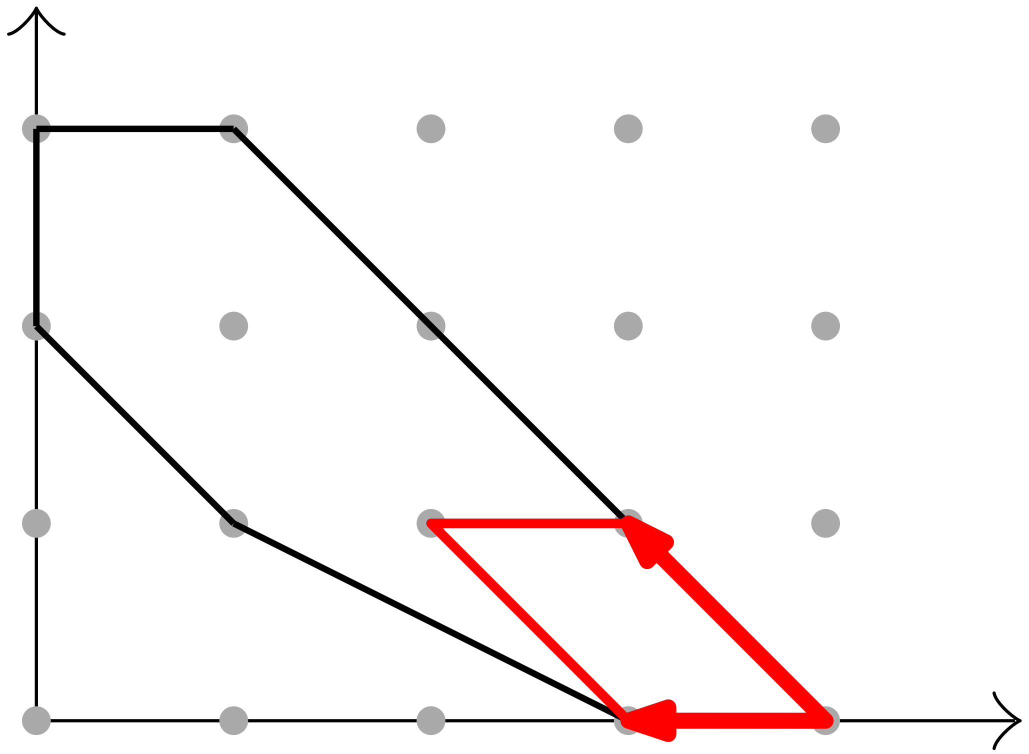

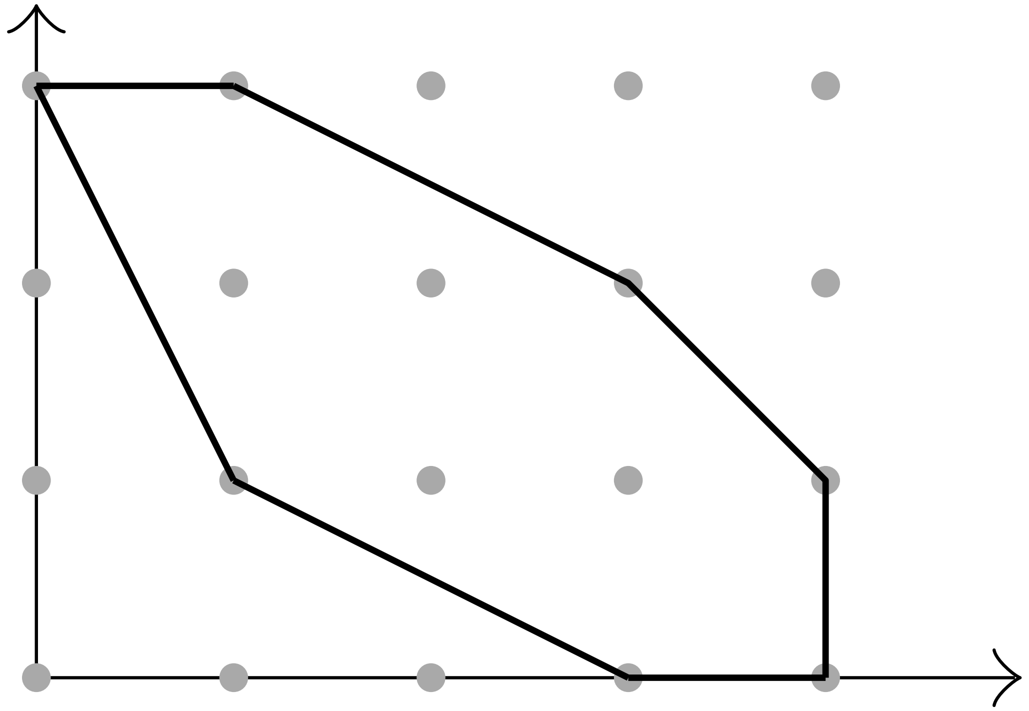

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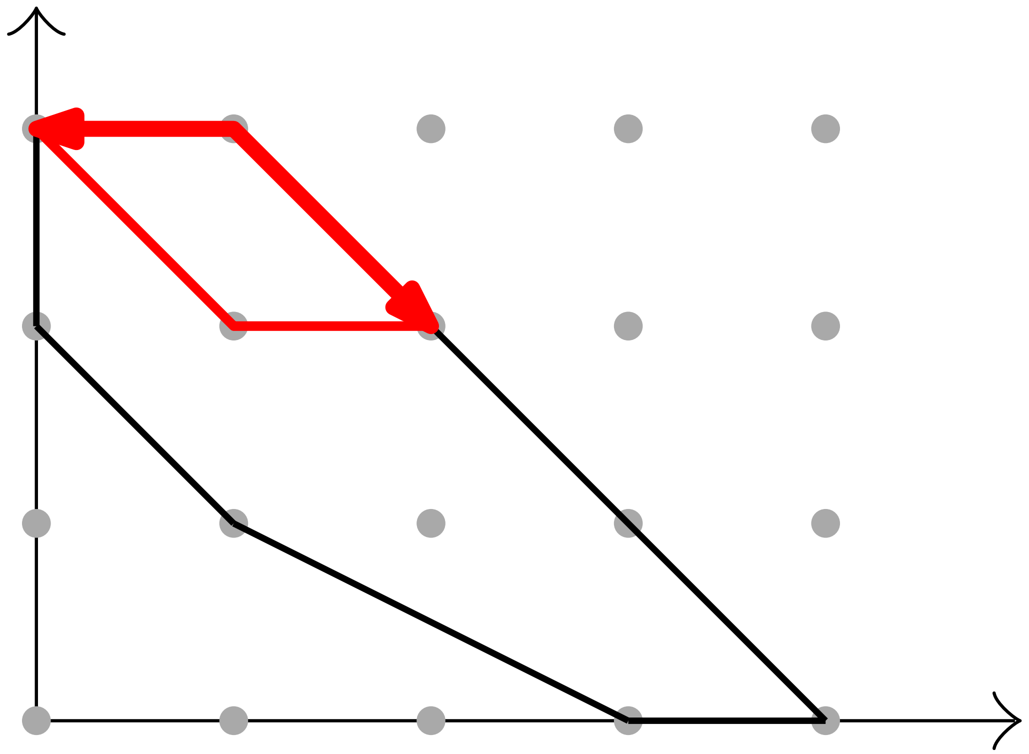

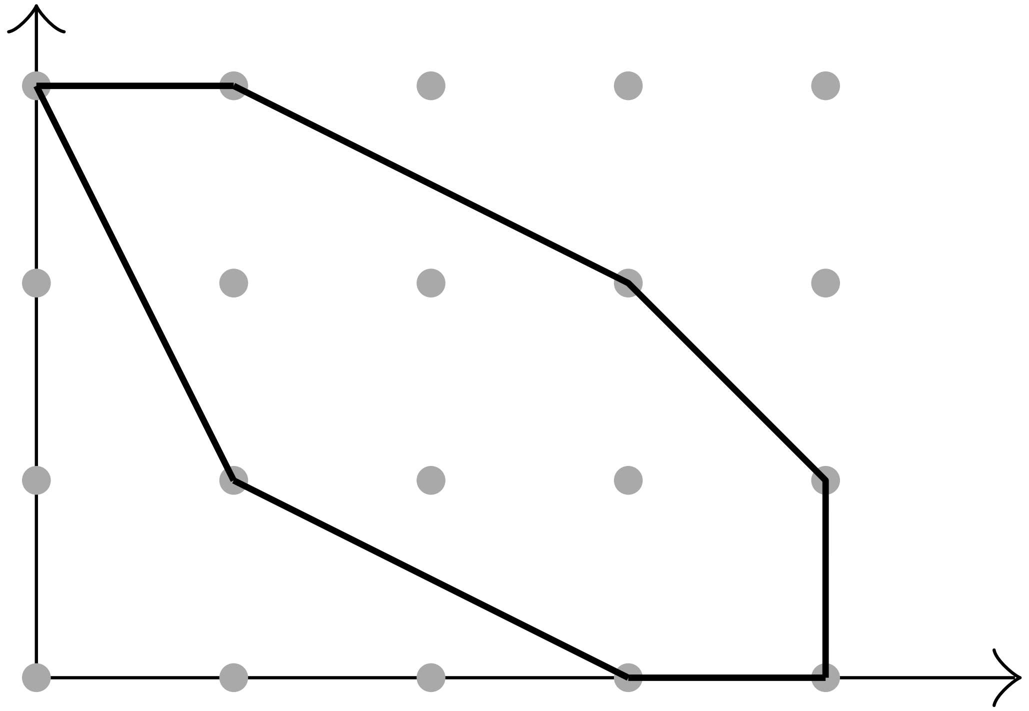

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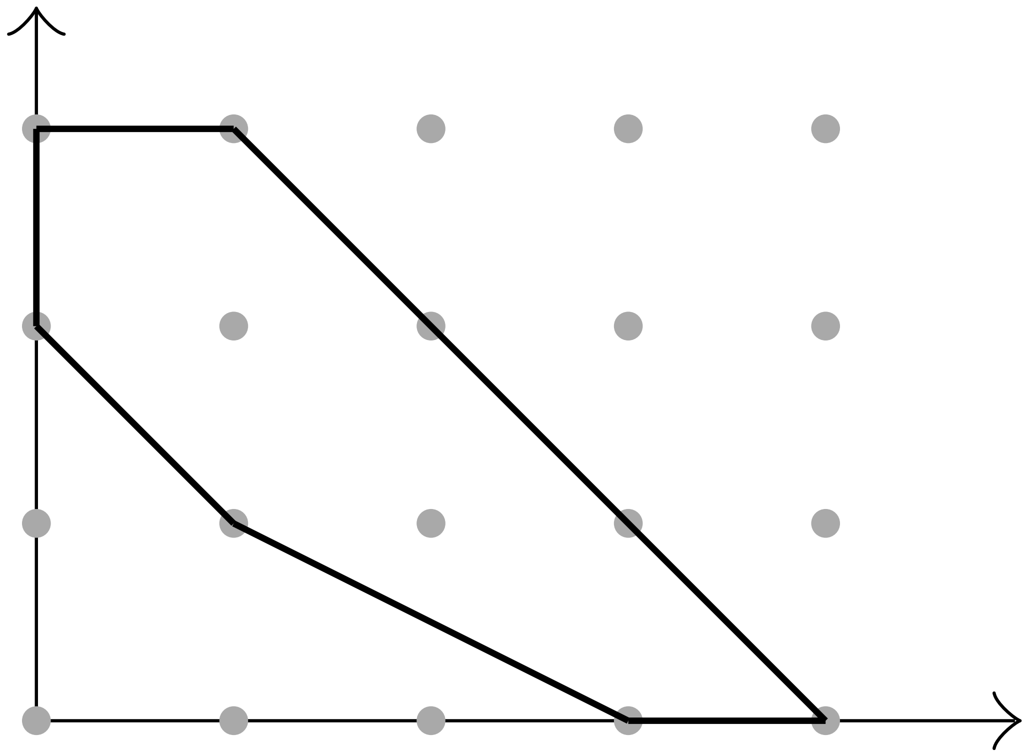

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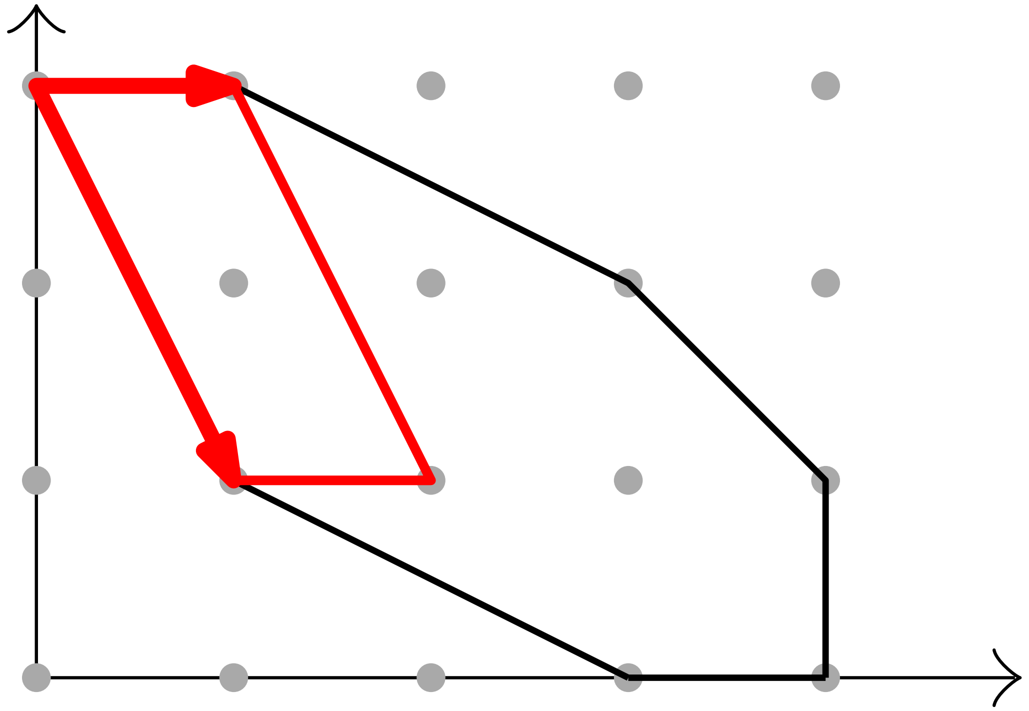

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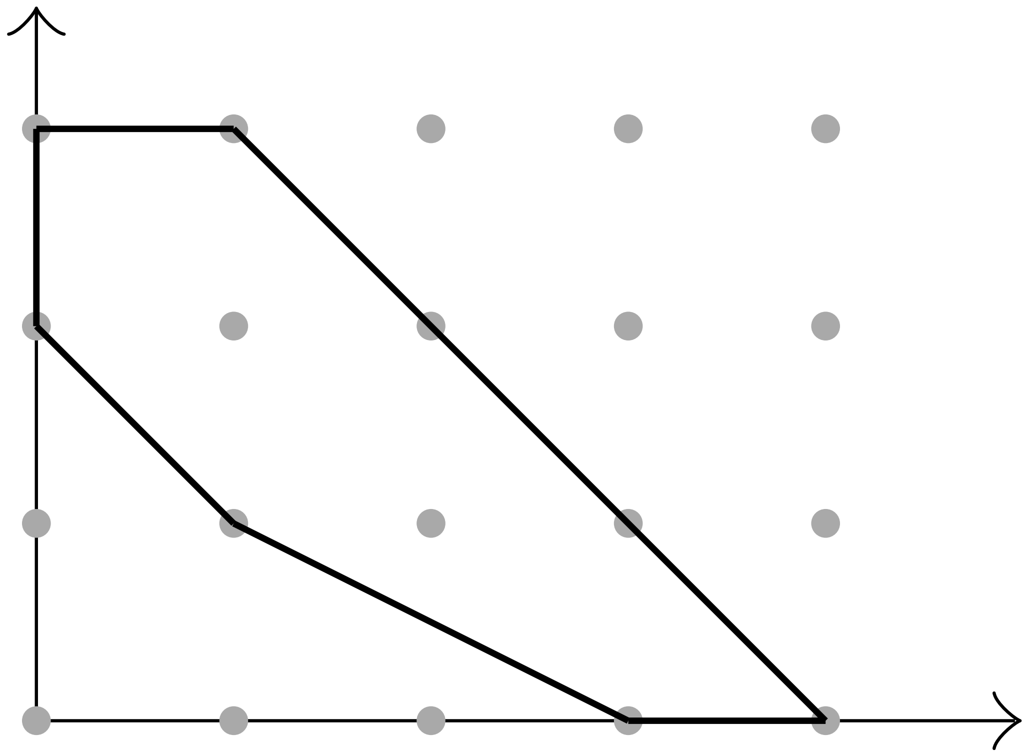

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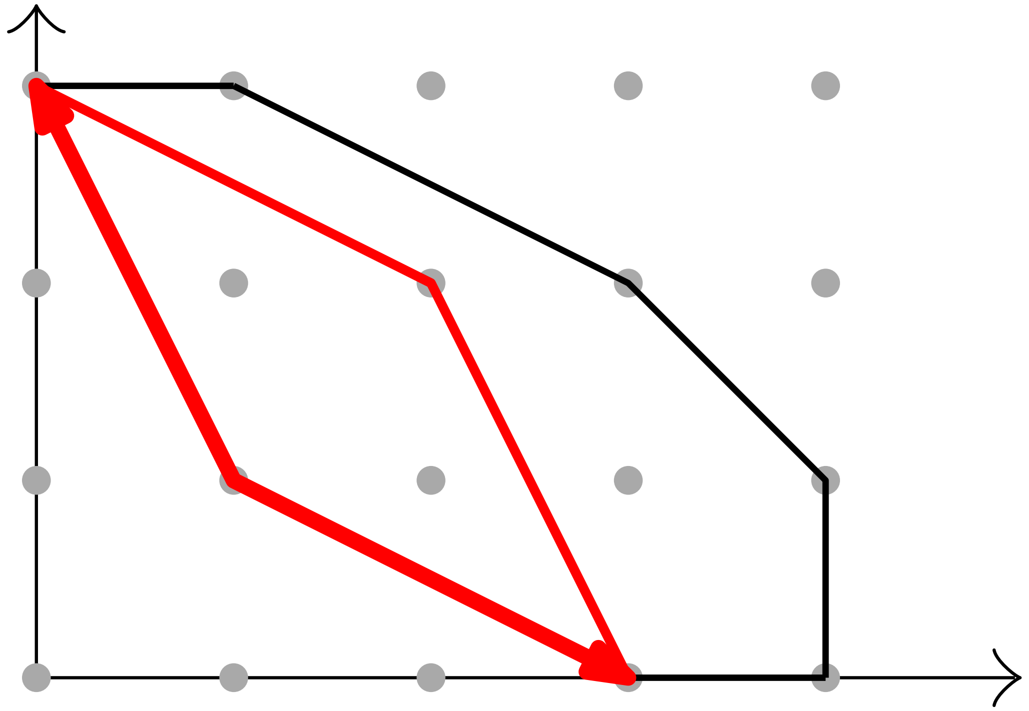


not smooth

Census of lattice polygons

V	A366409	A187015
1	1	1
2	1	2
3	1	3
4	3	7
5	2	6
6	4	13
7	4	13
8	6	27
9	5	26
10	7	44
⋮	⋮	⋮
196	66290	3413697413
197	65105	3595811439
198	69682	3791477384
199	76718	3992454863
200	78918	4208020815
⋮	⋮	
297	1687247	
298	1779013	
299	1833242	
300	1842802	

← entries in the On-Line Encyclopedia of Integer Sequences (OEIS)

For fixed d and V, there are finitely many d-dimensional lattice polytopes with volume V, up to unimodular equivalence.
[Jeff Lagarias, Günter Ziegler 1991]

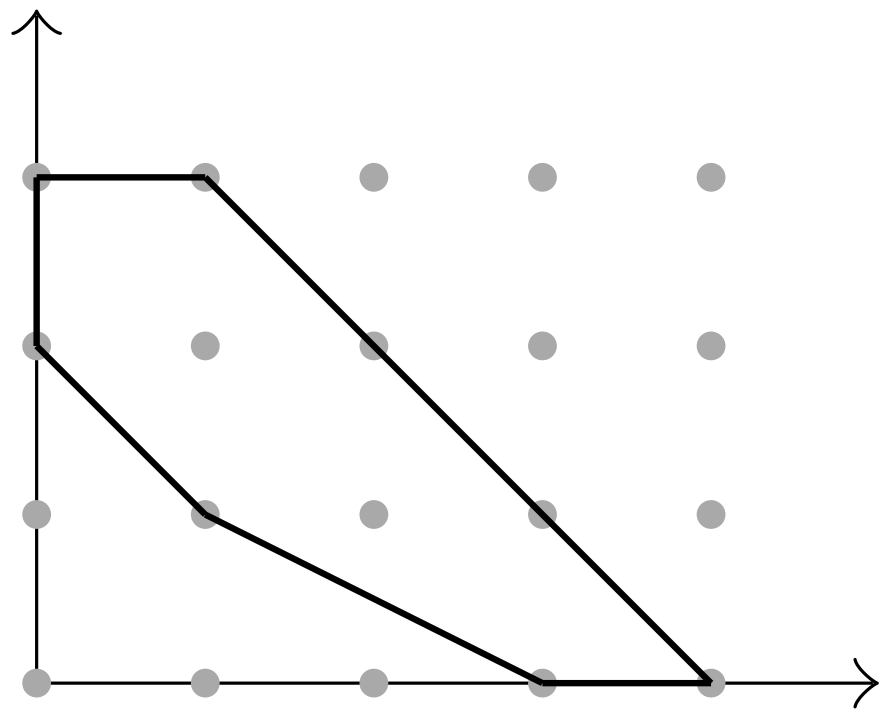
all

lattice polytopes with area $V/2$ [Balletti 2021 up to $V = 50$; Rote 2023]
Gabriele Balletti. Enumeration of lattice polytopes by their volume. (2021).

smooth lattice polygons with area $V/2$ [Rote 2023]

Census of lattice polygons

What to measure



$k = 6$ vertices

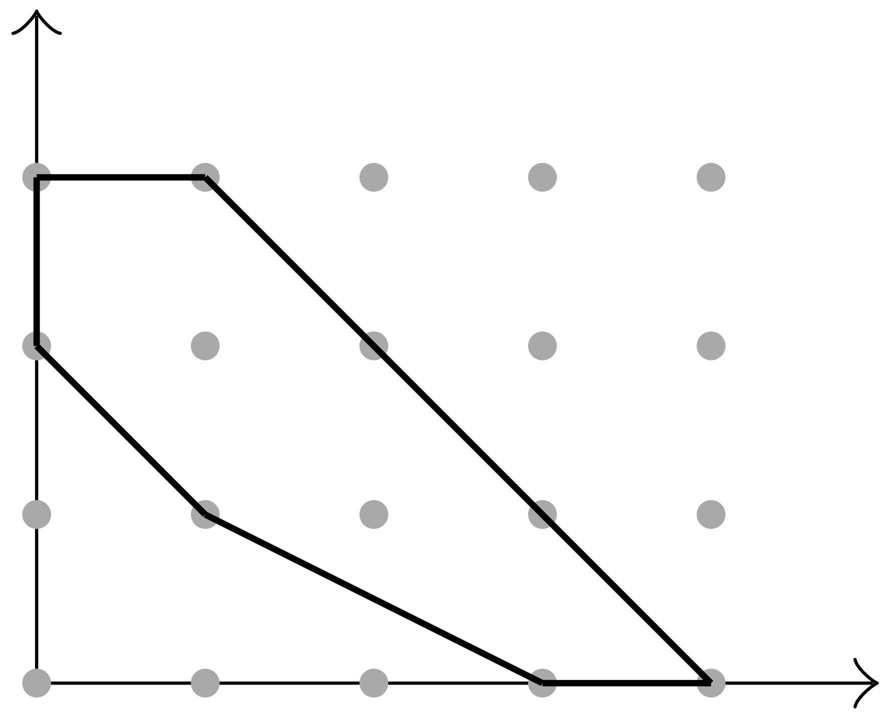
$B = 2$ additional points on the *boundary*

$I = 2$ *interior* lattice points

$n = k + B + I = 10$ lattice points in total

$V/2 = (k + B)/2 + I - 1 = 5 =$ area/ "volume" (Pick's formula)

What to measure



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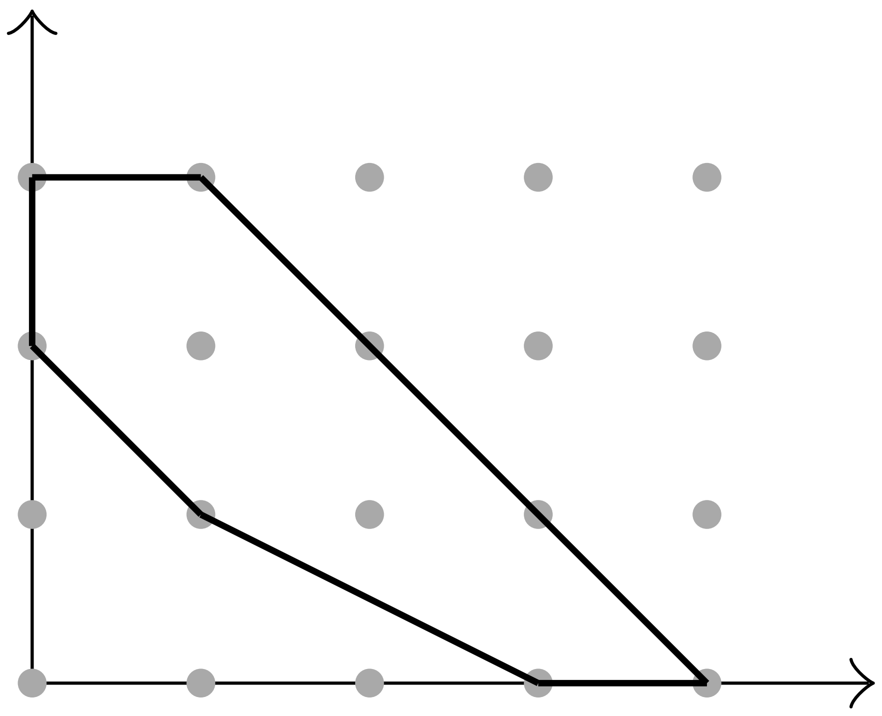
$V/2 = (k + B)/2 + I - 1 = 5 =$ area/ "volume" (Pick's formula)

OEIS A322343: "Number of equivalence classes of convex lattice polygons of genus n."

"genus" $= I =$ number of interior points

What to measure



$k = 6$

$B = 2$

$I = 2$

$n = k$

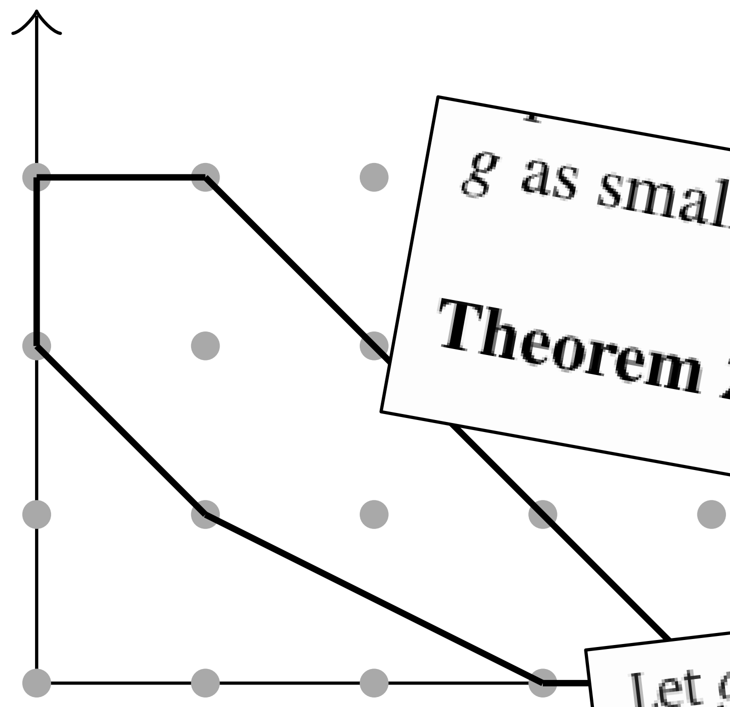
$V/2 =$

OEIS A322343: "Number of equivalence cl

"genus" $= I =$ number of interior points

```
# Every row contains five numbers
# V, k, B, I, N
# where N is the number of lattice polygons with
#   k vertices,
#   B lattice points on edges,
#   I interior lattice points,
#   and area V/2
# among all lattice polygons with area at most 200/2.
1 3 0 0 1
2 3 1 0 1
2 4 0 0 1
3 3 0 1 1
3 3 2 0 1
3 4 1 0 1
 ⋮
200 16 8 89 43
200 17 1 92 4088
200 17 3 91 646
200 17 5 90 11
200 18 0 92 26
200 18 2 91 2
```

What to measure



$k = 6$

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# Every row contains five numbers
# V, k, B, I, N
# where N is the number of lattice polygons with
#   k vertices,
```

lattice points on edges,

... ice points,

most 200/2.

3 3 2 0 1

3 4 1 0 1

⋮

200 16 8 89 43
200 17 1 92 4088
200 17 3 91 646
200 17 5 90 11
200 18 0 92 26
200 18 2 91 2

g as small as 3. Among the other consequences, we find the following.

Theorem 2 *The minimal genus of a lattice 15-gon is 45.*

Let $g(v)$ denote the least possible ... convex lattice v-gon. Values of this function are known ... It is also known that for the smallest not yet established case, it ... $) \leqslant 17$. The purpose of this paper is to prove that $g(11) = 17$.

OEIS A322343: "Nu...

"genus" $= I =$ number of interior points

Quantitative (polygonal) Helly numbers for the integer lattice $\mathbb{Z}^2$

OEIS A298562: $g(\mathbb{Z}^2, m) =$ the maximum k such that there exists a lattice polygon with k vertices containing exactly $m + k$ lattice points (in its interior or on the boundary)

G. Averkov, B. González Merino, I. Paschke, M. Schymura, and S. Weltge, Tight bounds on discrete quantitative Helly numbers (2017). for $m \le 30$.

$$\boxed{m = B + I}$$

m	$g(\mathbb{Z}^2, m)$
0	4
1	6
2	6
3	6
4	8
5	7
6	8
7	9
8	8
9	8

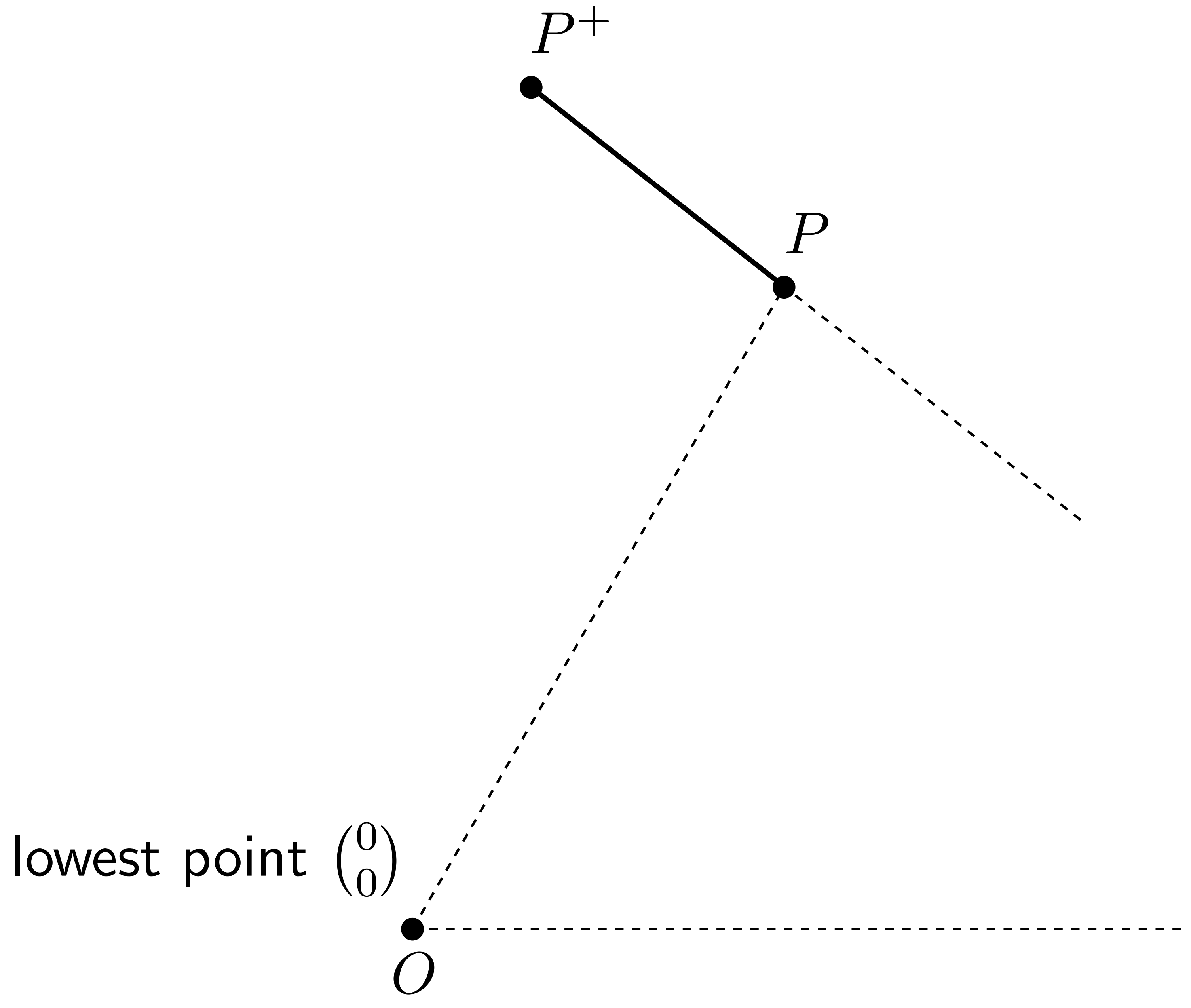
m	$g(\mathbb{Z}^2, m)$
10	10
11	9
12	9
13	10
14	10
15	10
16	10
17	11
18	11
19	12

m	$g(\mathbb{Z}^2, m)$
20	12
21	12
22	11
23	11
24	12
25	12
26	12
27	13
28	12
29	12

$\cdots$

m	$g(\mathbb{Z}^2, m)$
191	23
192	23
193	23
194	23
195	23
196	23
197	23
198	23
199	24
200	23

Dynamic programming in two dimensions

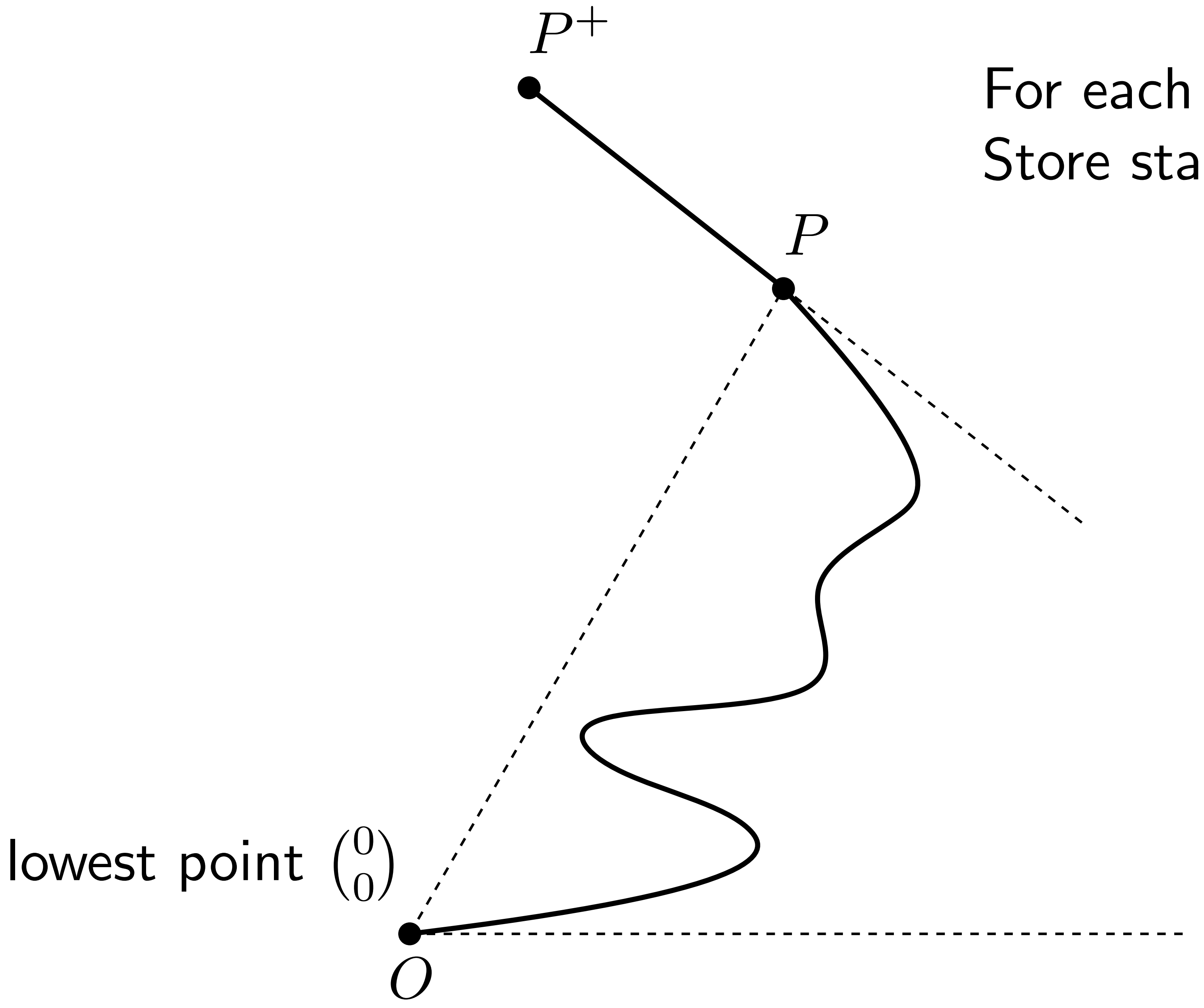


Finding <u>minimum</u> area k-gons. David Eppstein, Mark Overmars, Günter Rote, and Gerhard Woeginger (1992)

<u>Counting</u> convex polygons in planar point sets. $O(kN^3)$ time, $O(kN^2)$ space

vs. <u>Enumerating</u> Joseph Mitchell, Günter Rote, Gopalakrishnan Sundaram, and Gerhard Woeginger (1995)

Dynamic programming in two dimensions



For each PP^+, consider all lattice polygons ending in PP^+.
Store statistics about the quantities that you care for:

- For each k, the smallest area of a convex k-gon $O \dots PP^+$

- For each V, the number of convex polygons $O \dots PP^+$ of area V

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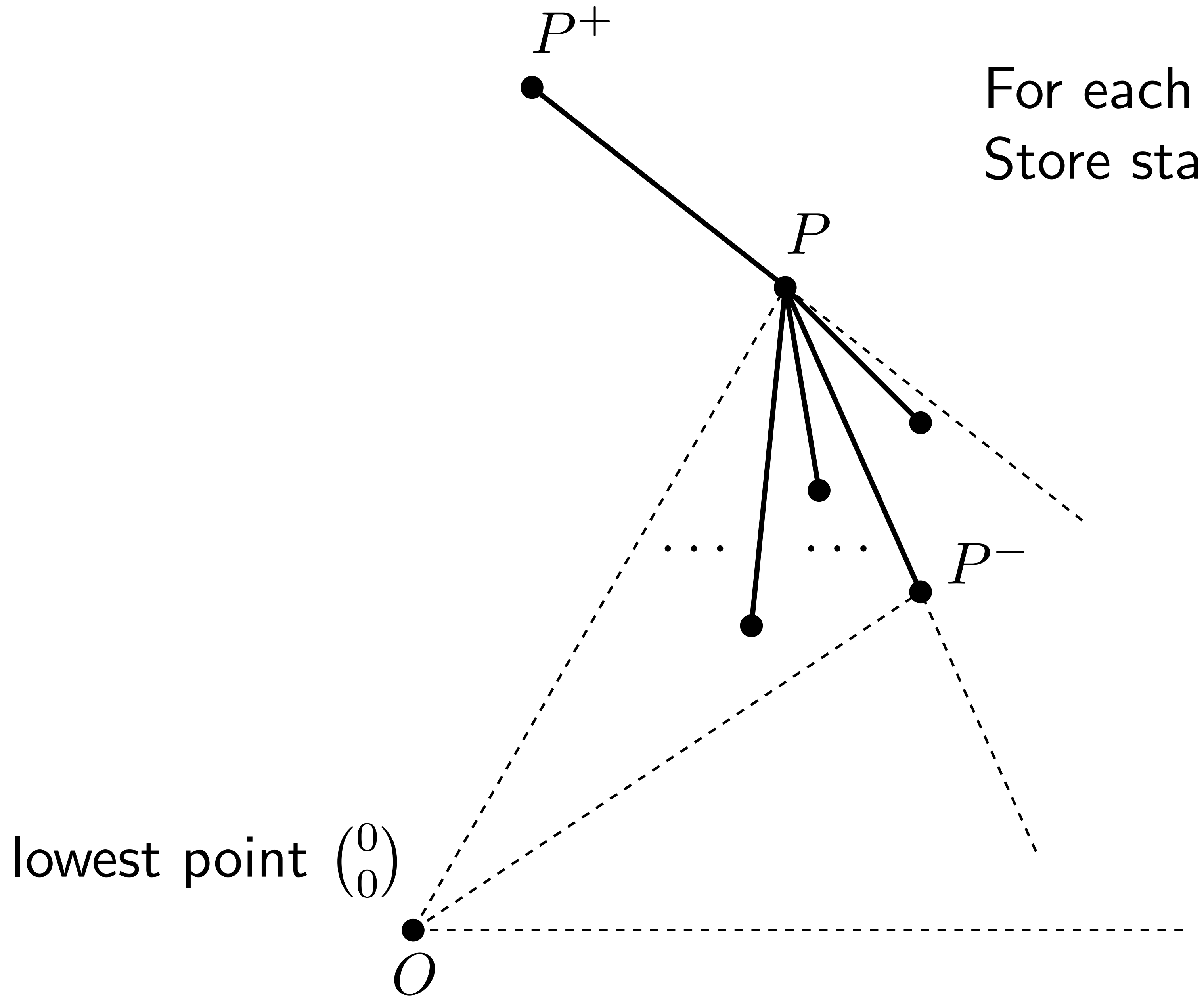

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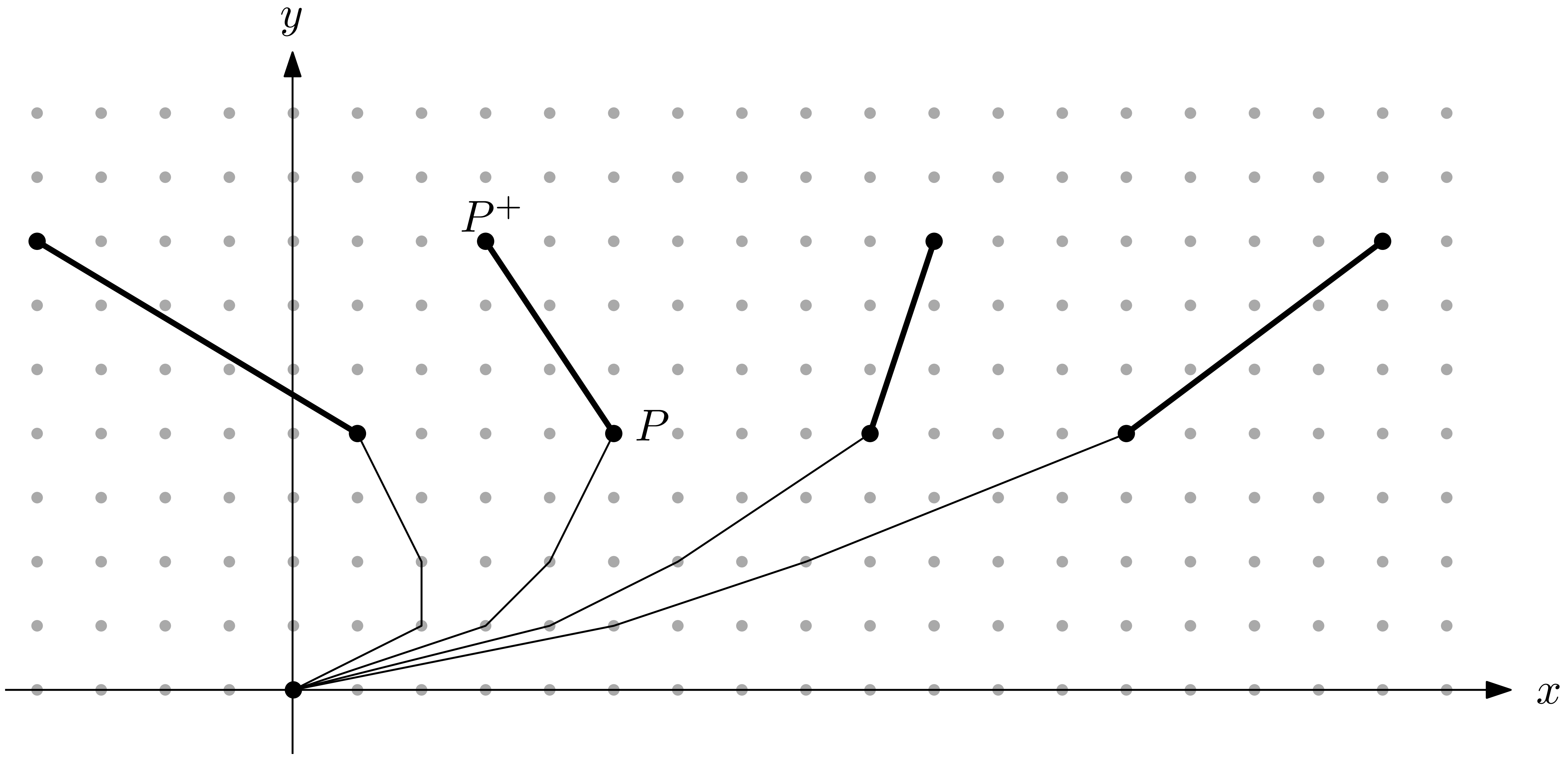
Normalize by horizontal shearings

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \pm y \\ y \end{pmatrix}$$



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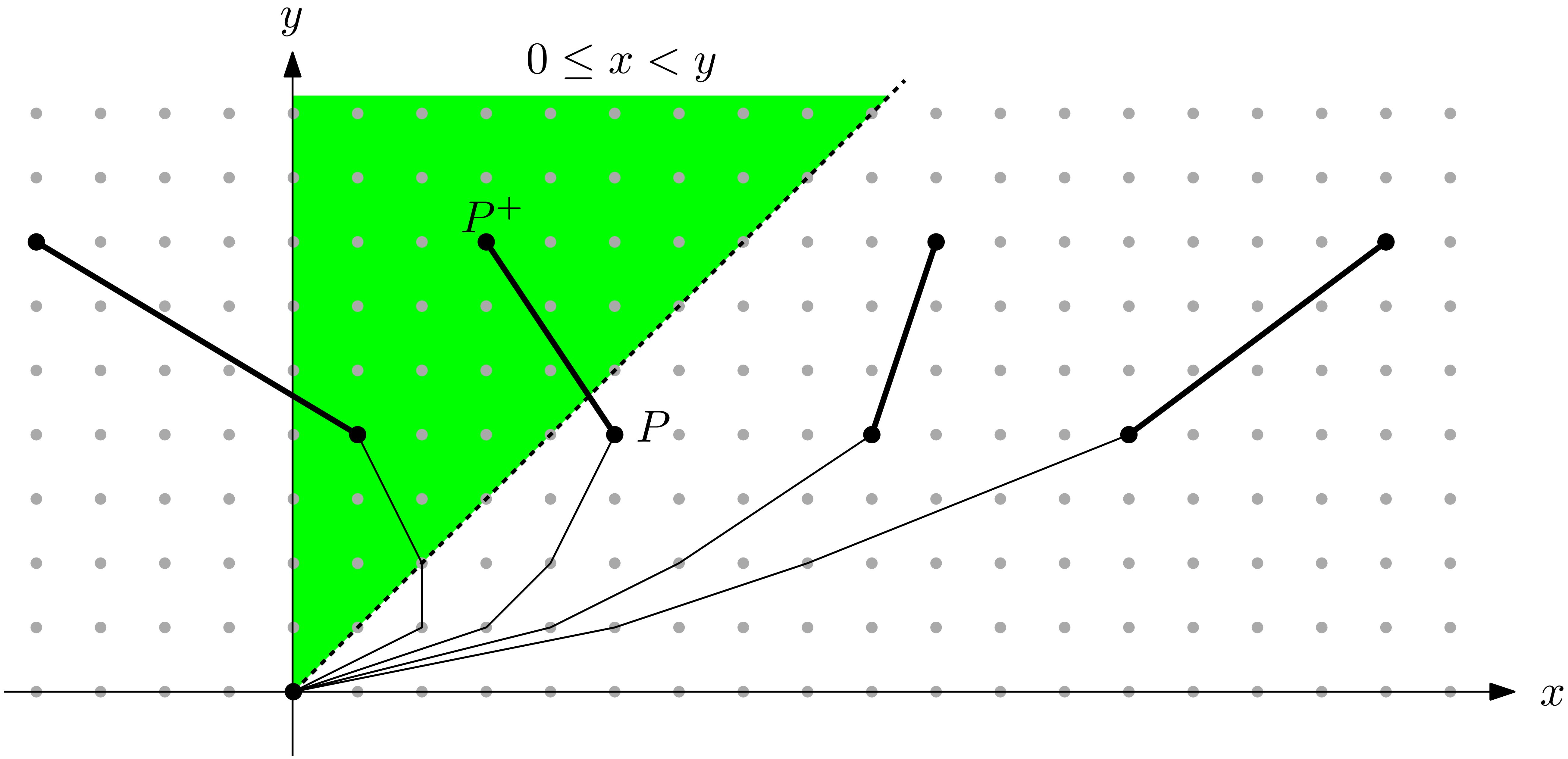


Upper bound for the height of smallest k-gons

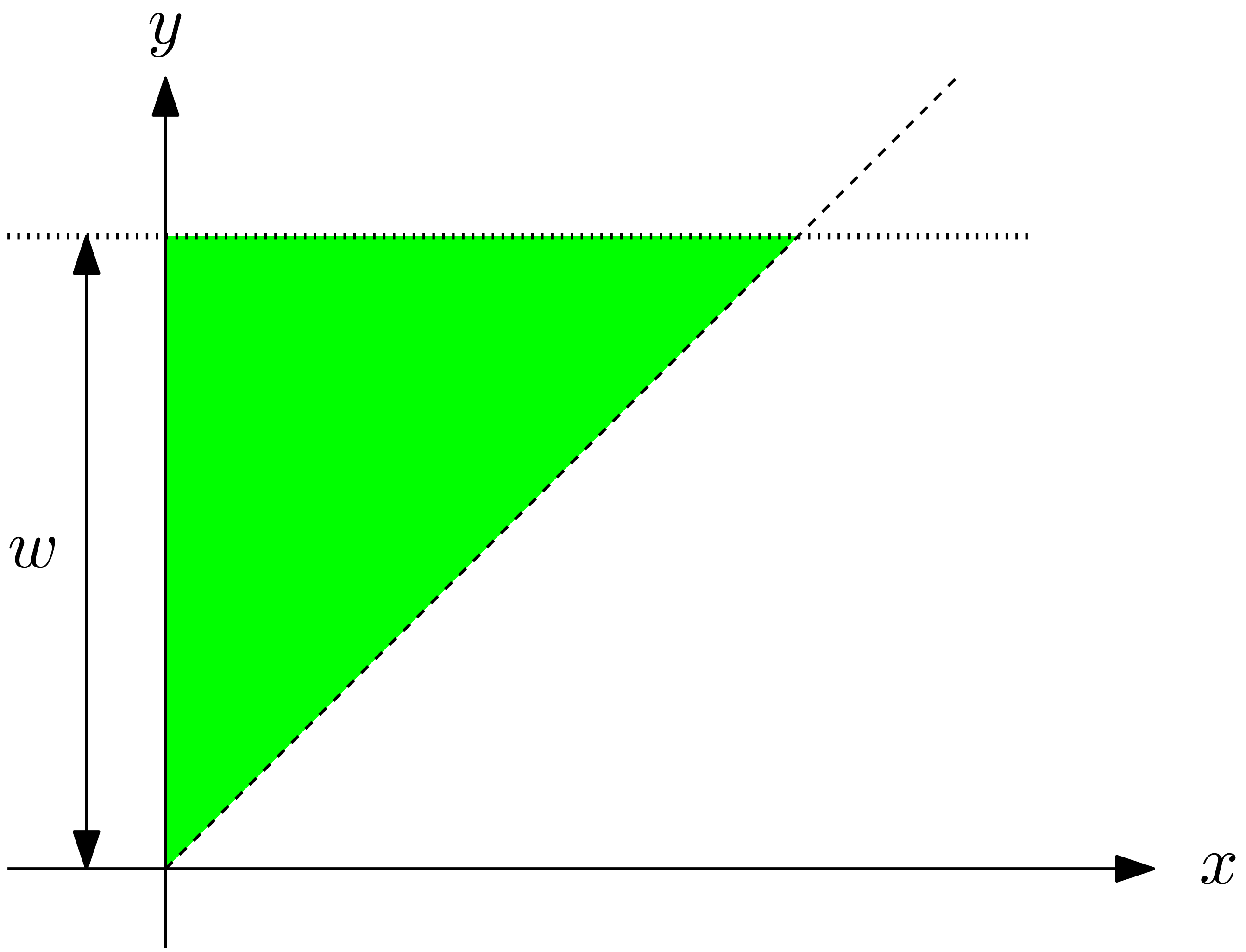
Lemma:

- A convex lattice polygon P of lattice width w has area at least ~~$w^2/3$~~. $\frac{3}{8}w^2$
- [If k is even, P can be assumed to be centrally symmetric, and then it has area at least $w^2/2$.]

L. Fejes Tóth, E. Makai jr. (1974), F. Cools, A. Lemmens (2017)

Lattice width w $\rightarrow$ A unimodular transformation brings P into the strip $0 \le y \le w$.

If a k-gon of area V is found:
$\rightarrow$ terminate as soon as $y > \sqrt{3V}$



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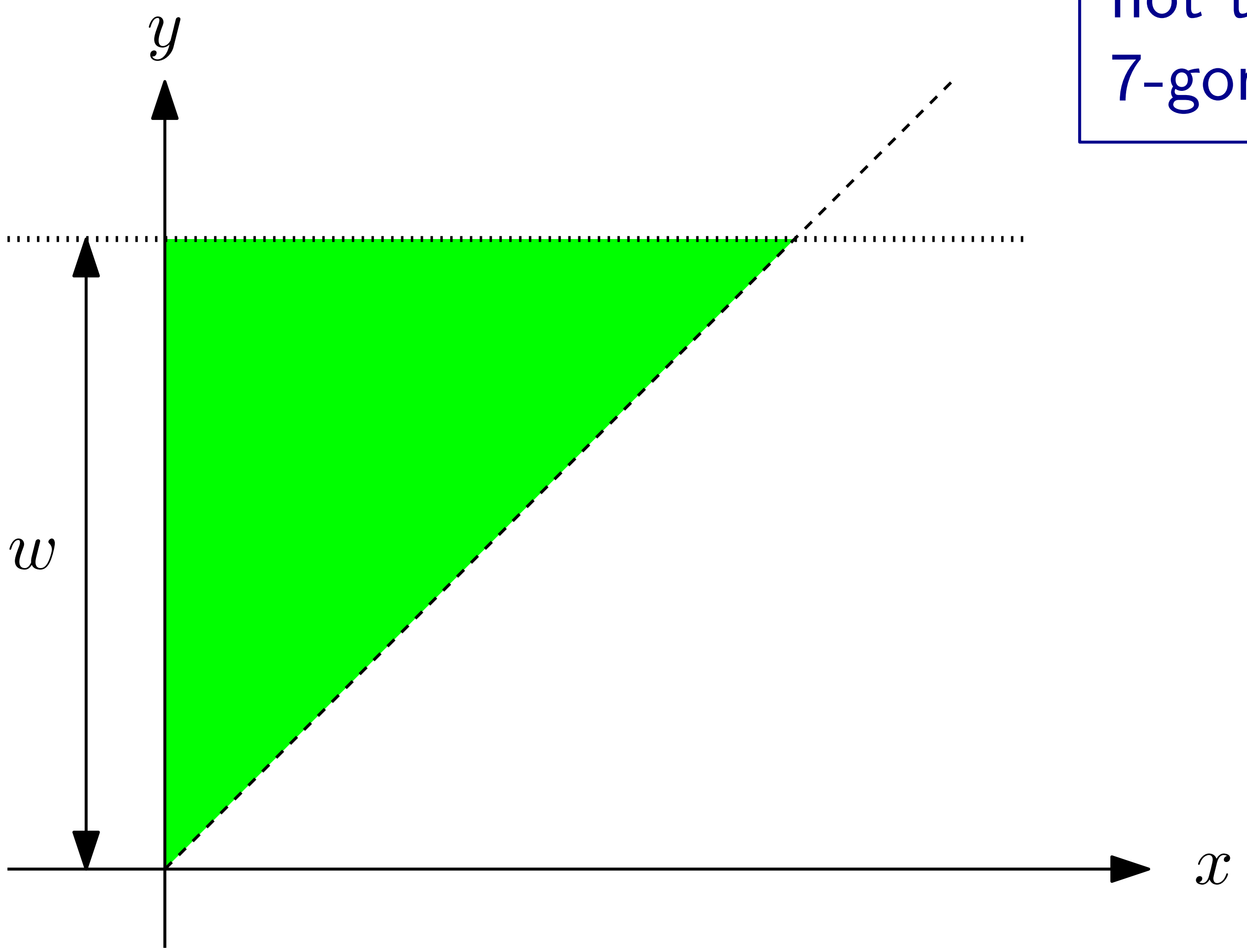
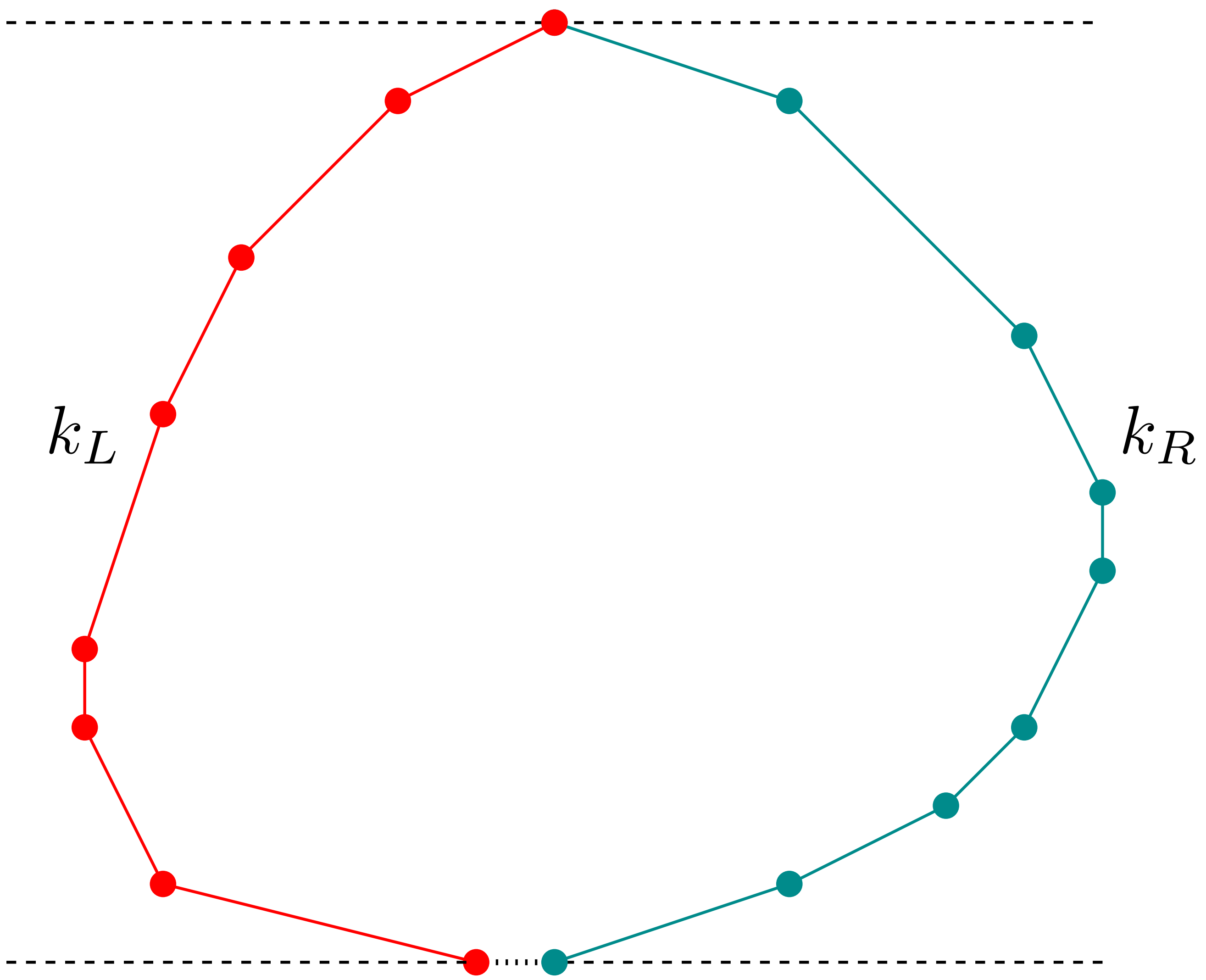
Lattice width w → A unimodular transformation brings P into the strip $0 \le y \le w$.

not true for optimal 5-gons, 7-gons, and 11-gons

If a k-gon of area V is found:
→ terminate as soon as $y > \sqrt{3V}$

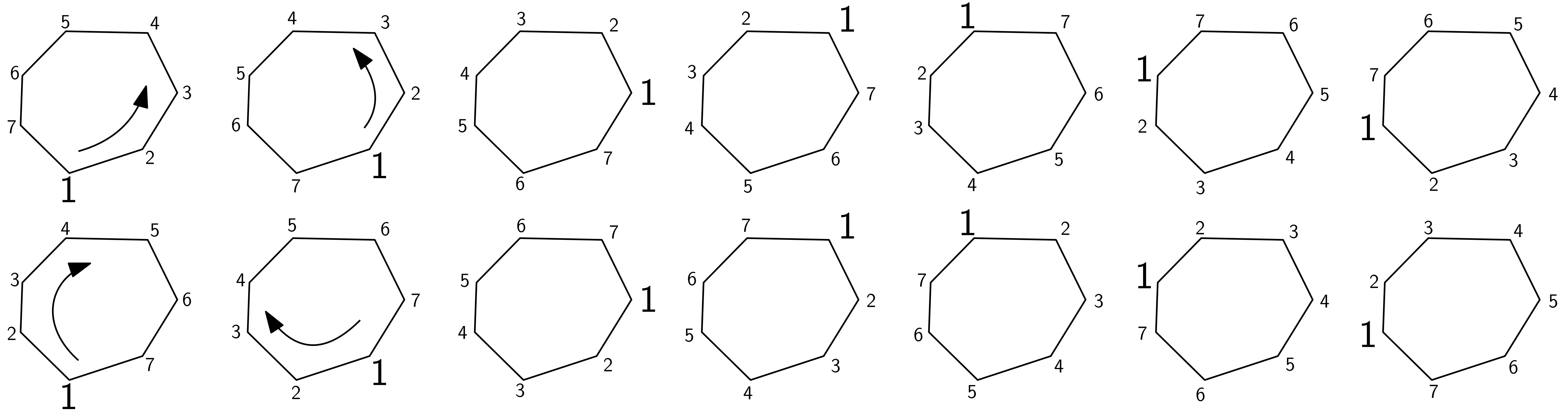


Putting together the solution



OPEN QUESTION:
Can we assume that $|k_L - k_R| \leq 1$?

Overcounting and Symmetries

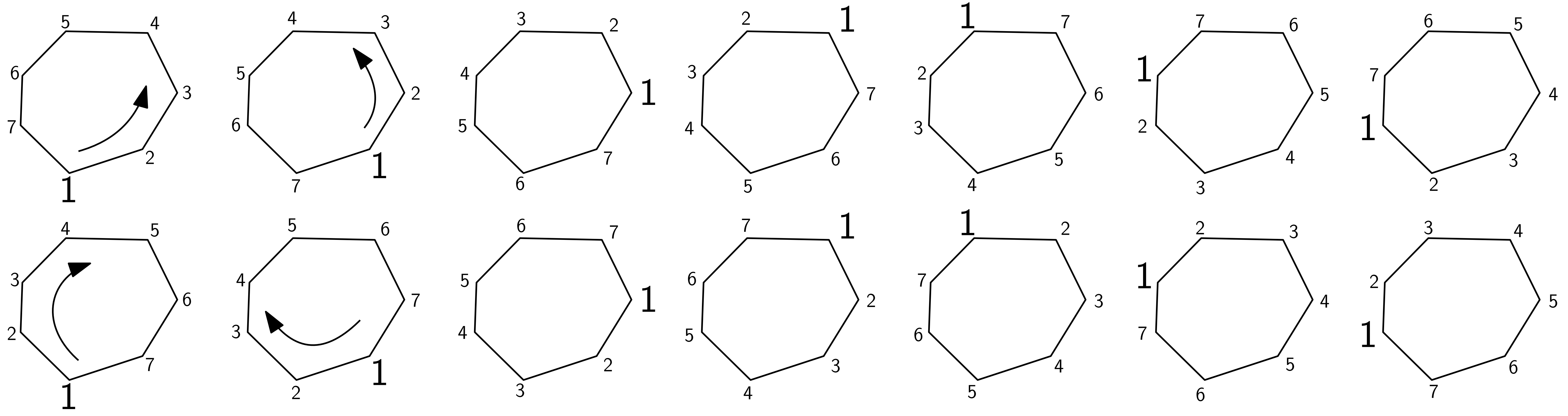


Dihedral group D_{2k} of order $2k$: k "rotations" and k "reflections" $g \in D_{2k}$

Burnside's lemma:

$$\#\text{orbits} = \frac{1}{|D_{2k}|} \sum_{g \in D_{2k}} \#(\text{polygons fixed by } g)$$

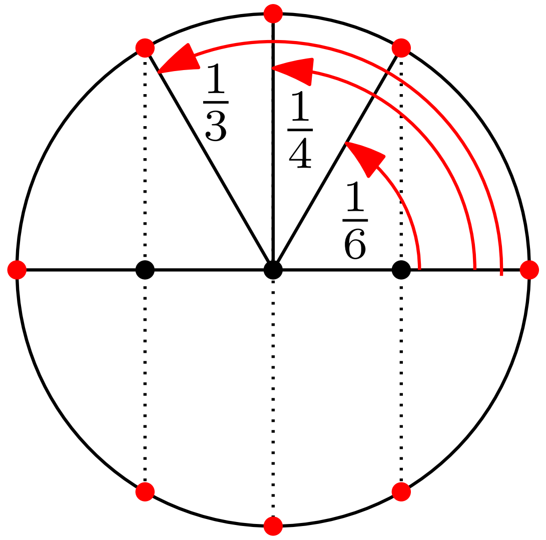
Overcounting and Symmetries



"Rotations" of order r:

$\binom{x}{y} \mapsto M\binom{x}{y} + t$, $M \in \mathbb{Z}^{2\times 2}$, $\det M = +1$, $M^r = I$

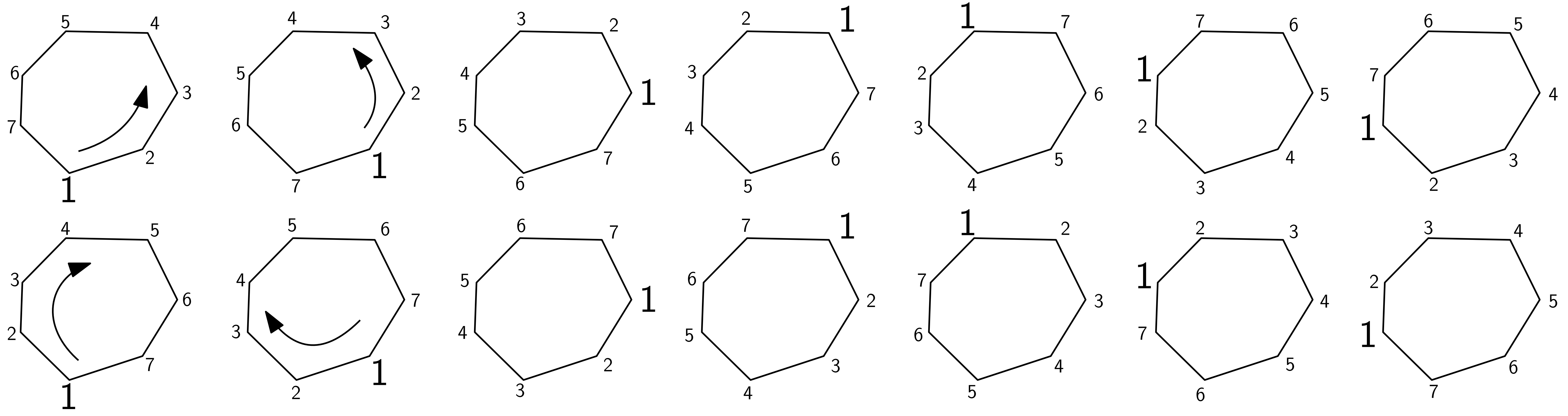
$$M = S \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} S^{-1}$$



order r	α	$\operatorname{tr} M = 2\cos\alpha \in \mathbb{Z}$
1	$2\pi \cdot 1$	2
2	$2\pi \cdot \frac{1}{2}$	-2
3	$2\pi \cdot \frac{1}{3}$, $2\pi \cdot \frac{2}{3}$	-1
4	$2\pi \cdot \frac{1}{4}$, $2\pi \cdot \frac{3}{4}$	0
6	$2\pi \cdot \frac{1}{6}$, $2\pi \cdot \frac{5}{6}$	1

(cf. the crystallographic restriction)

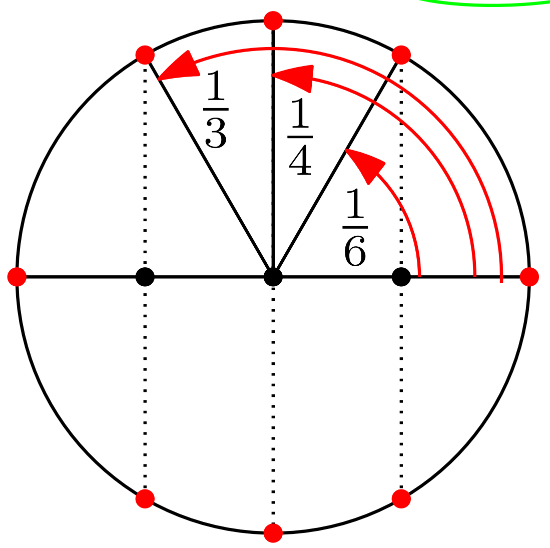
Overcounting and Symmetries



“Rotations” of order r:

$\binom{x}{y} \mapsto M\binom{x}{y} + t$, $M \in \mathbb{Z}^{2\times 2}$, $\det M = +1$, $M^r = I$

$$M = S \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} S^{-1}$$



order r	α	$\operatorname{tr} M = 2\cos\alpha \in \mathbb{Z}$
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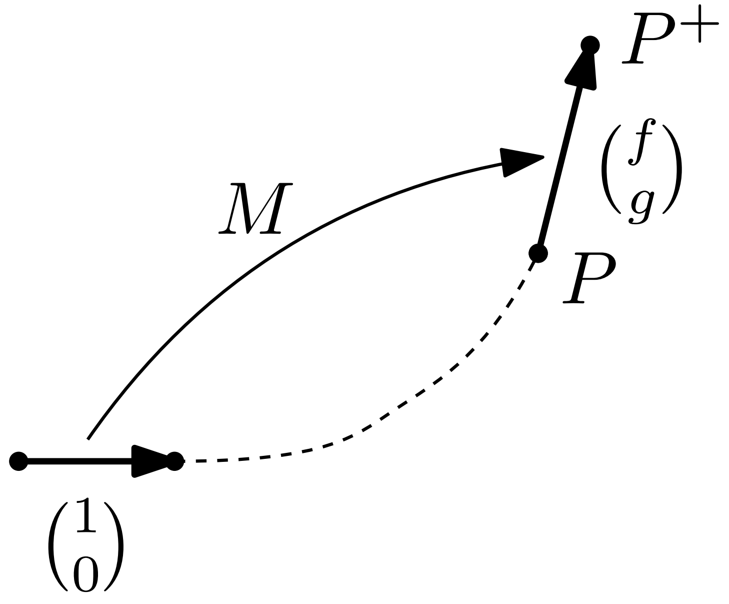
Overcounting and Symmetries

$M\binom{1}{0} = \binom{f}{g}$

Can this map be iterated so that $M^r = I$?



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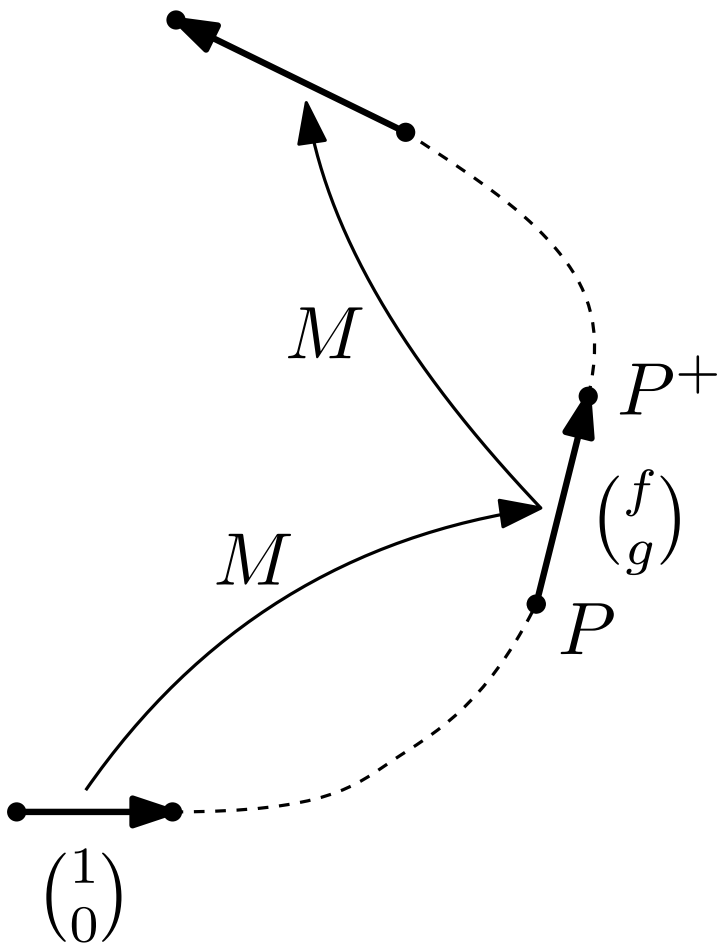

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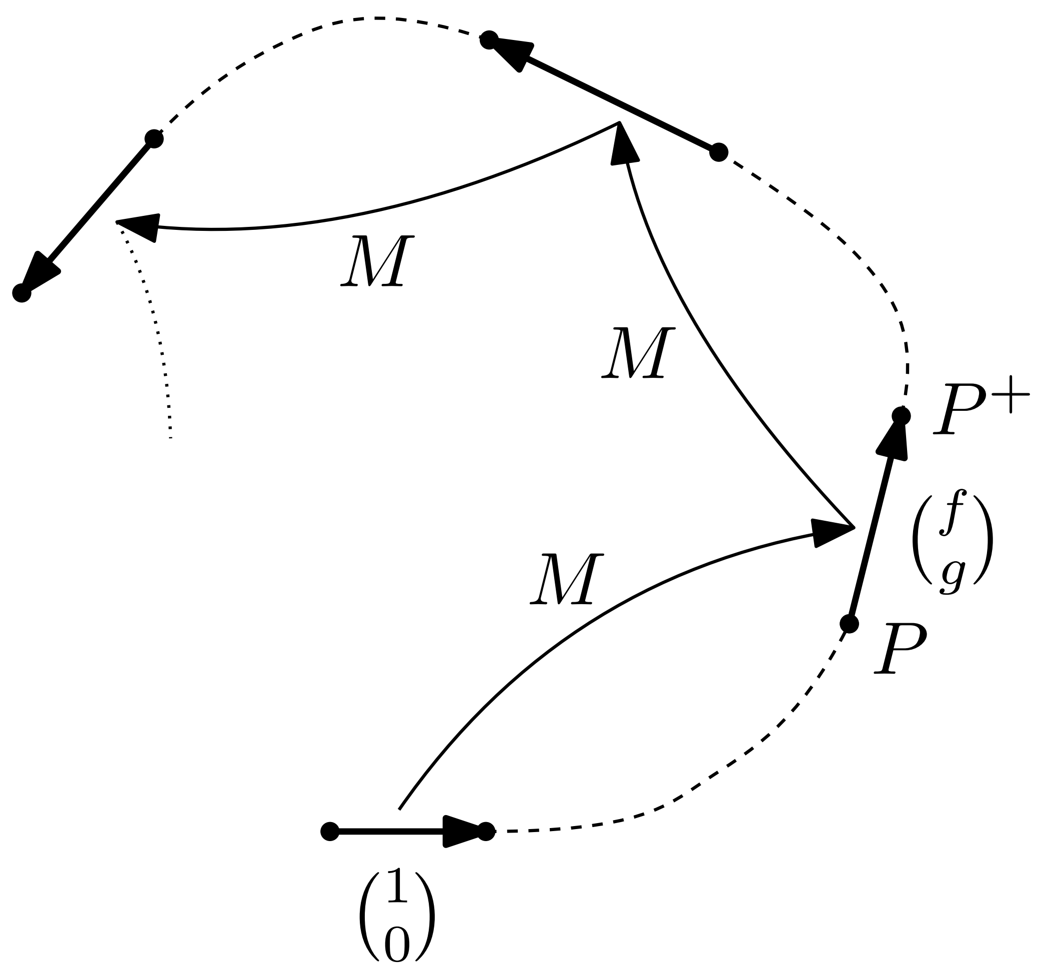

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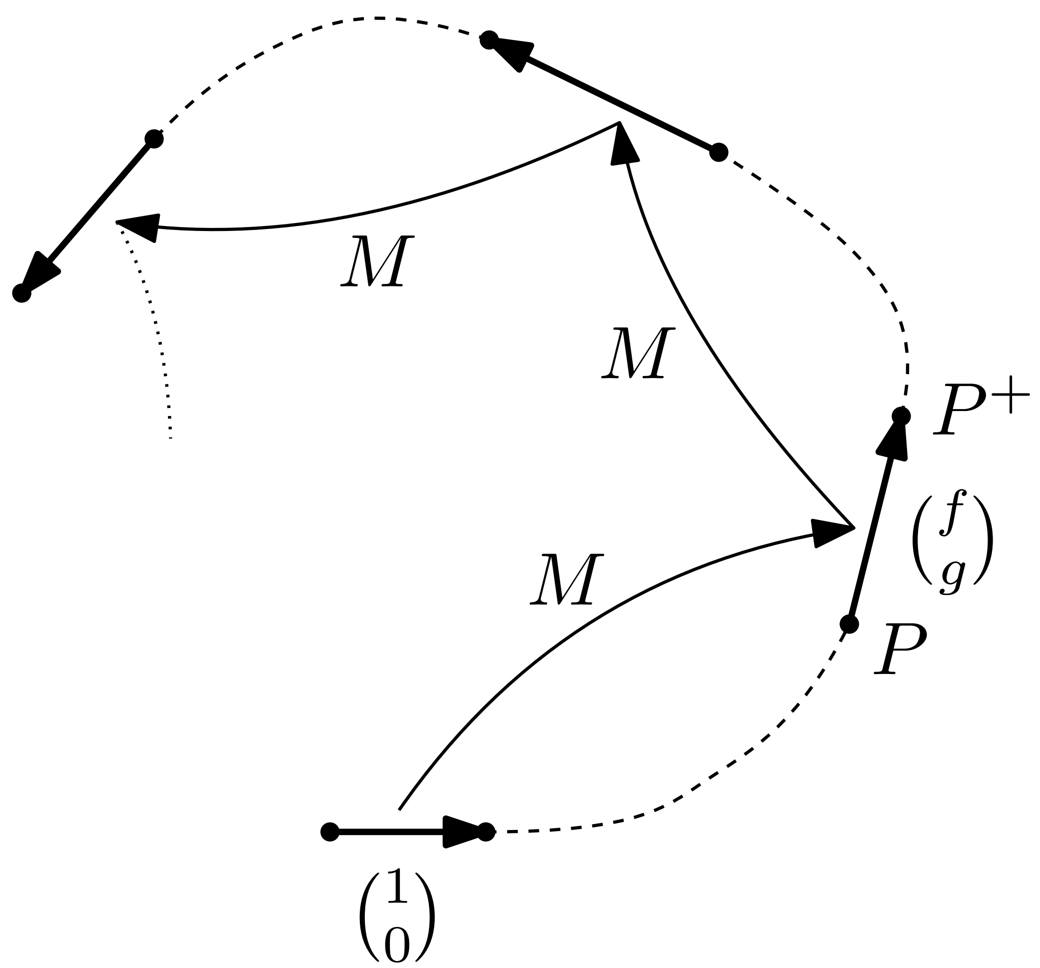

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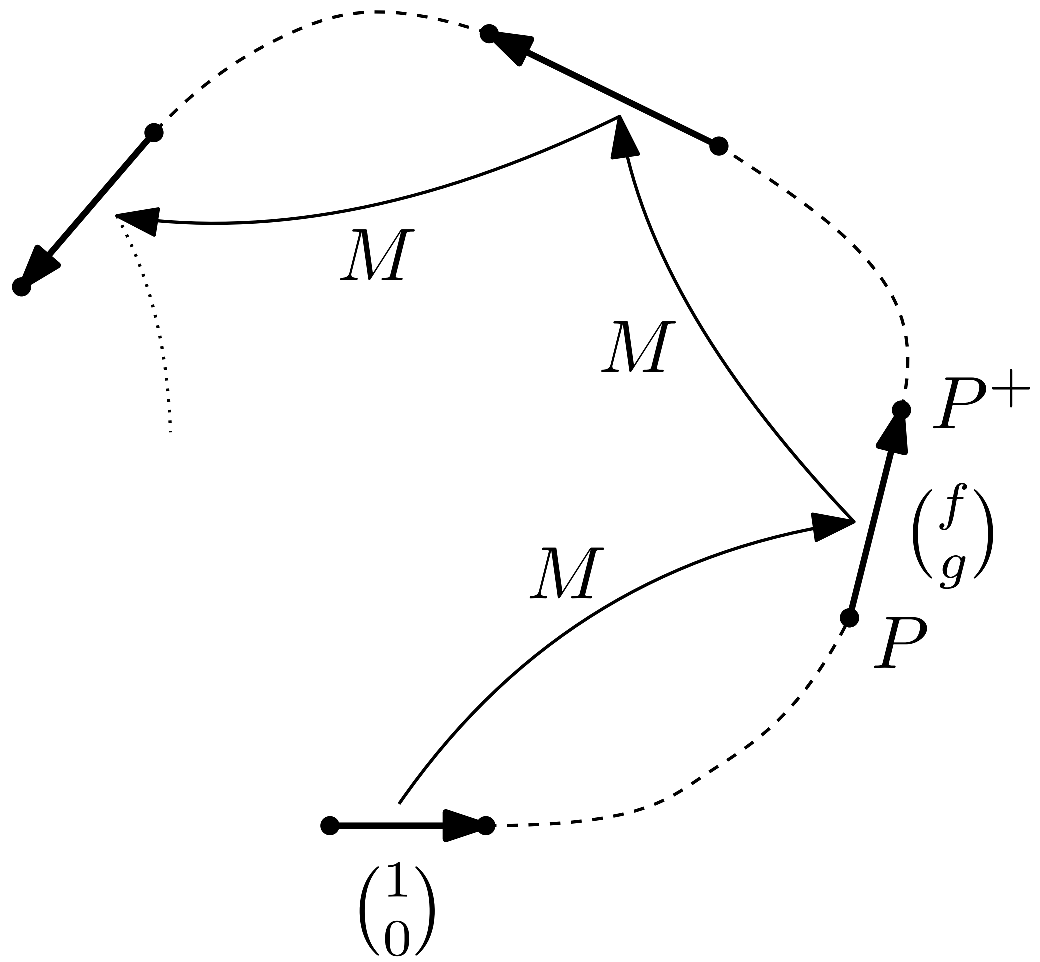

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$\text{tr } M = -1, 0, +1$ (three possibilities)

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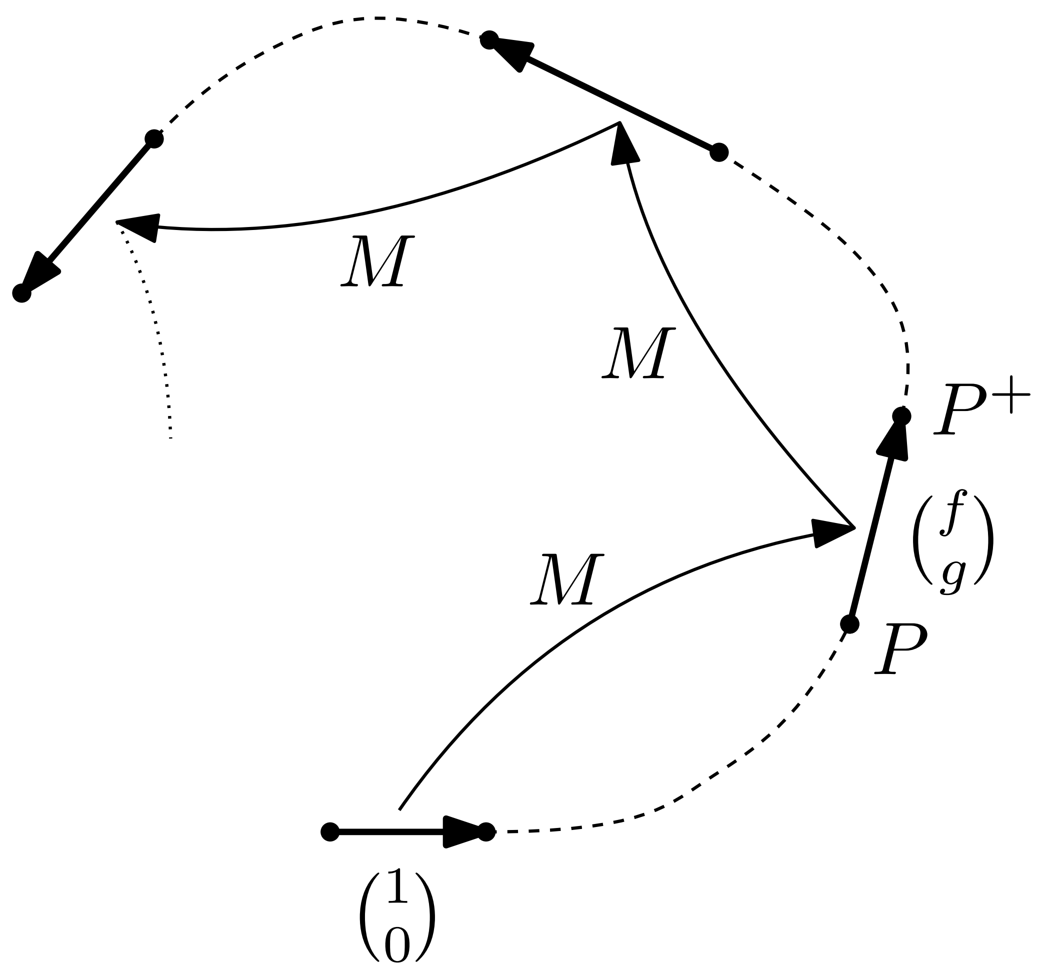

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$\det M = 1, \quad M \in \mathbb{Z}^{2\times 2}$!

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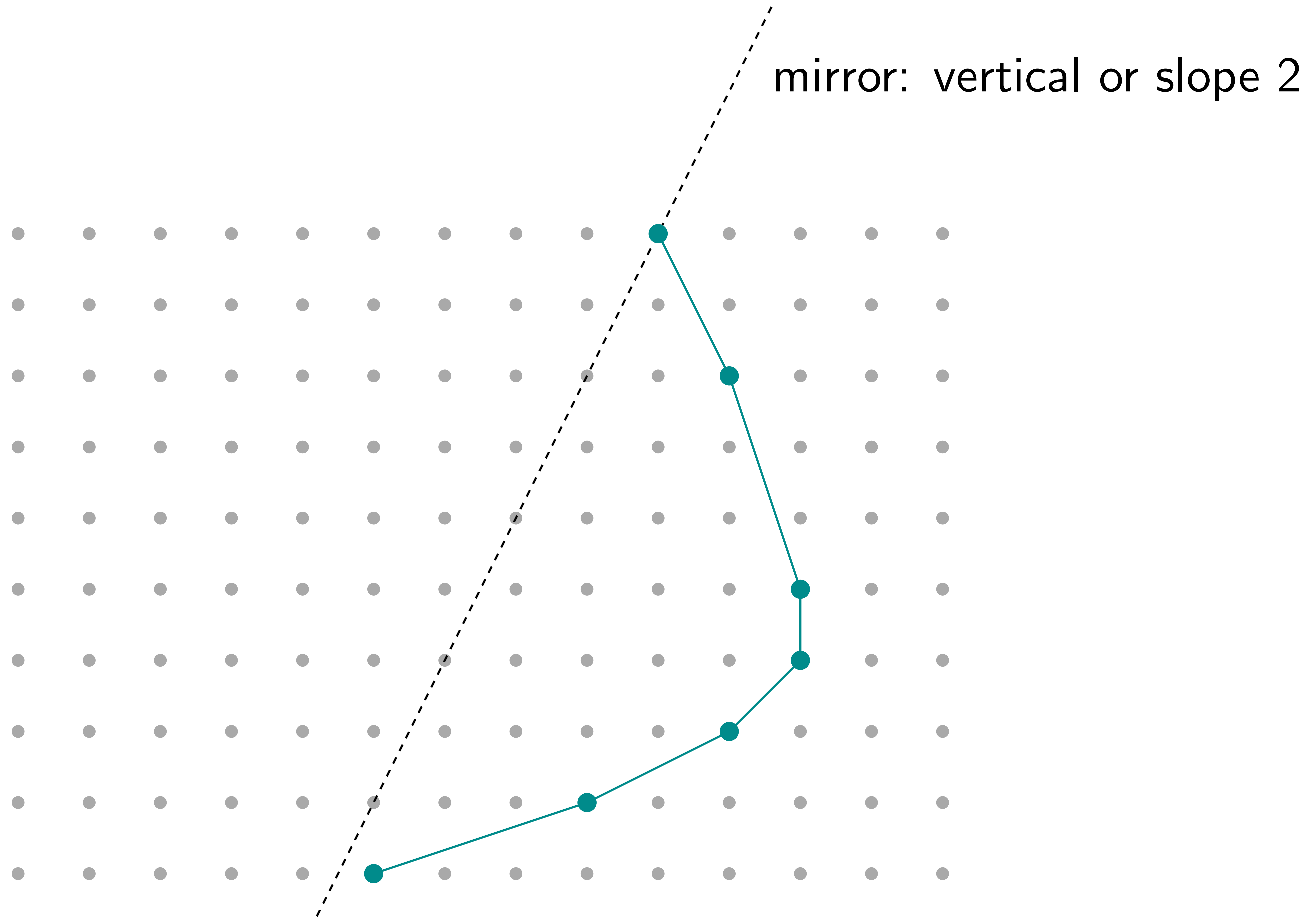
Overcounting and Symmetries

"Reflections": $\binom{x}{y} \mapsto \binom{-x}{y}$ or $\binom{x}{y} \mapsto \binom{y-x}{y}$



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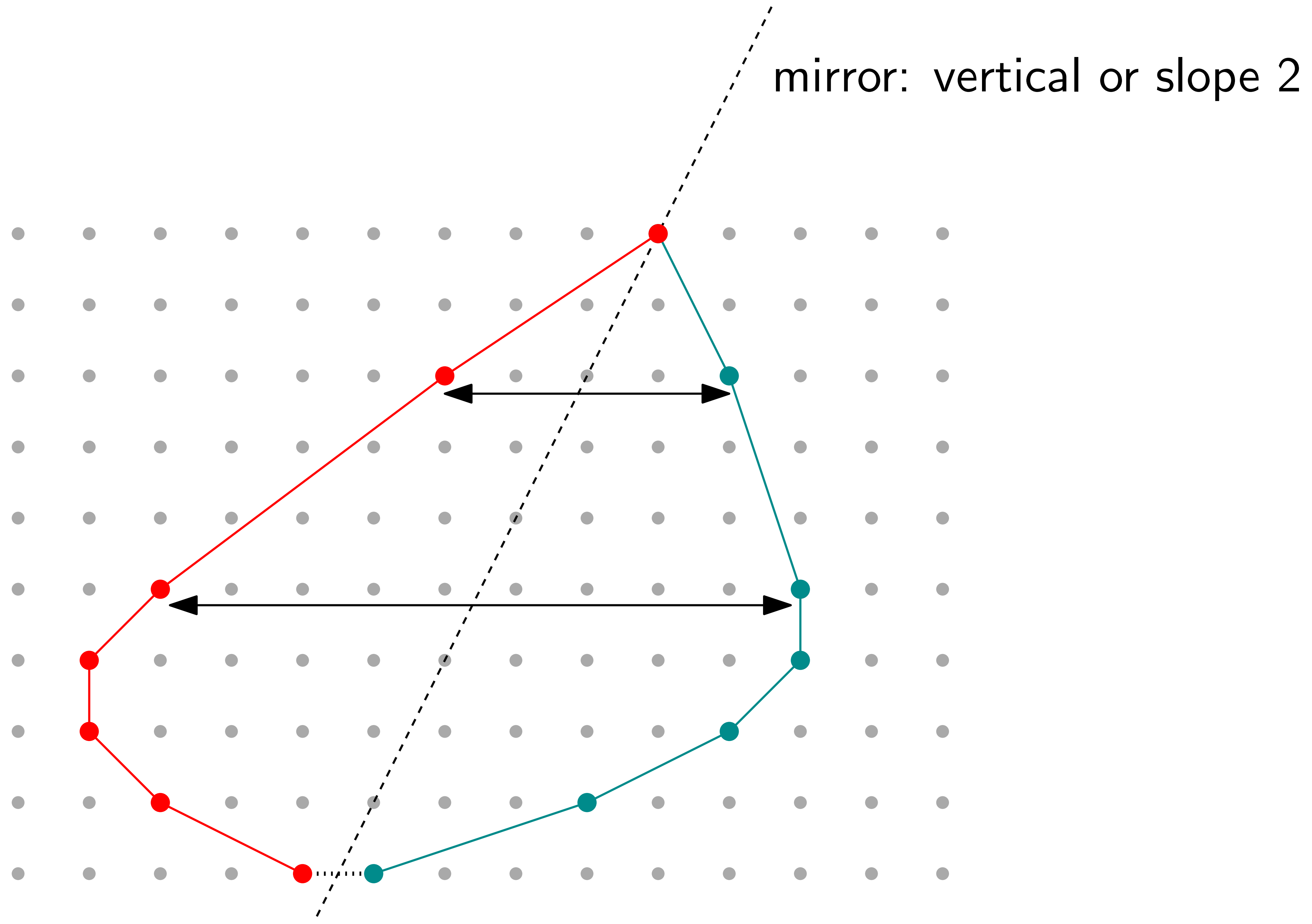
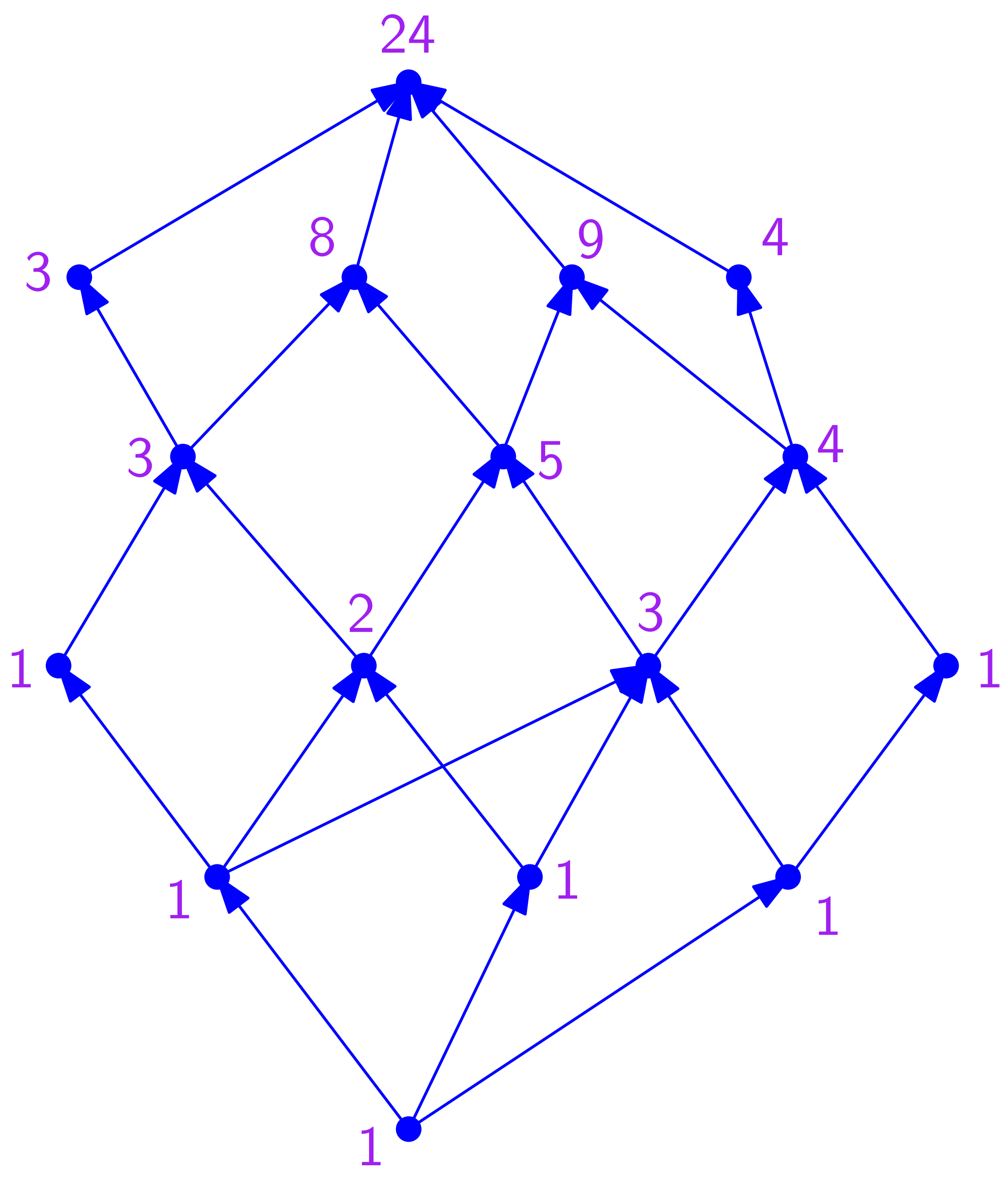


Enumerate ALL / generate a RANDOM lattice polygon with given parameters

Abstract model as a directed acyclic graph:

nodes $\equiv$ subproblems $\equiv$ edges PP^+

source-sink paths $\equiv$ solutions $\equiv$ polygons



Enumerate ALL / generate a RANDOM lattice polygon with given parameters

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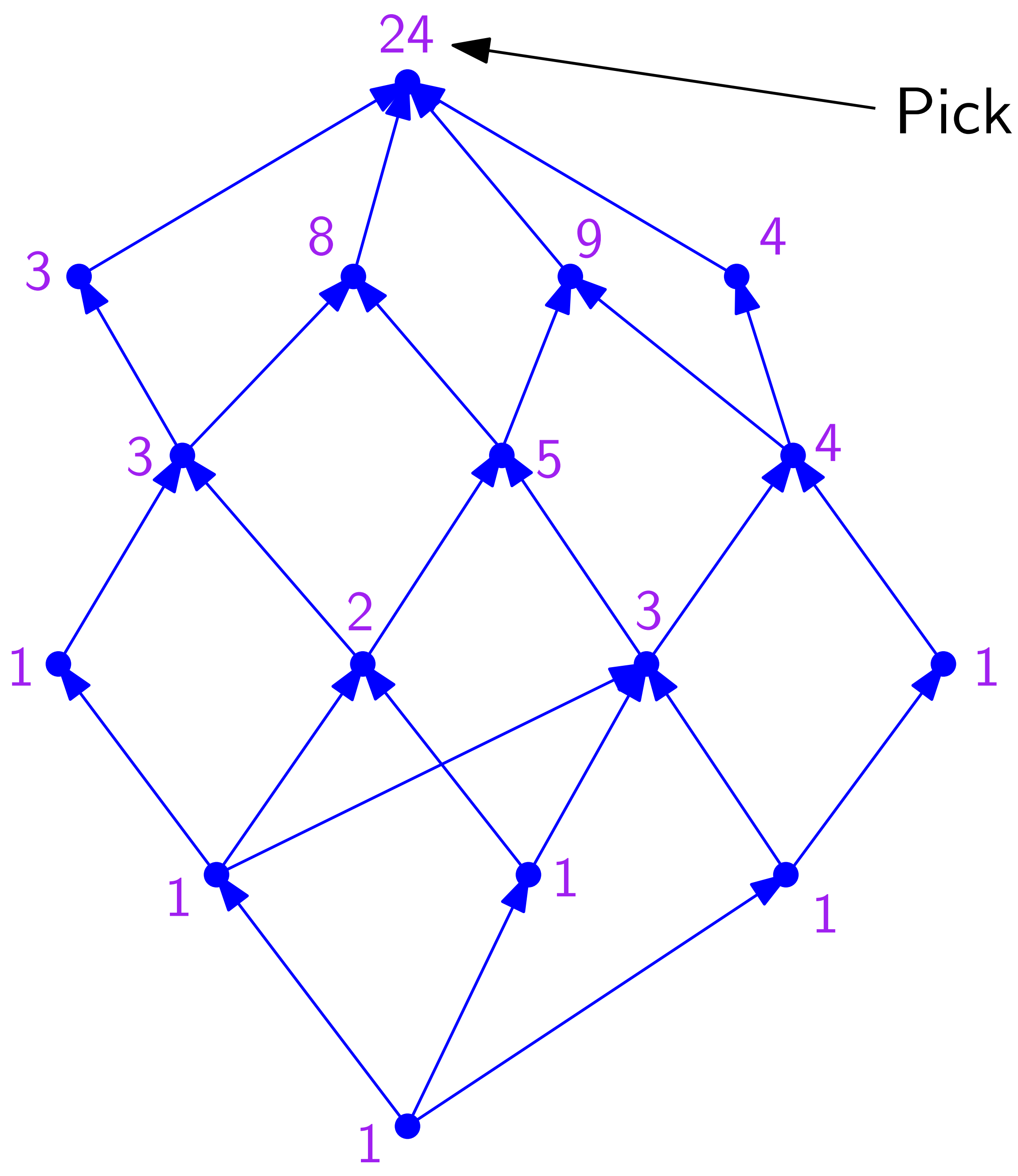

Pick a random number r between 1 and 24 and find the r-th solution.

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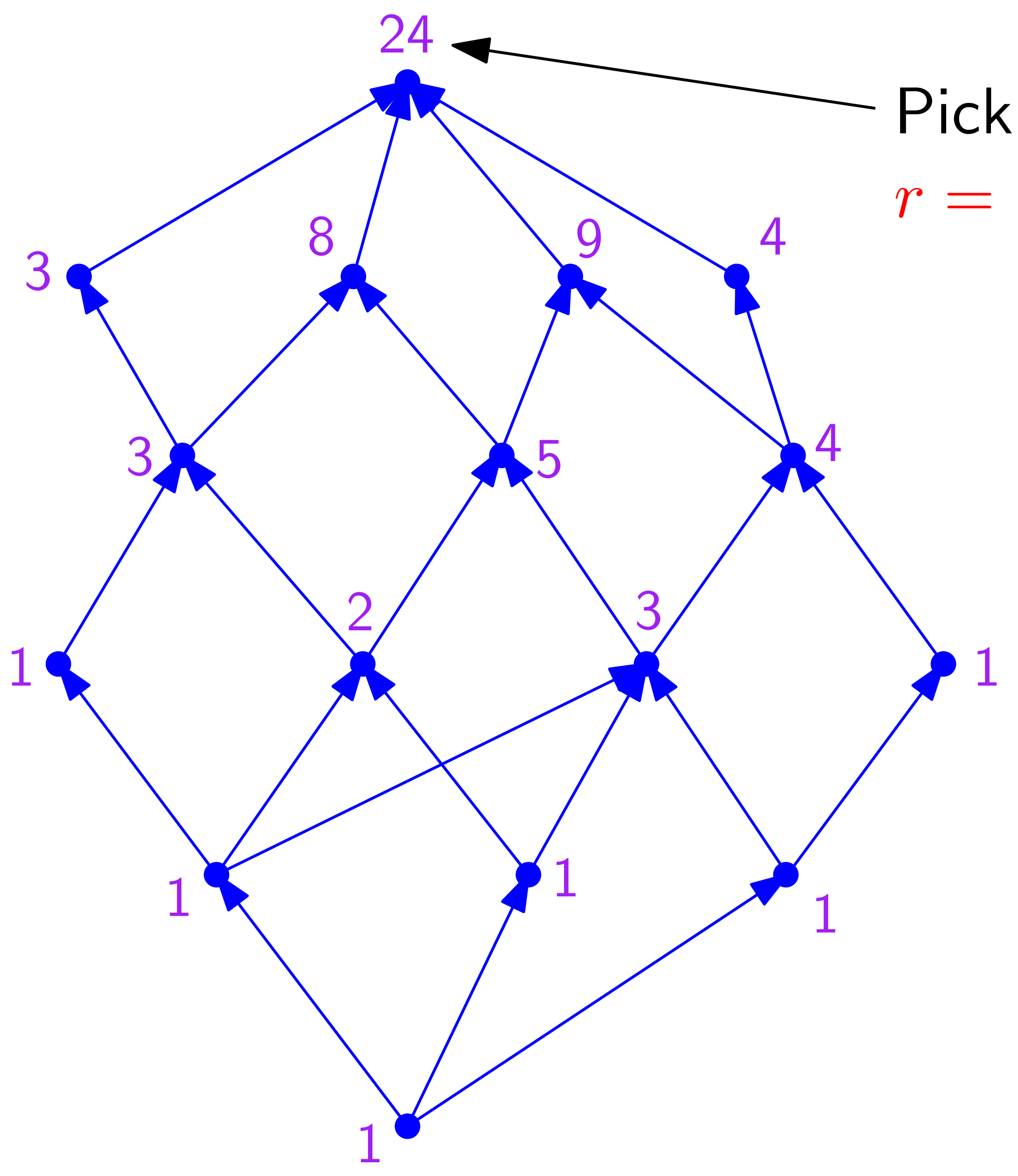

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$r = 16$

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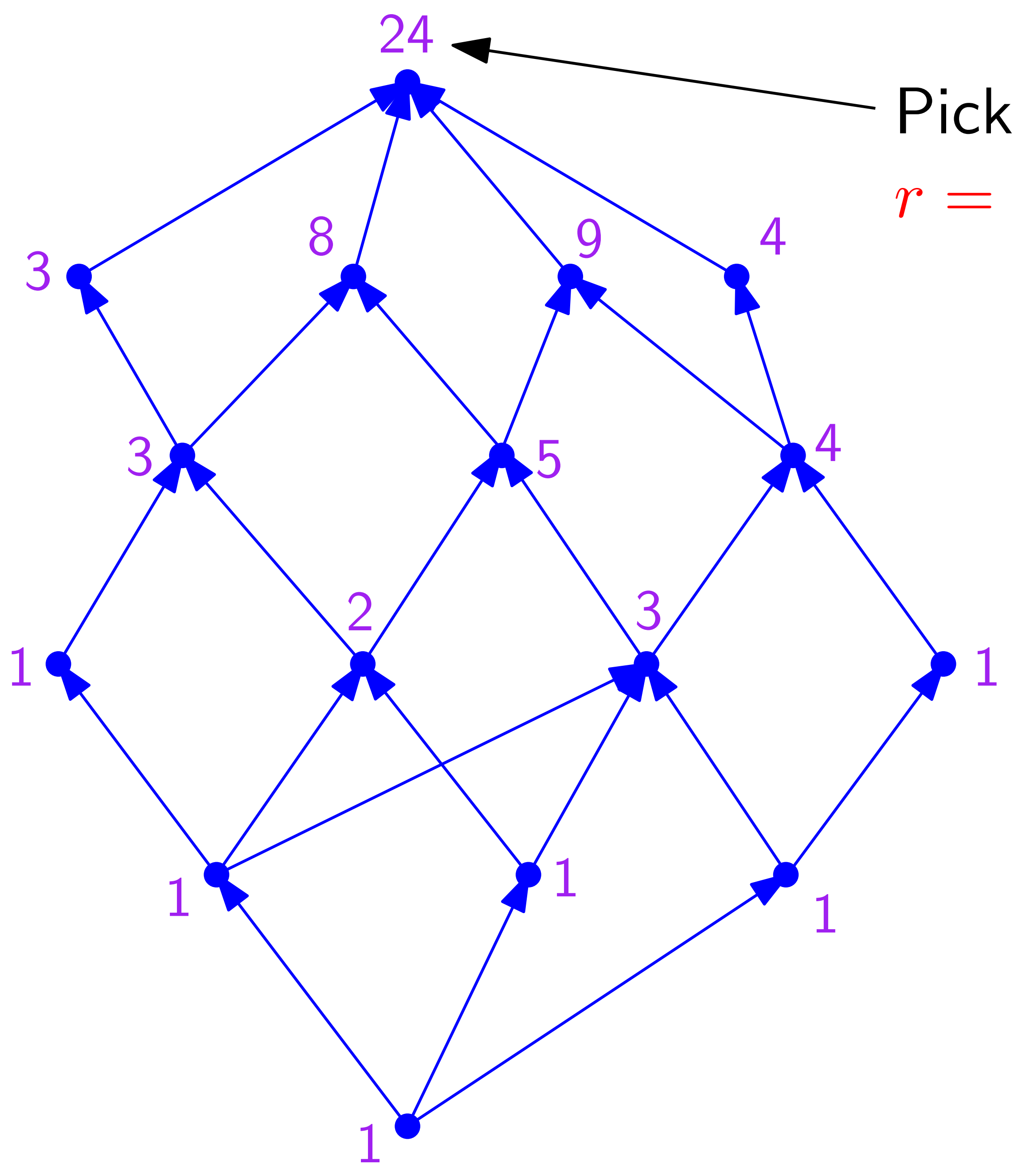

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$r = 16 = 3 + 8 + 5$

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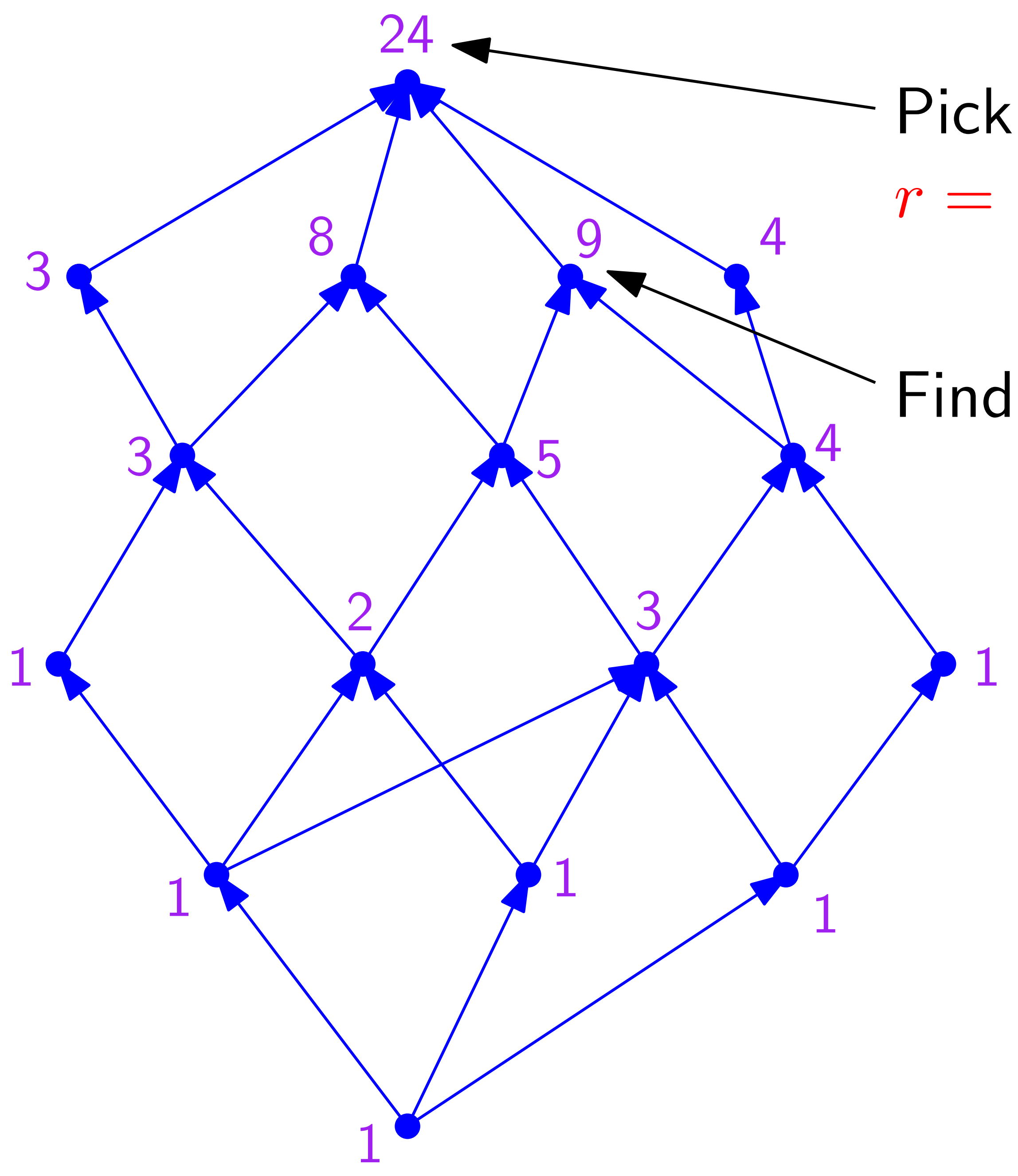

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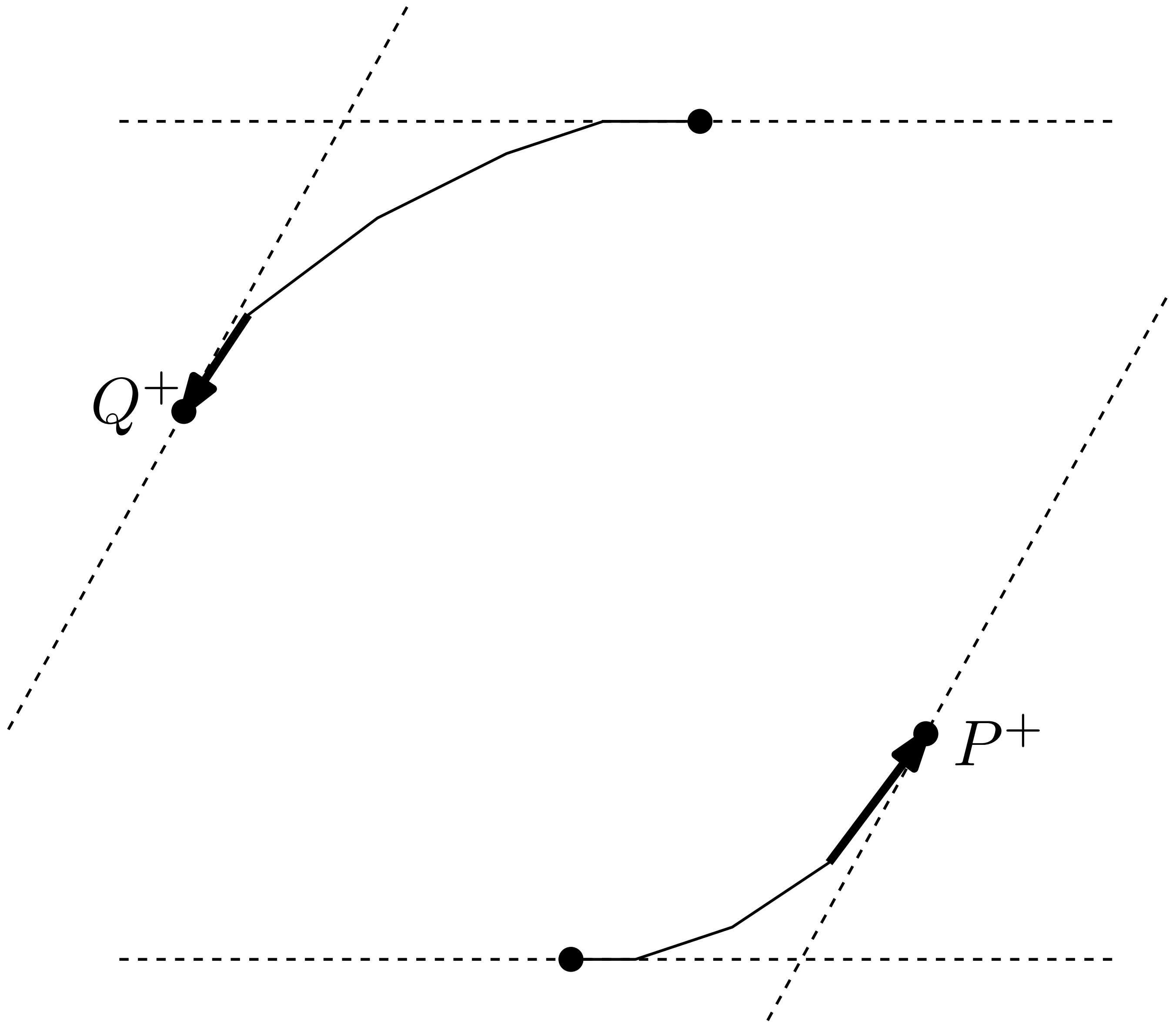
$r = 16 = 3 + 8 + 5$

Find the 5-th solution leading to this node.

Taking the lattice width into account?

OEIS A322348: *Maximal lattice width* of a convex lattice polygon containing I lattice points in its interior ("of genus I").

2 ($n = 0$),
3, 2, 4, 4, 4, 5, 4, 4, 5, 6,
5, 6, 6, 6, 7, 6, 6, 7, 8, 7,
8, 8, 8, 8, 8, 8, 8, 9, 8, 9



cf. F. Cools, A. Lemmens (2017): Characterization of *minimal* polygons with given lattice width.