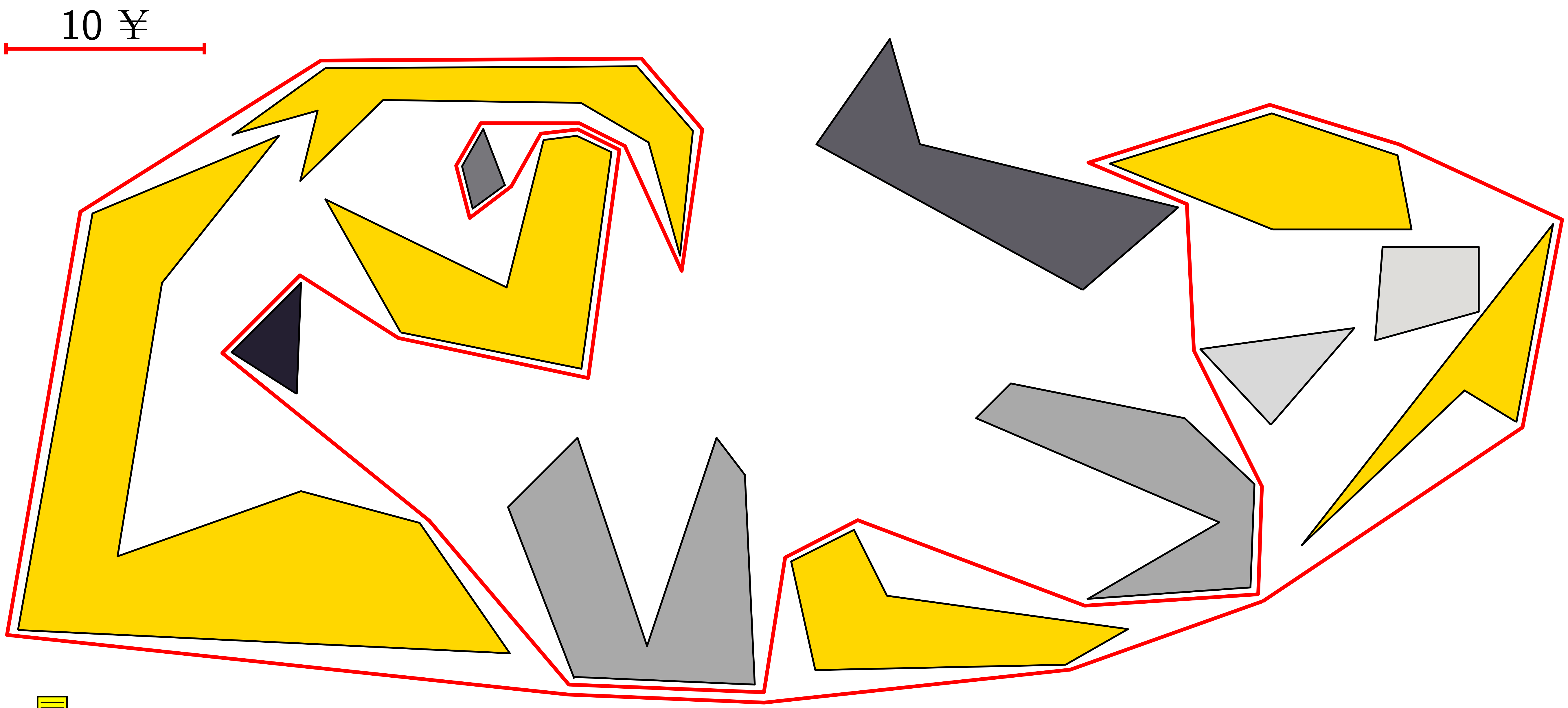
# Finding a Shortest Curve that Separates Few Objects from Many

Therese Biedl, Éric Colin de Verdière, Fabrizio Frati, Anna Lubiw, Günter Rote



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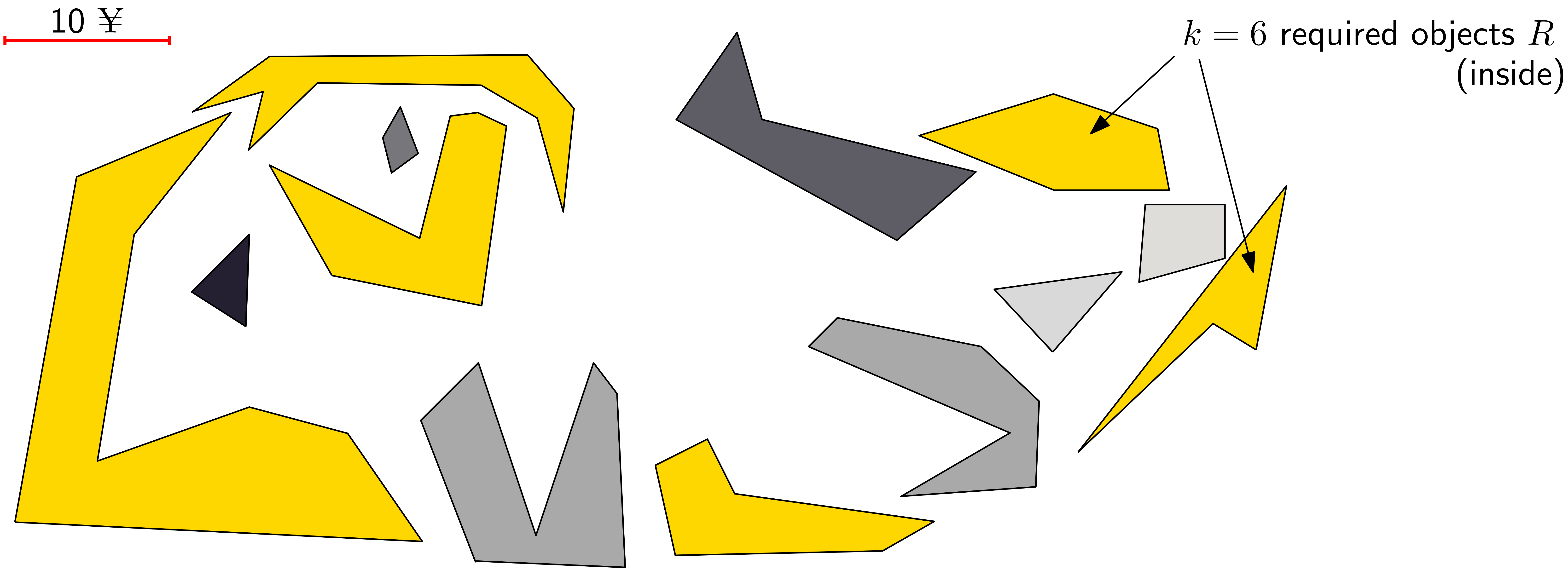

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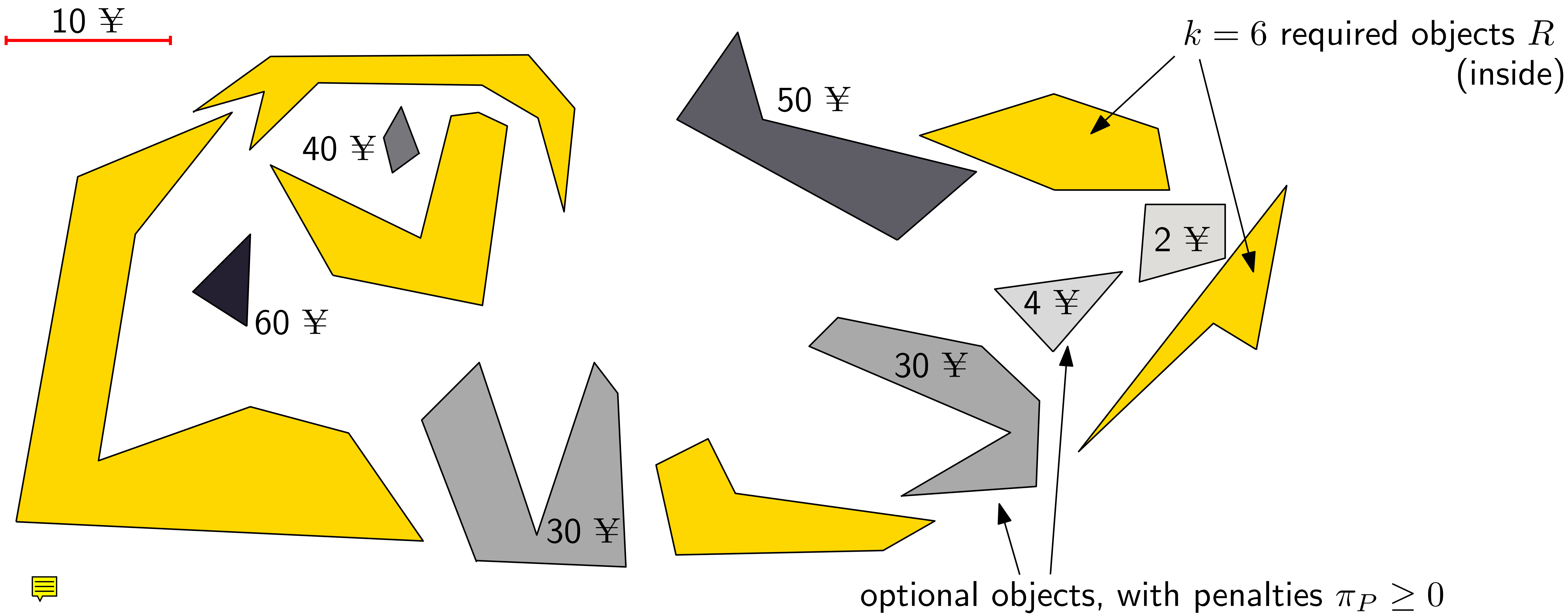

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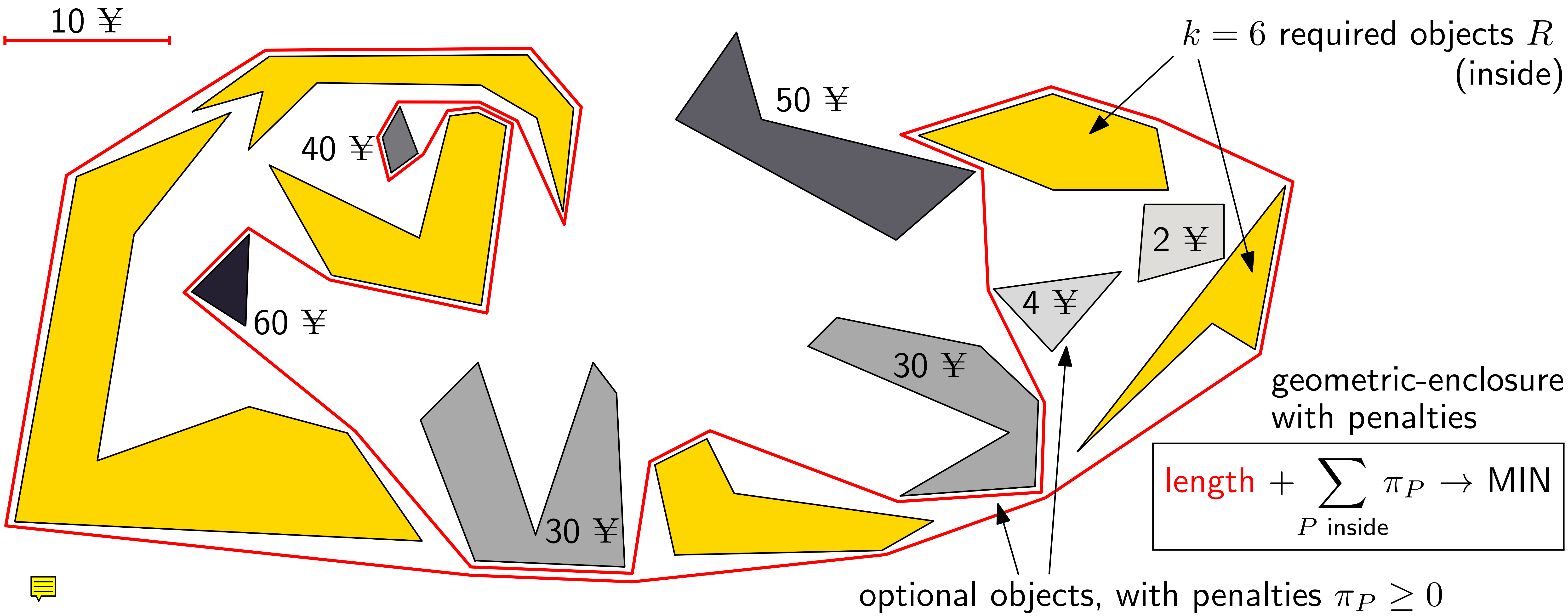


geometric-enclosure with penalties

$$\text{length} + \sum_{P \text{ inside}} \pi_P \to \text{MIN}$$

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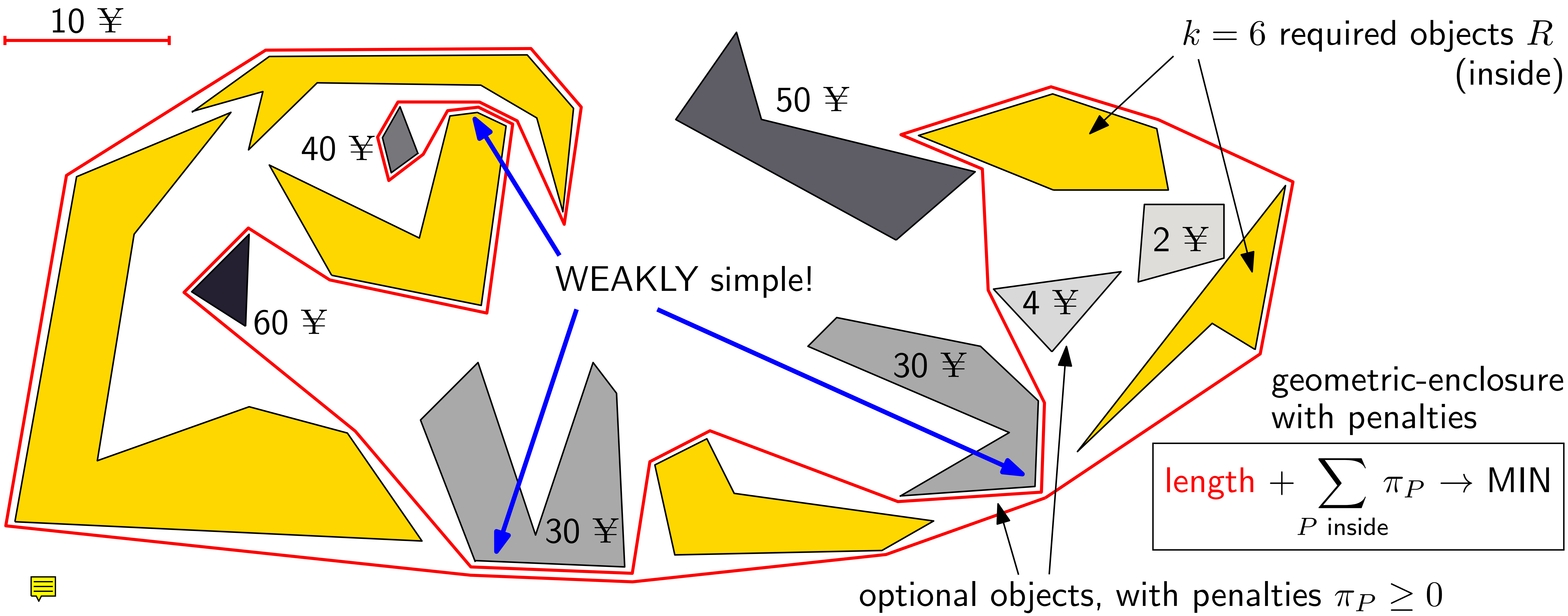


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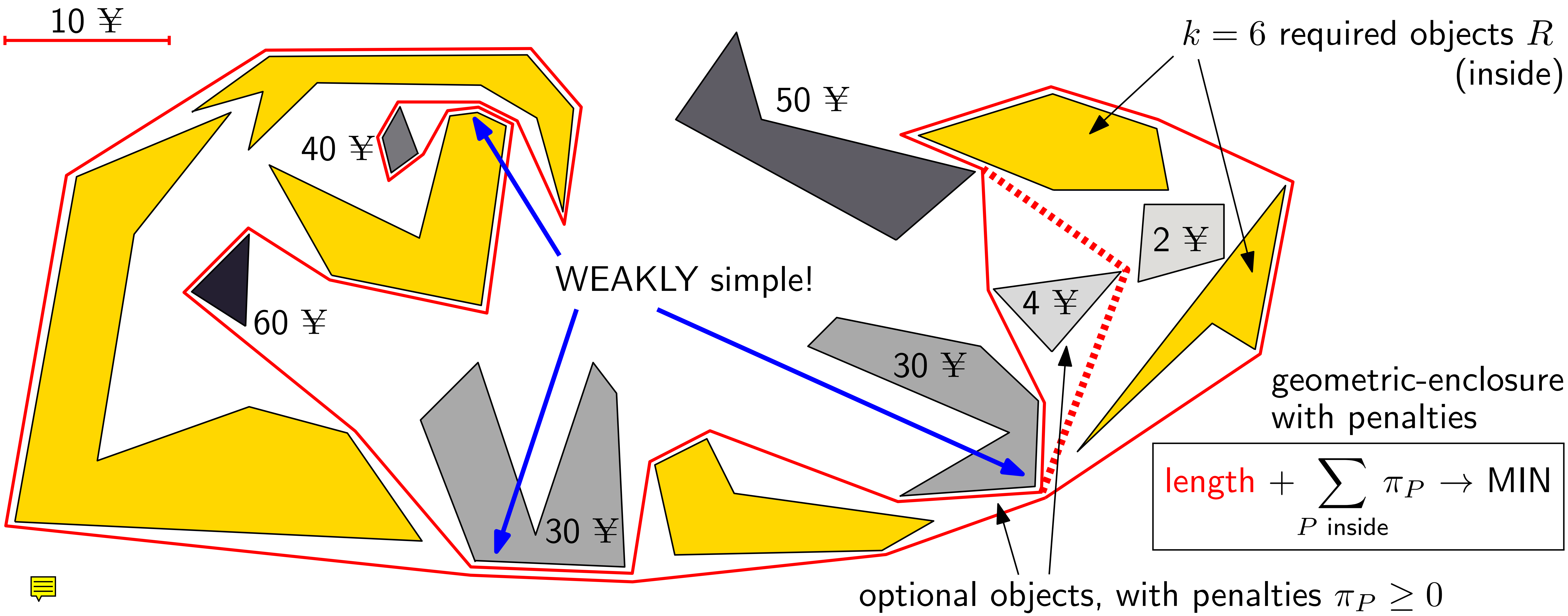

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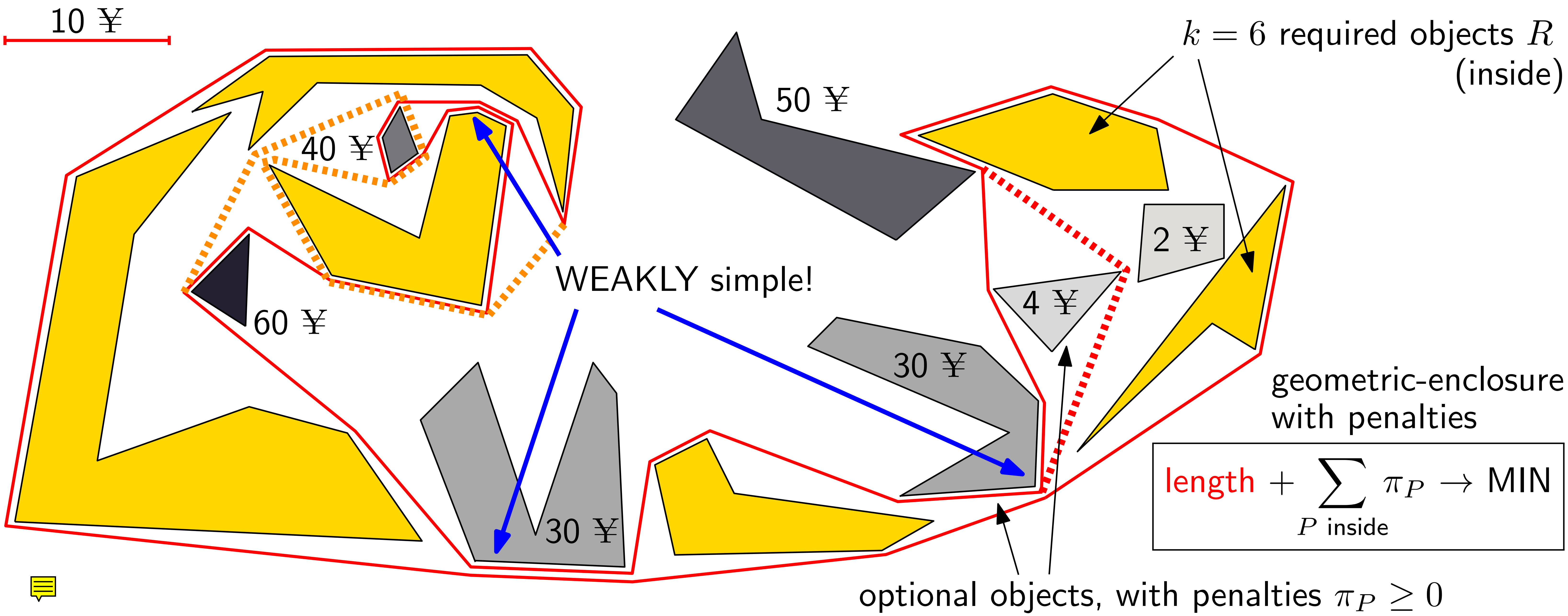

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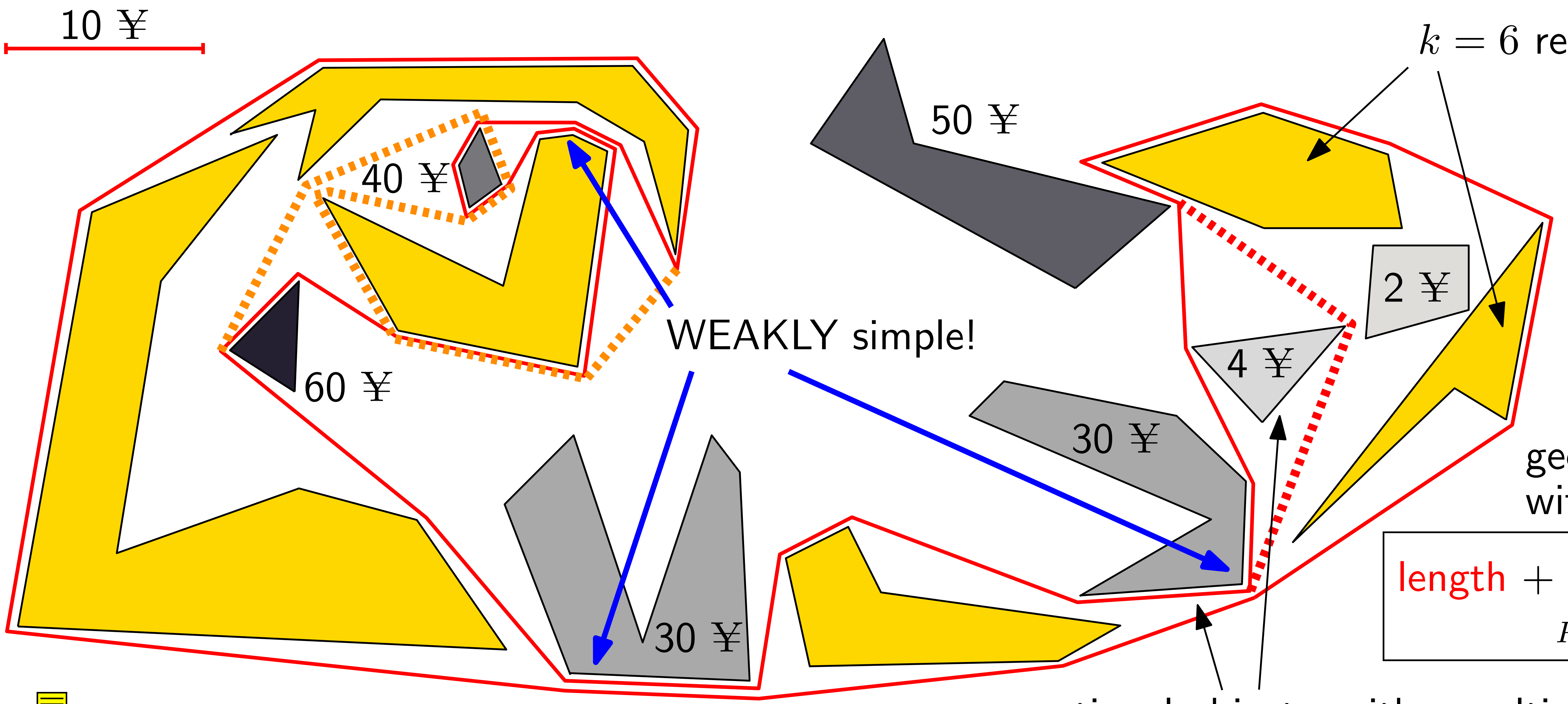
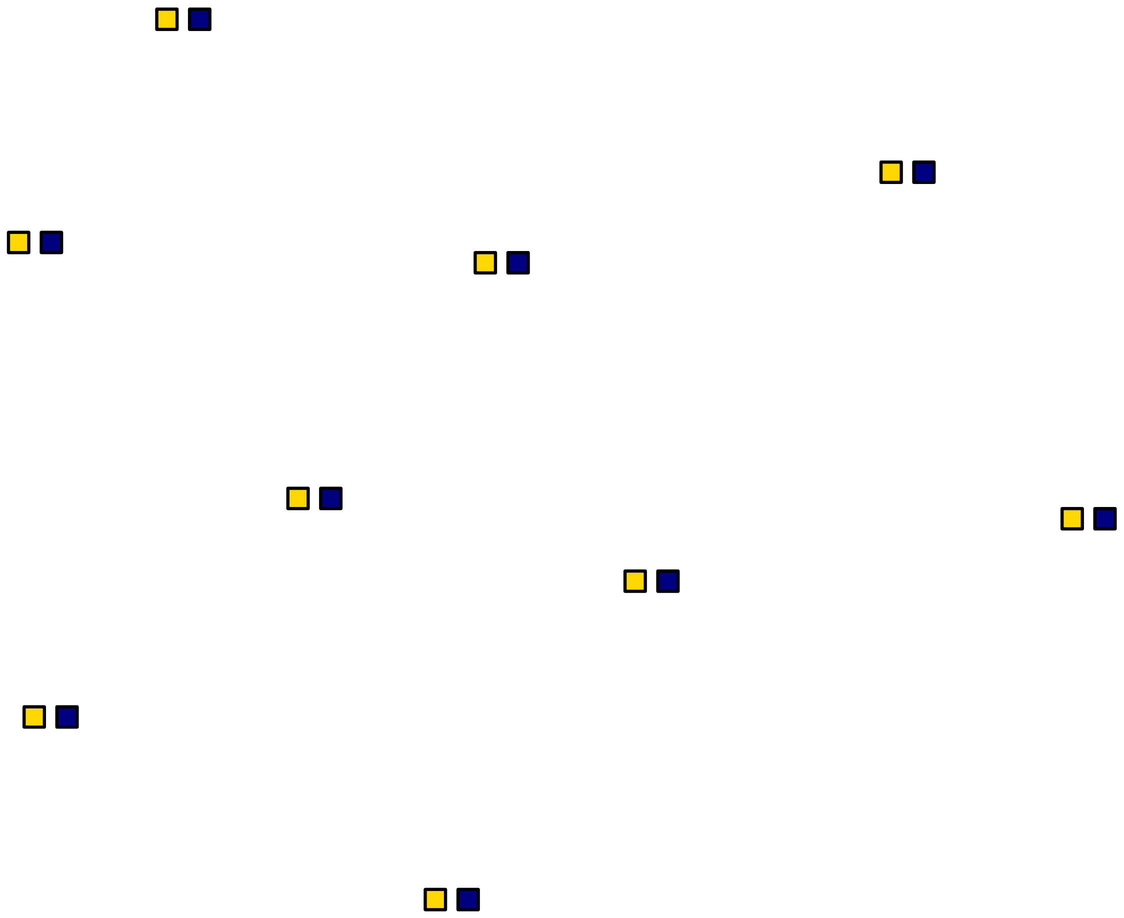



$O(3^k n^3)$ time
$O(2^k n^2)$ space

geometric-enclosure with penalties

$$\text{length} + \sum_{P \text{ inside}} \pi_P \rightarrow \text{MIN}$$

# Large number $k$ of required objects



NP-hard (from the Traveling Salesperson Problem)
[Eades and Rappaport 1993]

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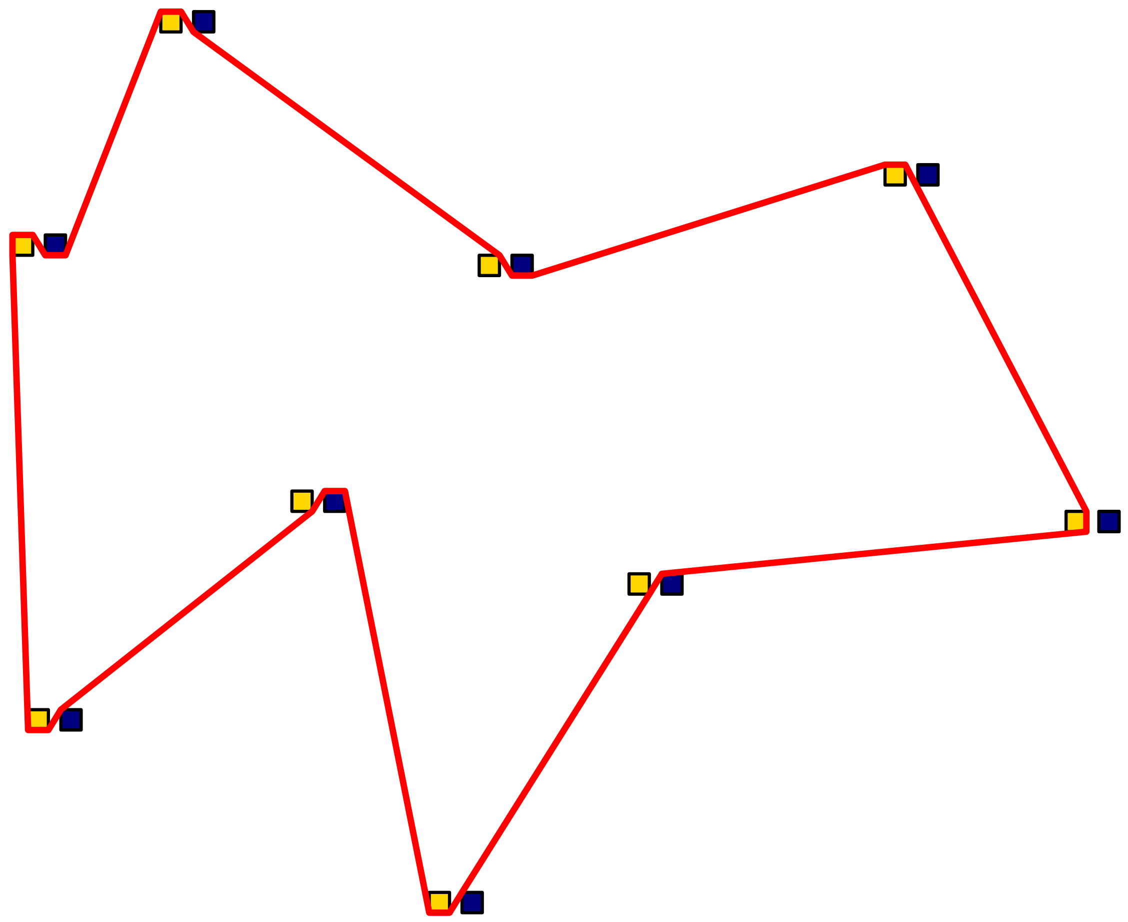

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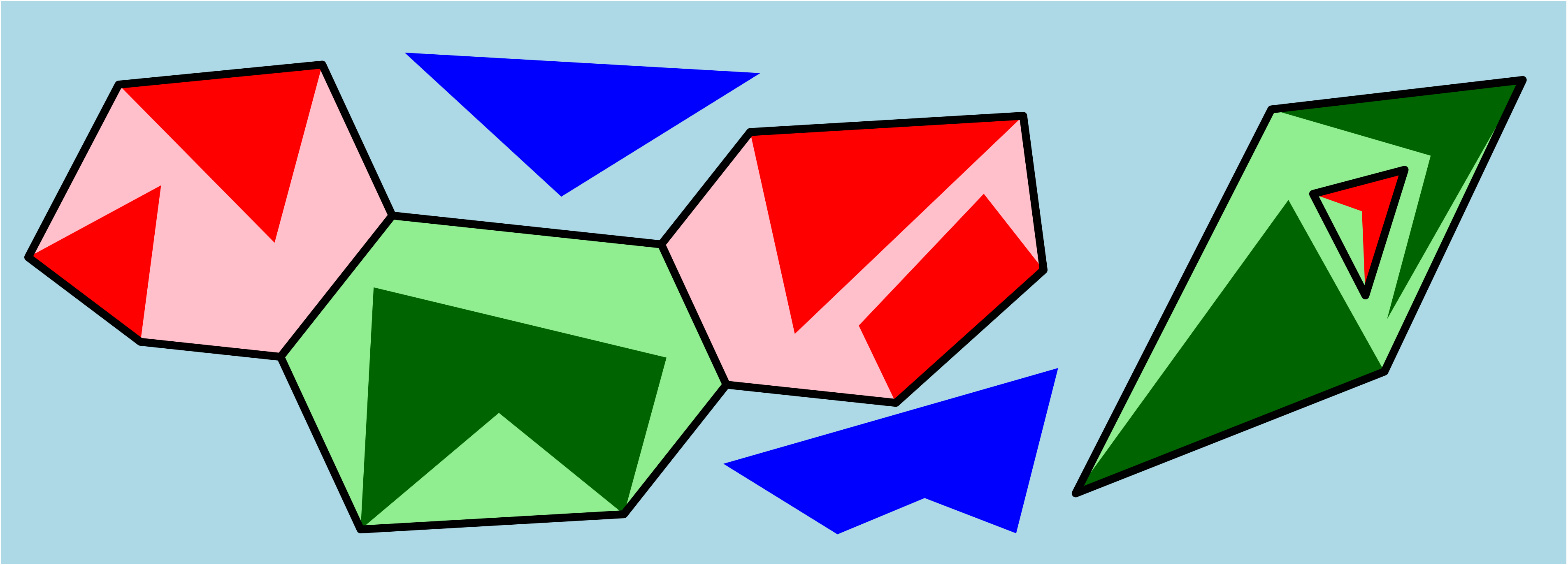
# Overview

- Problem definition ✓
- Related enclosure/separation problems:
  - separate $k$ *unspecified points* from the rest (OPEN for polygon objects)
  - systems of *fences* [ Abrahamsen, Giannopoulos, Löffler, Rote 2020 ]
  - geometric knapsack



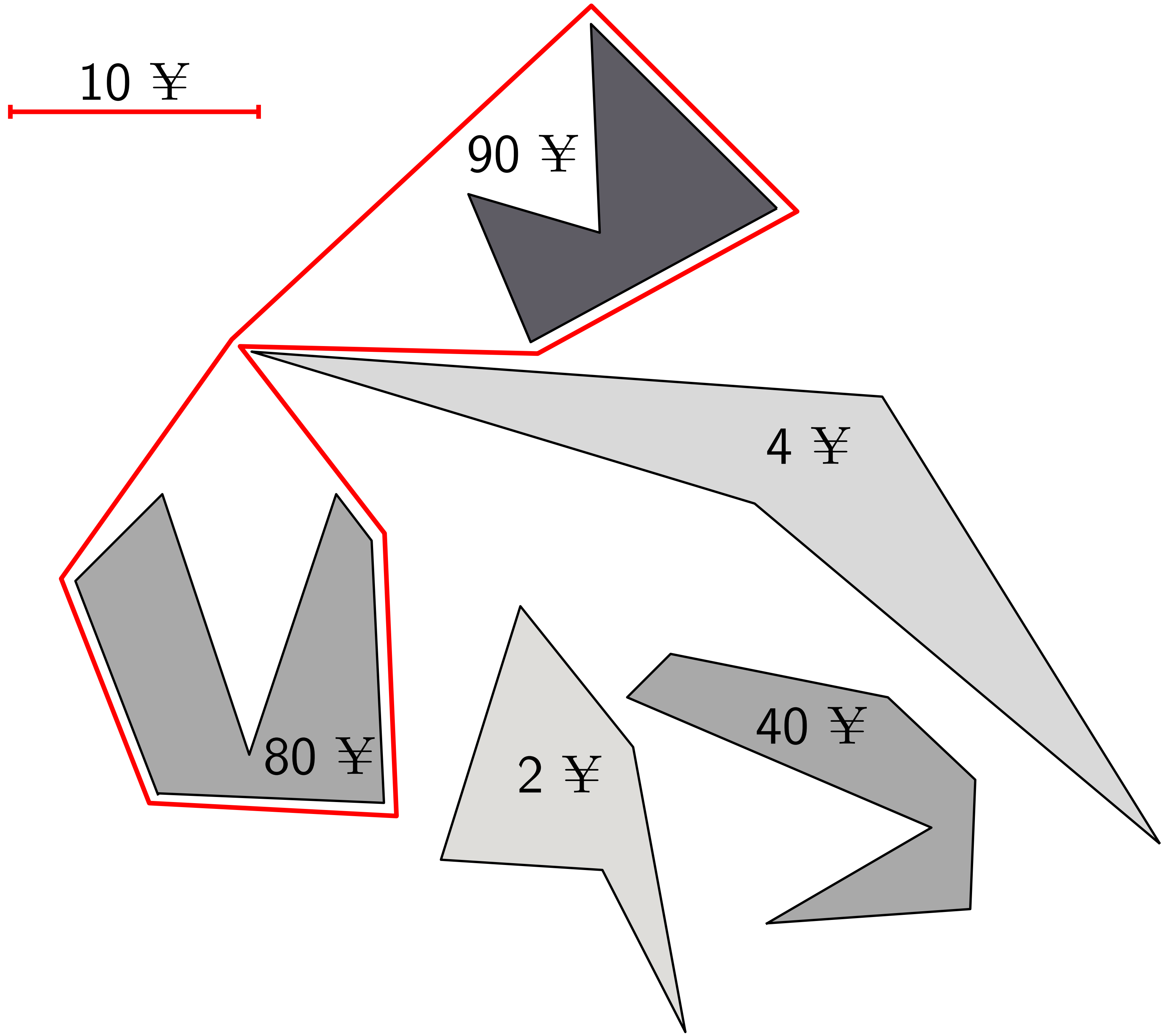
- Dynamic programming algorithm
  - definition of subproblems
  - dynamic programming recursion
  - tricky part: self-intersections
- Variations and improvements
  - the inverted problem (inside $\leftrightarrow$ outside)
  - speedup $3^k n^5 \to 3^k n^3$
  - PLANE-GRAPH-enclosure with penalties
- weakly simple immersed polygons (WSiMPs)
- ETH $\Rightarrow$ $2^k$ lower bound

# Related problem: Geometric knapsack

[Arkin, Khuller, Mitchell 1993]: $O(n^2 \cdot \#\text{visibility-edges}) = O(n^4)$

rewards

$$\text{MAX} \sum_{P \text{ inside}} \pi_P - \text{length}$$



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penalties

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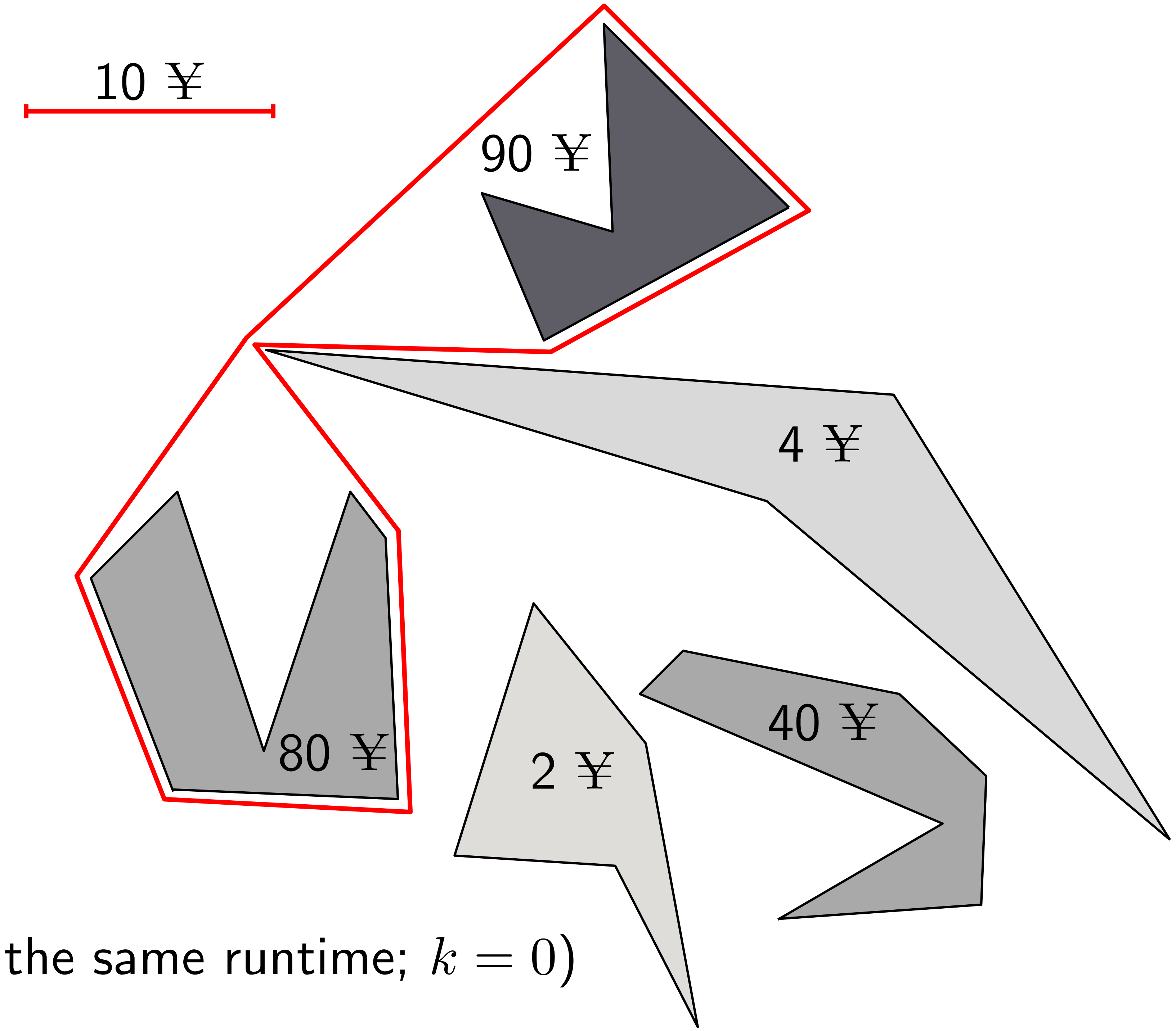


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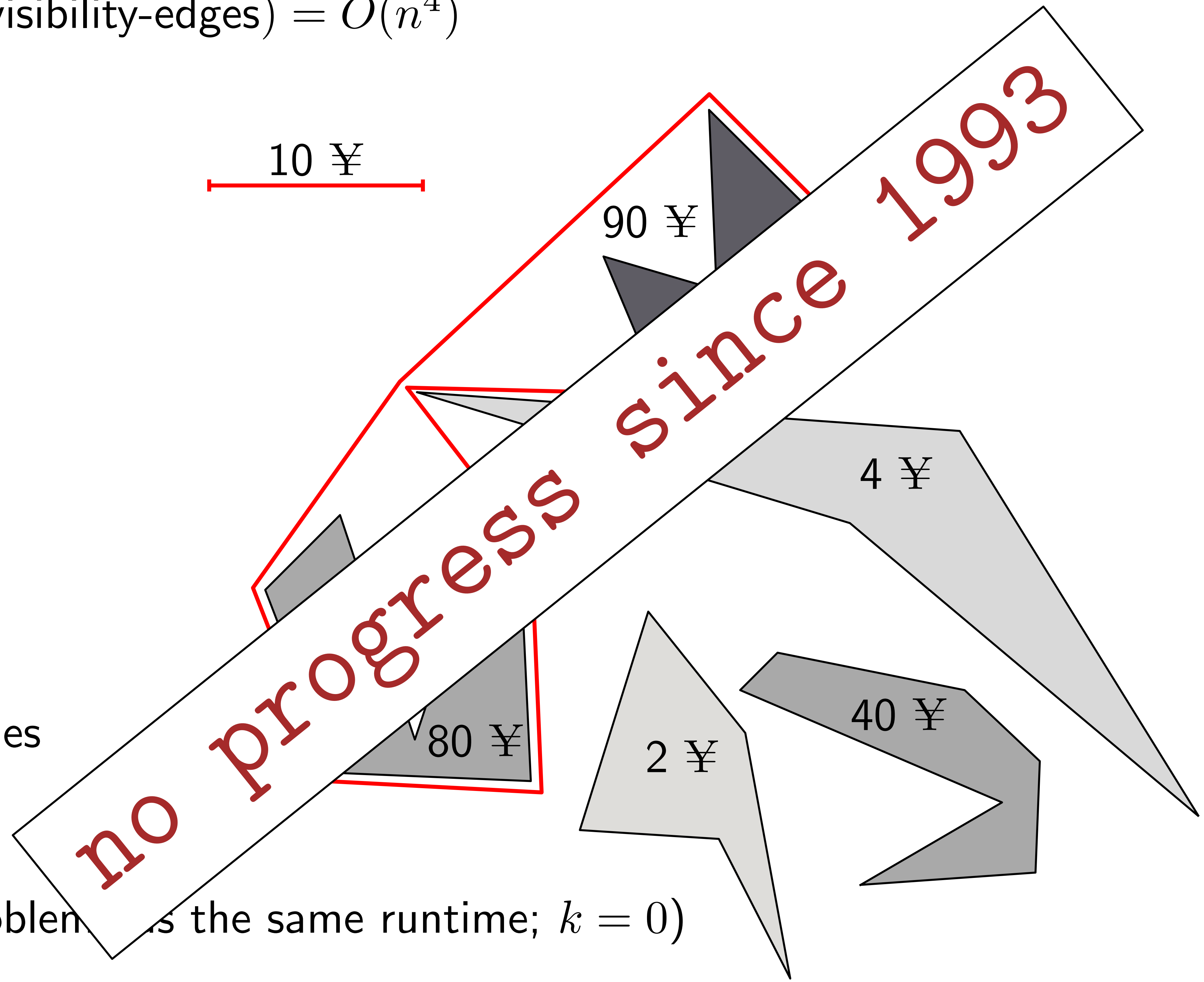


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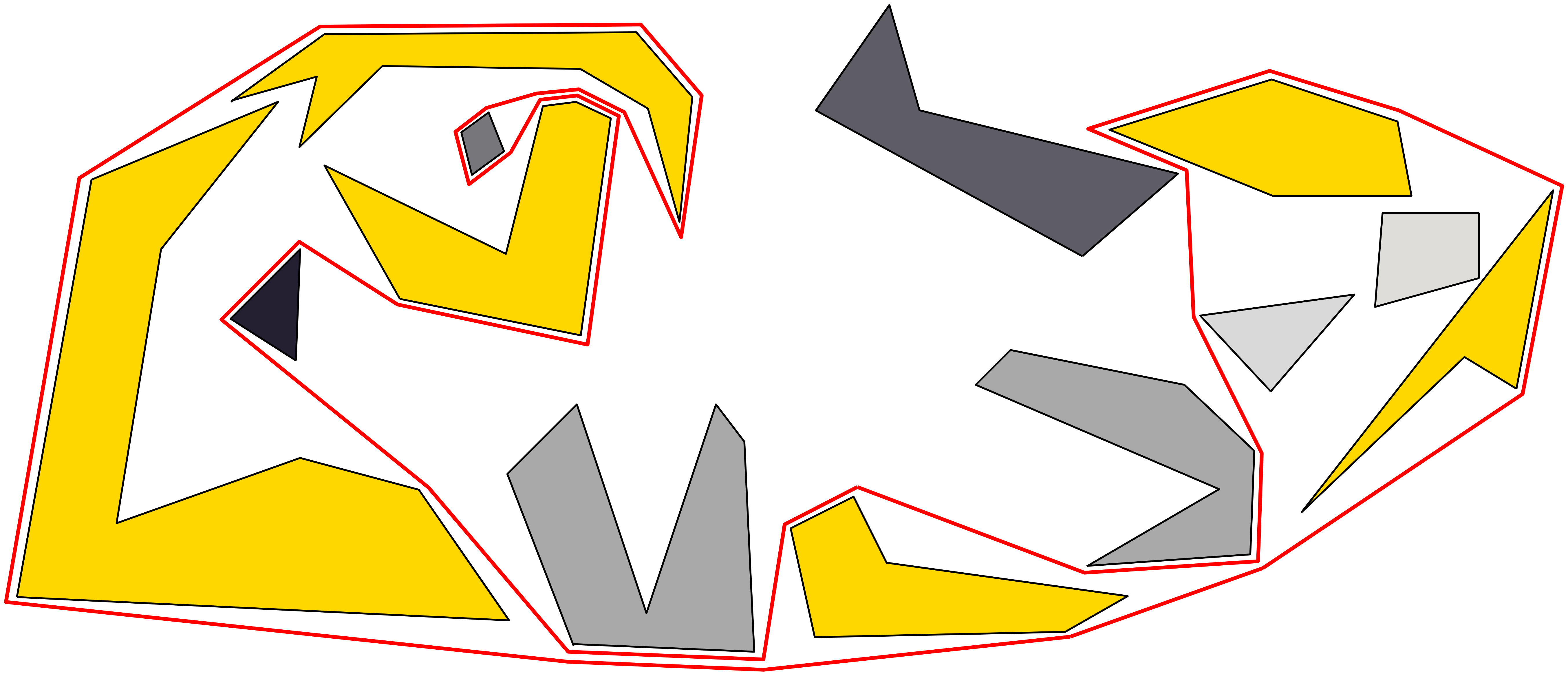

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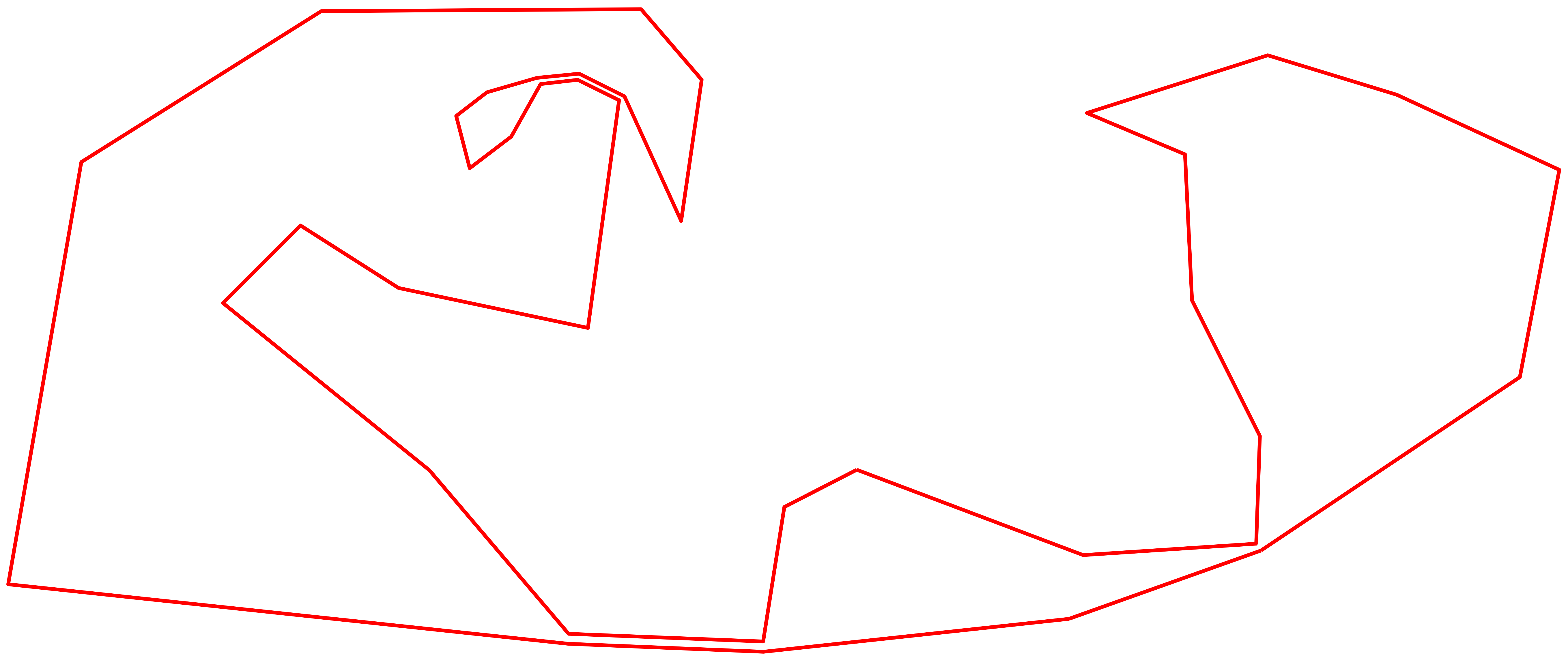

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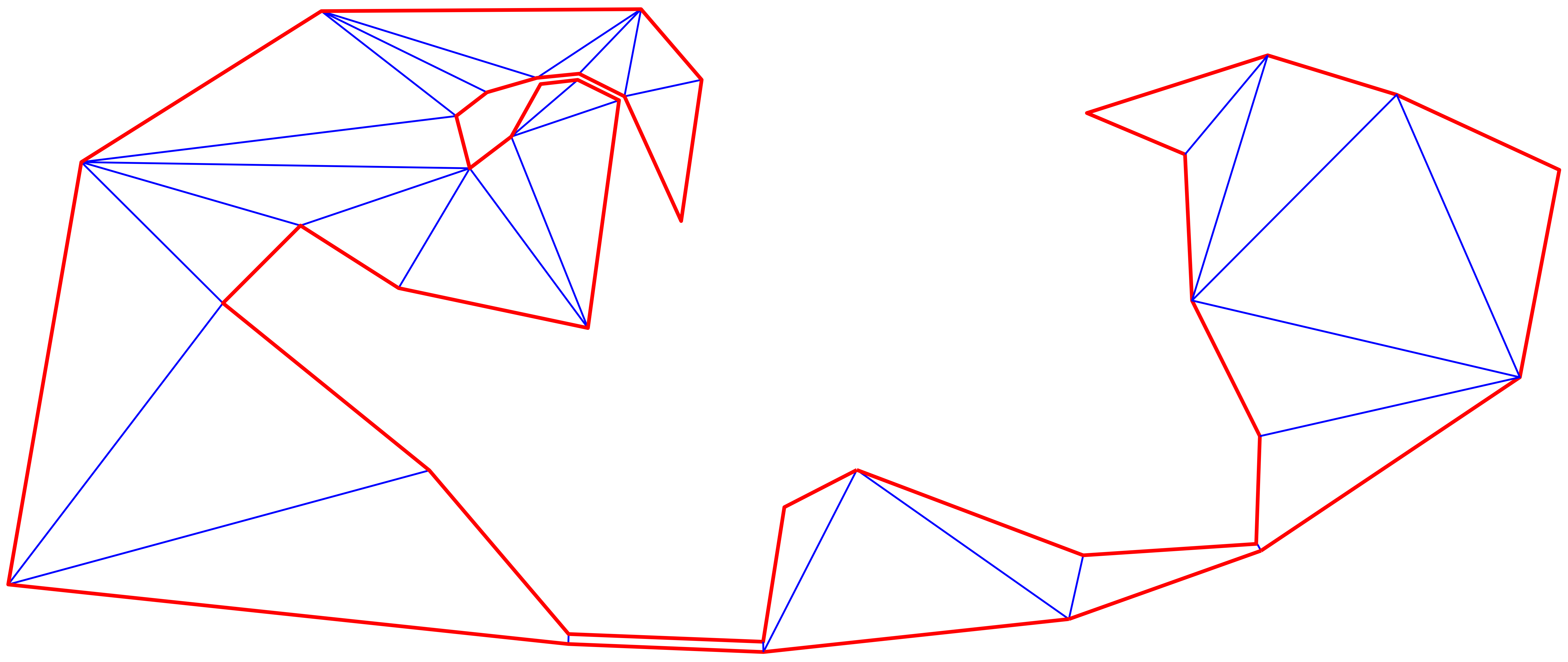

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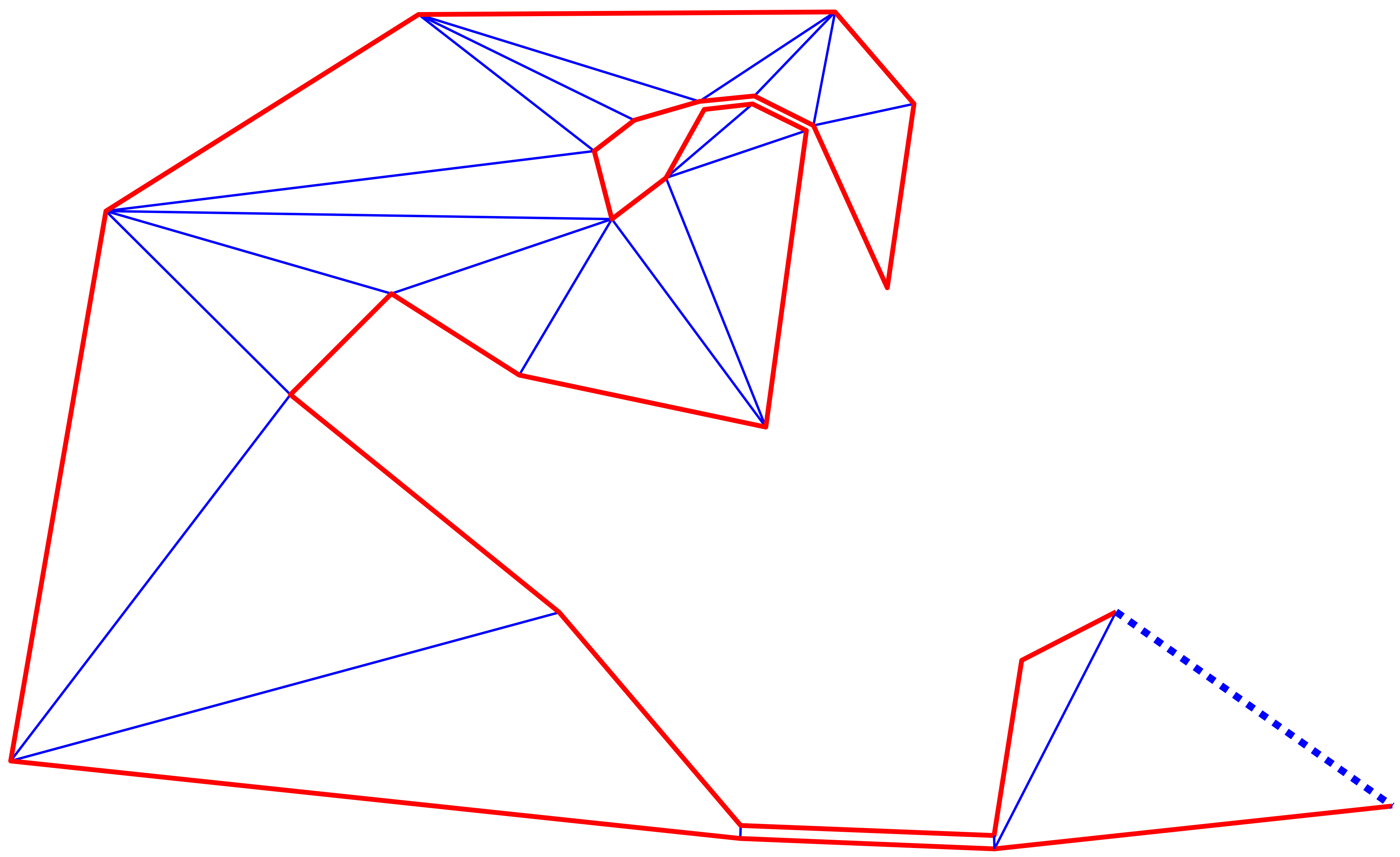

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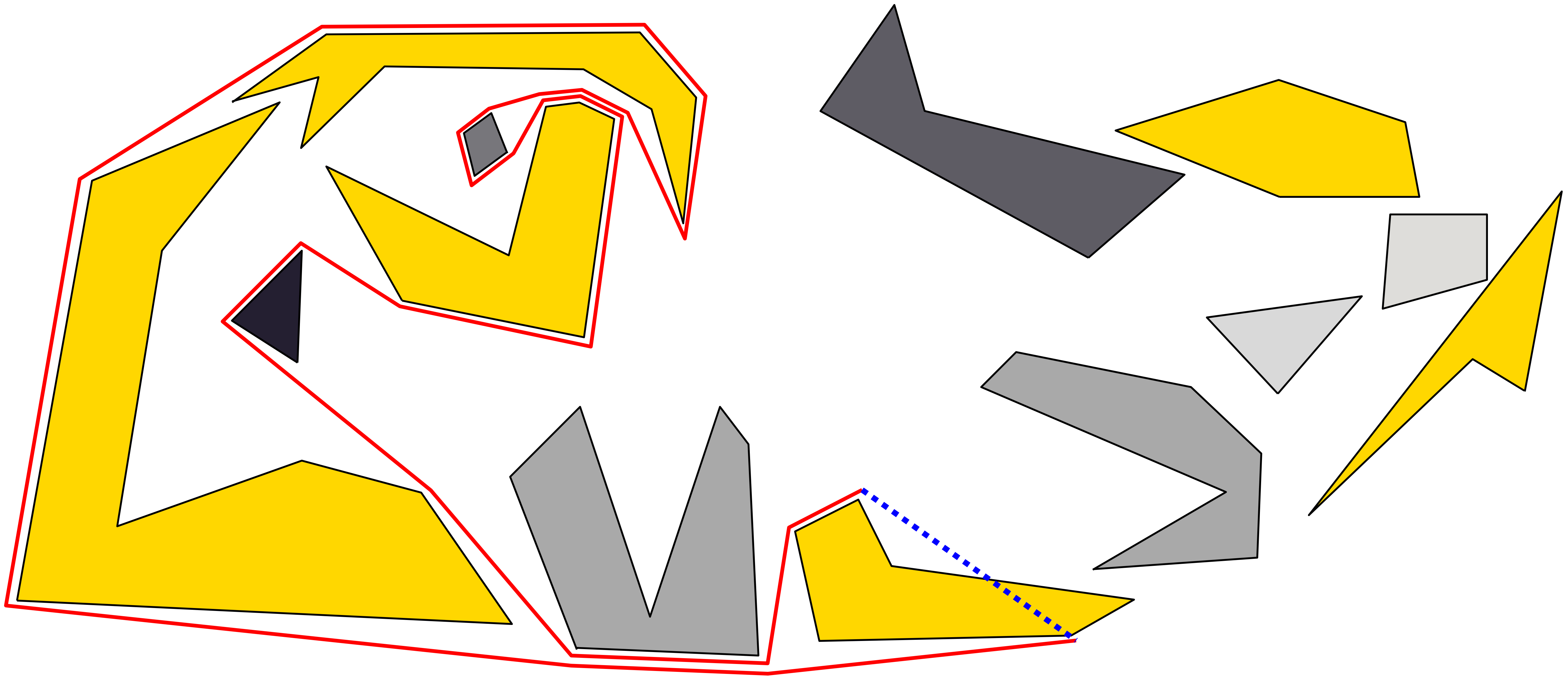

*Mouth* of a partial solution may cut through objects.

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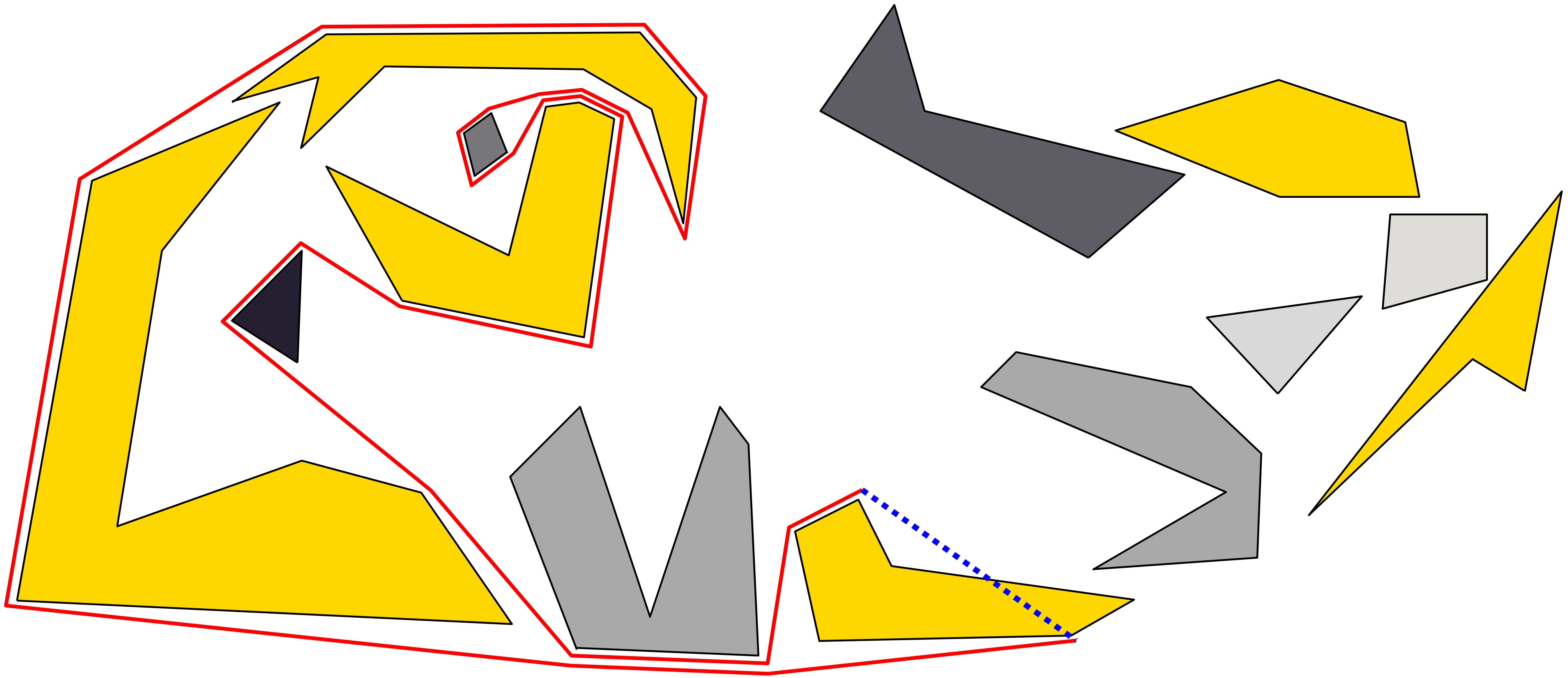

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We want to keep track of subset $B \subseteq R$ of required objects contained in a partial solution ($2^k$ choices).

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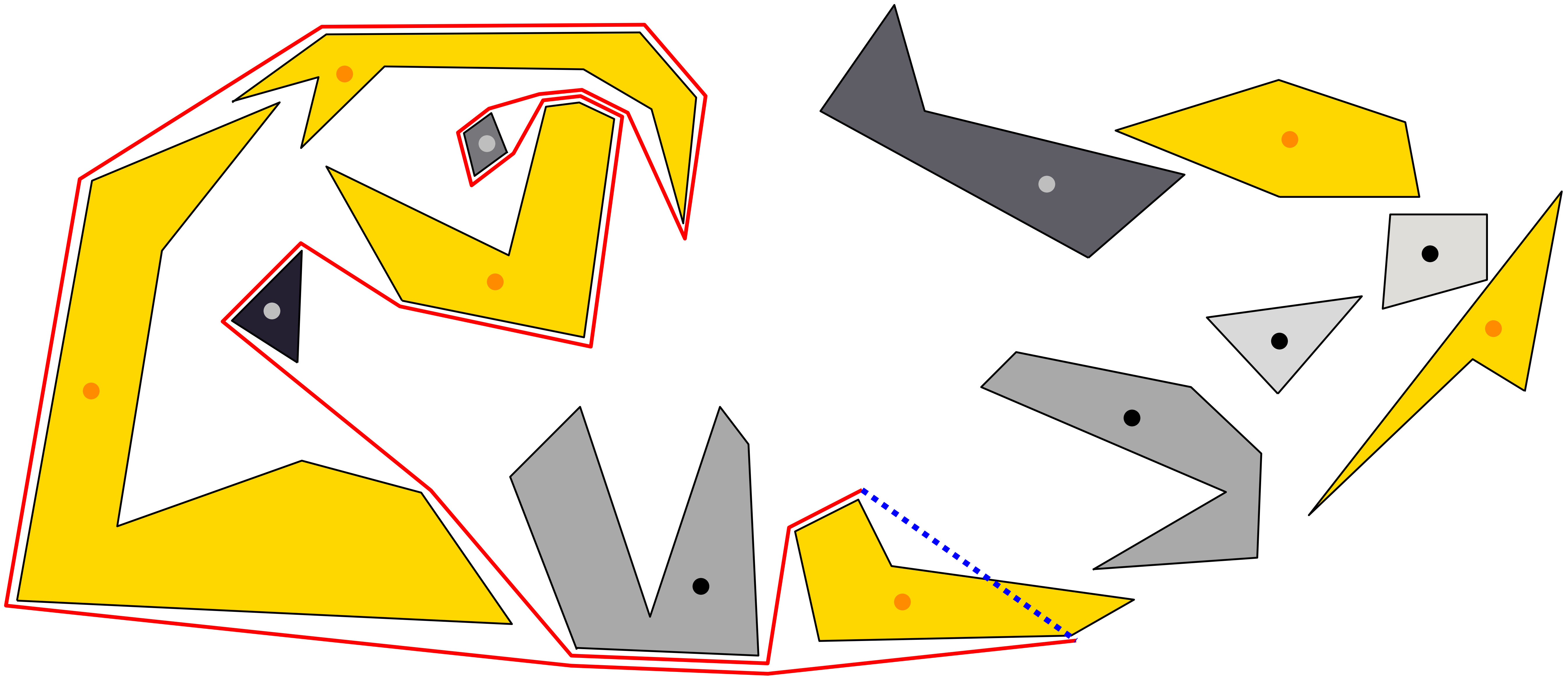

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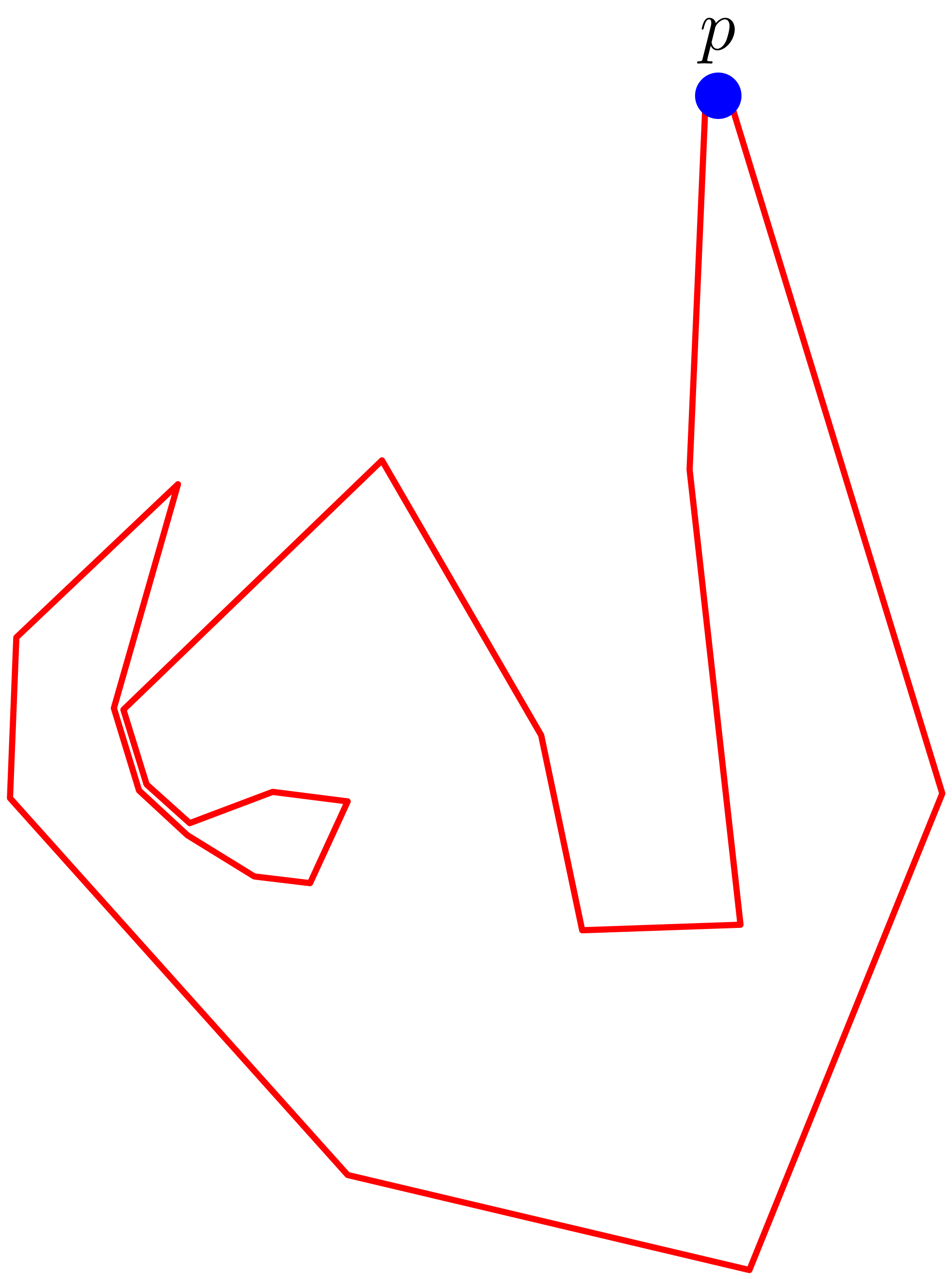
We want to keep track of subset $B \subseteq R$ of required objects contained in a partial solution ($2^k$ choices).

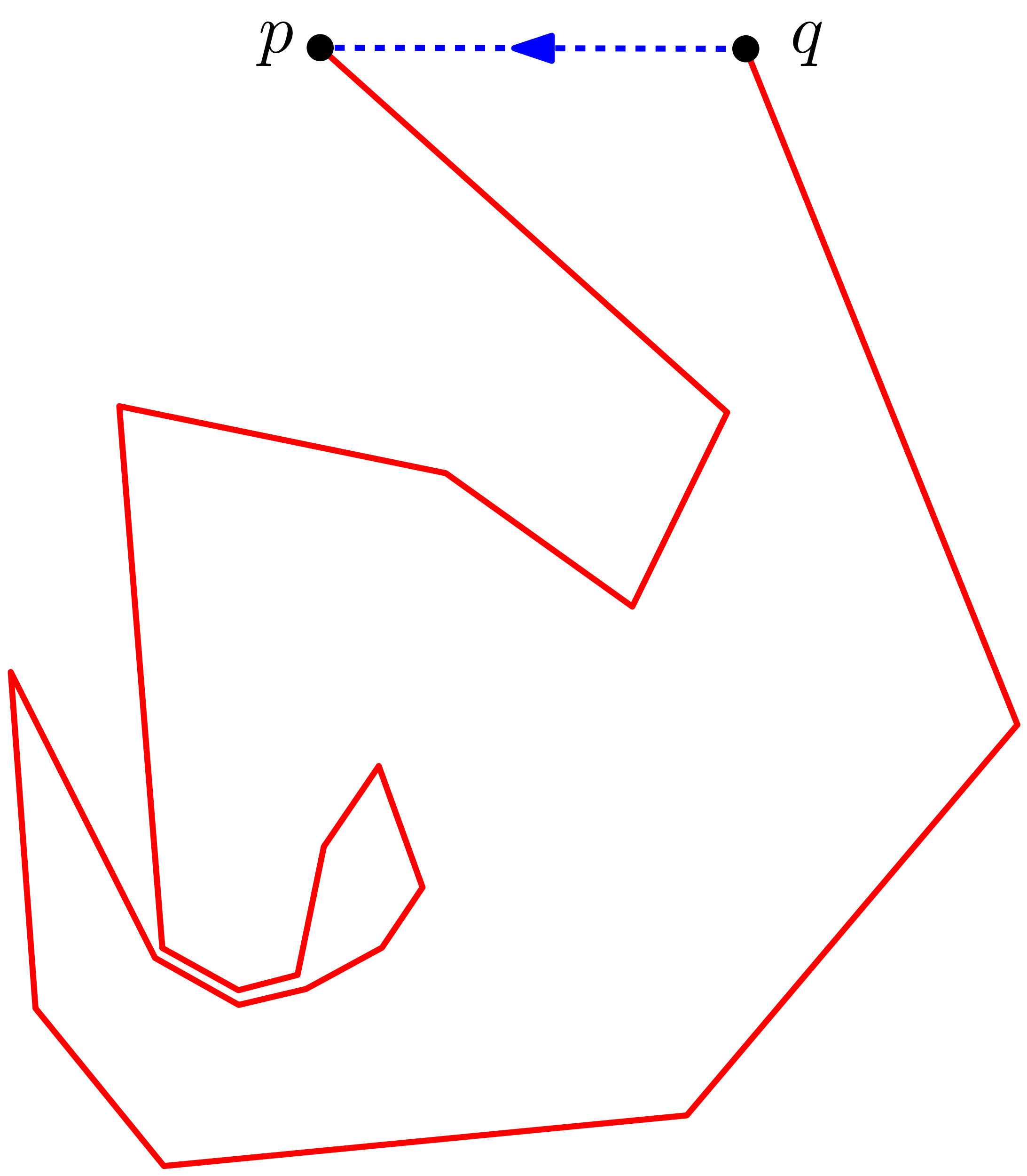
Choose a *reference point* for each object.

# Types of subproblems

<u>*Closed*</u> problems $C(p)$




Open problems $M(pq)$ with a <u>*mouth*</u> $qp$




$0 \le t \le 6n$

$B \subseteq R$ ($2^k$ choices)

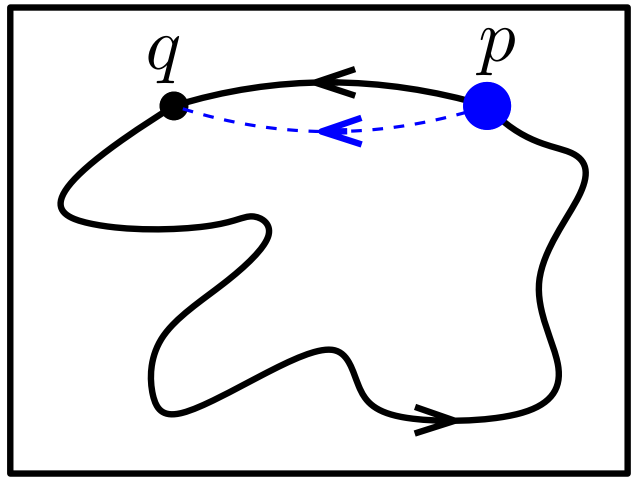
$C(p, t, B), M(pq, t, B) =$

least-cost solution with:

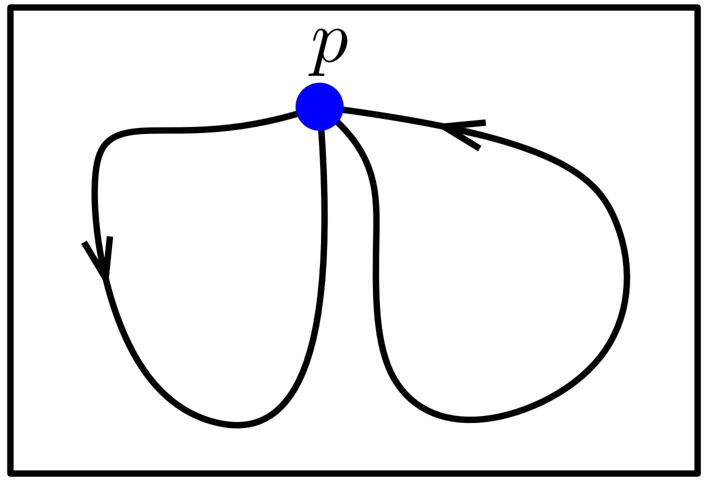
- at most $t$ edges
- enclosing exactly $B$ among the objects in $R$

# Recursion

$C(p, t, B)$:

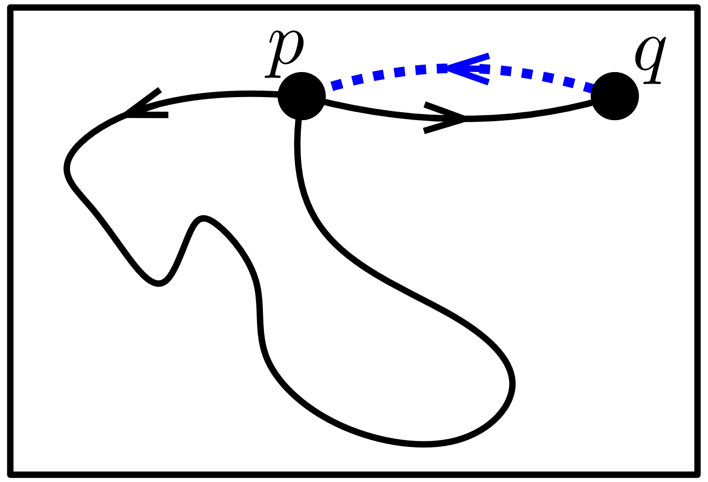



$w_{pq} + M(qp, t-1, B)$
$pq$ a free-space edge




$C(p, t_1, B_1) + C(p, t_2, B_2)$
$B = B_1 \sqcup B_2,\ t = t_1 + t_2$

$M(pq, t, B)$:

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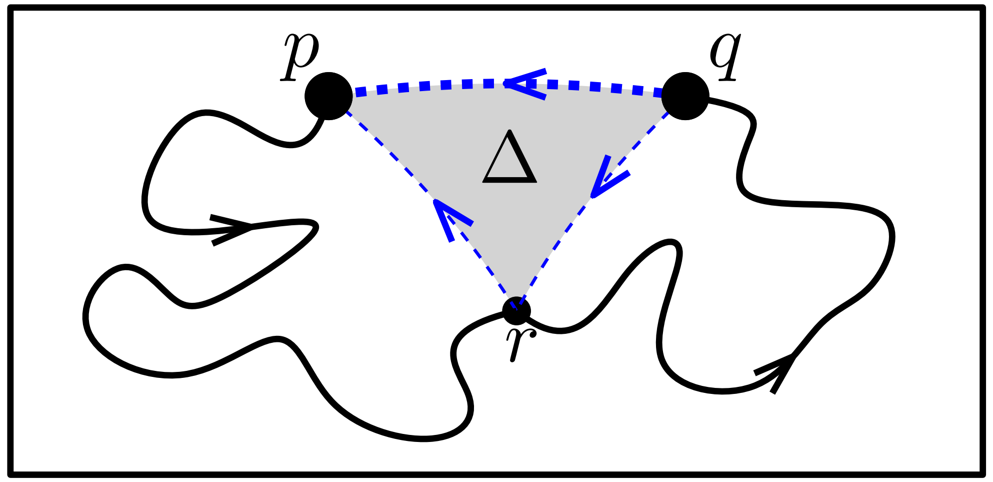


$M(pr, t_1, B_1) + M(rq, t_2, B_2) + \pi(\Delta)$
$B = B_1 \sqcup B_2 \sqcup R(\Delta),\ t = t_1 + t_2$

# Recursion

$$M(pq,t,B) := \min \begin{cases} C(p,t-1,B) + w_{pq} \text{ , if } pq \text{ is a free-space edge} \\ \min\{\, M(pr,t_1,B_1) + M(rq,t_2,B_2) + \pi(\Delta) \;\big|\; r,\ t = t_1 + t_2,\ B = B_1 \sqcup B_2 \sqcup R(\Delta) \,\} \end{cases}$$

$$(\Delta = pqr)$$

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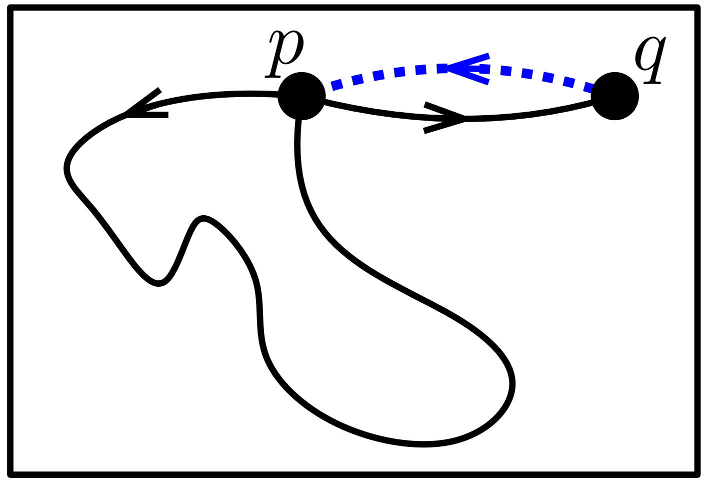


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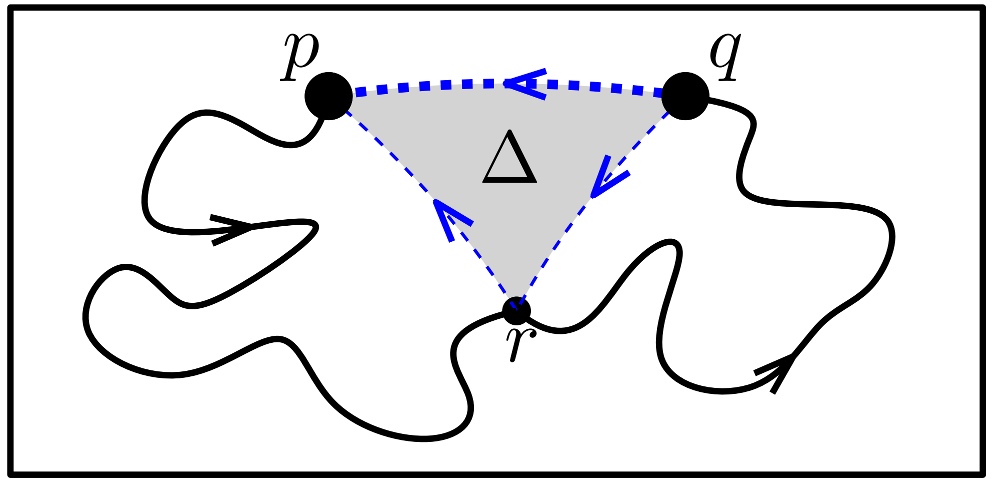


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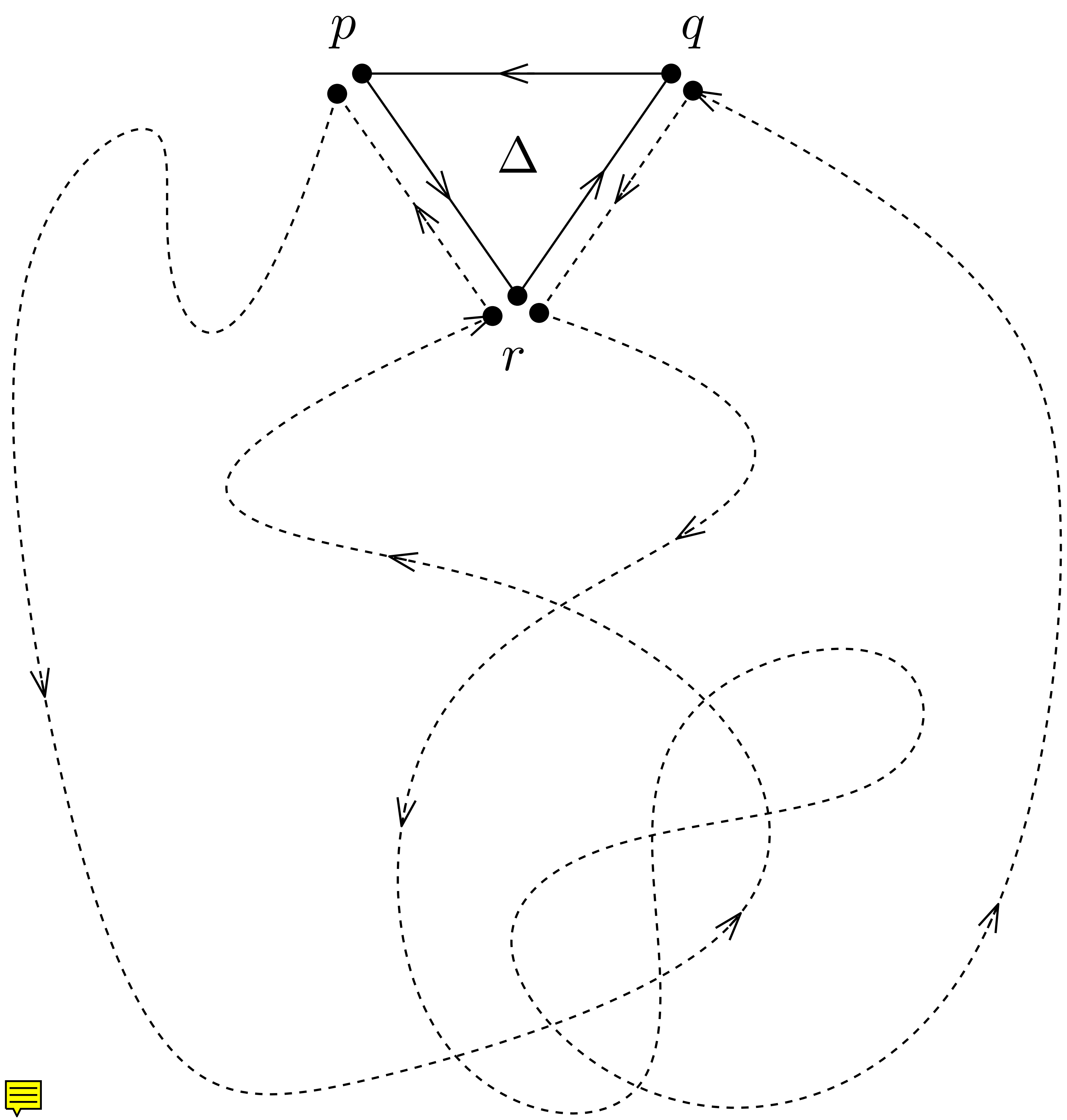
# Self-intersections




The DP algorithm cannot control self-intersections.

Winding numbers to the rescue!

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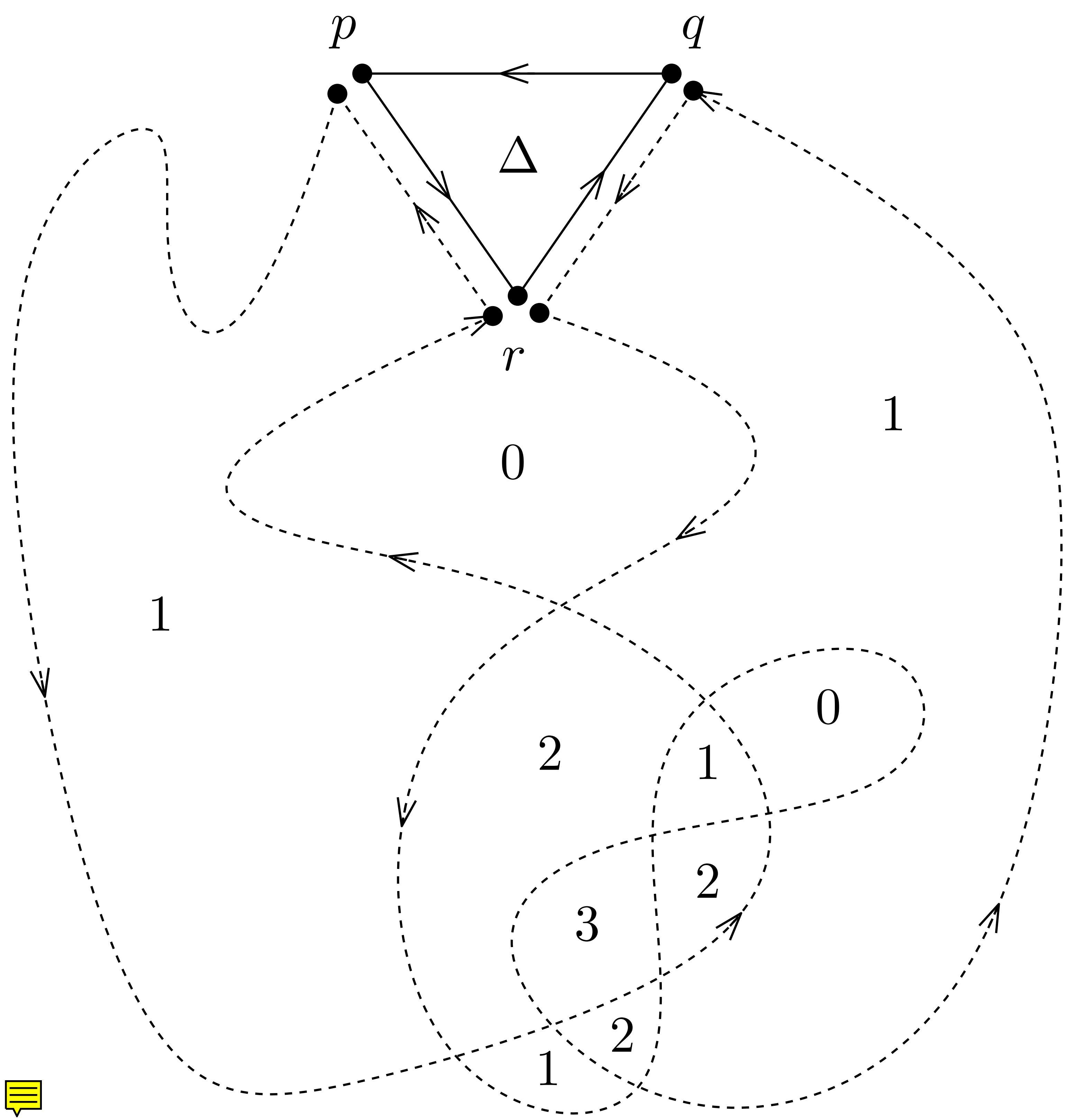


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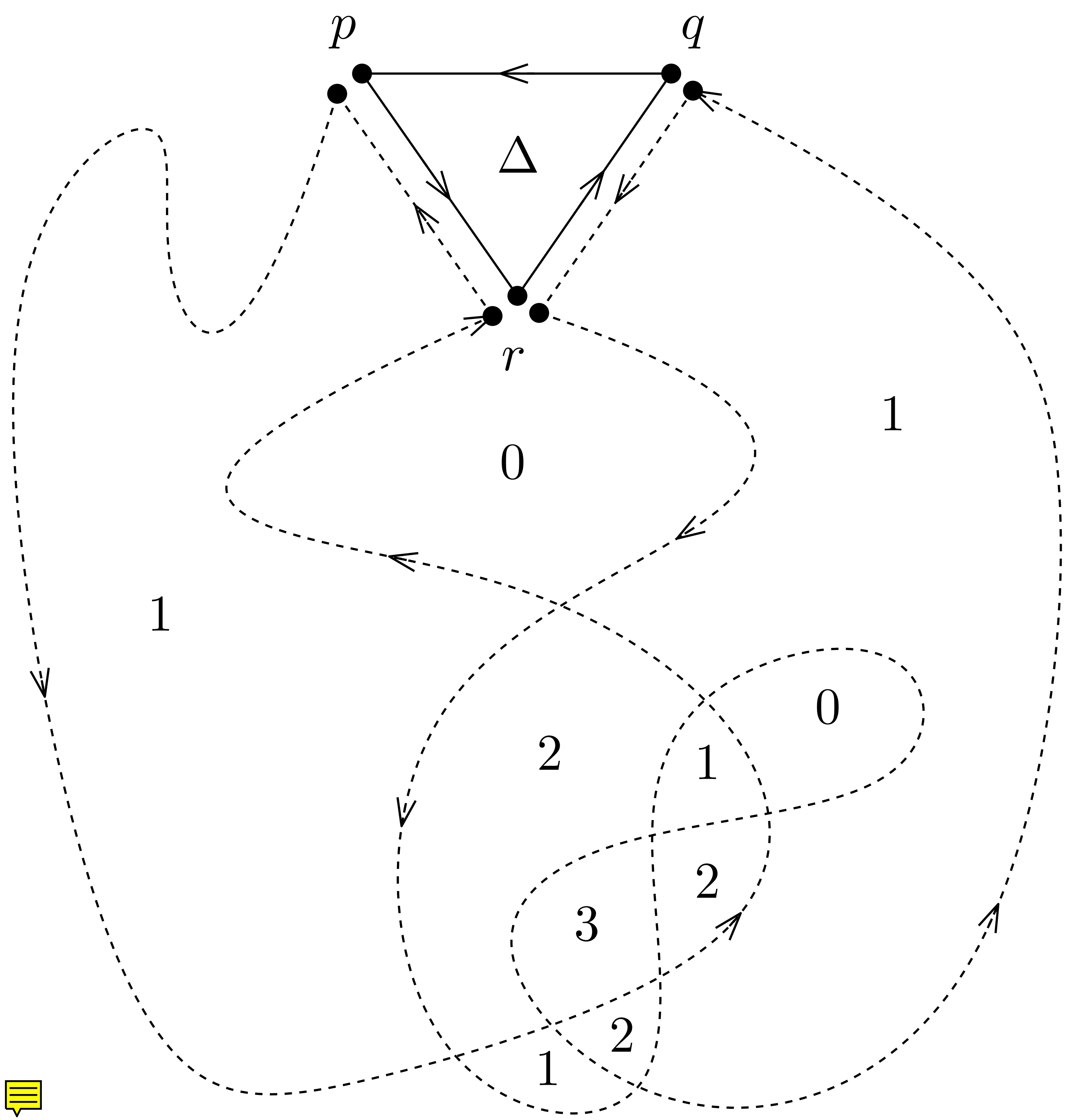


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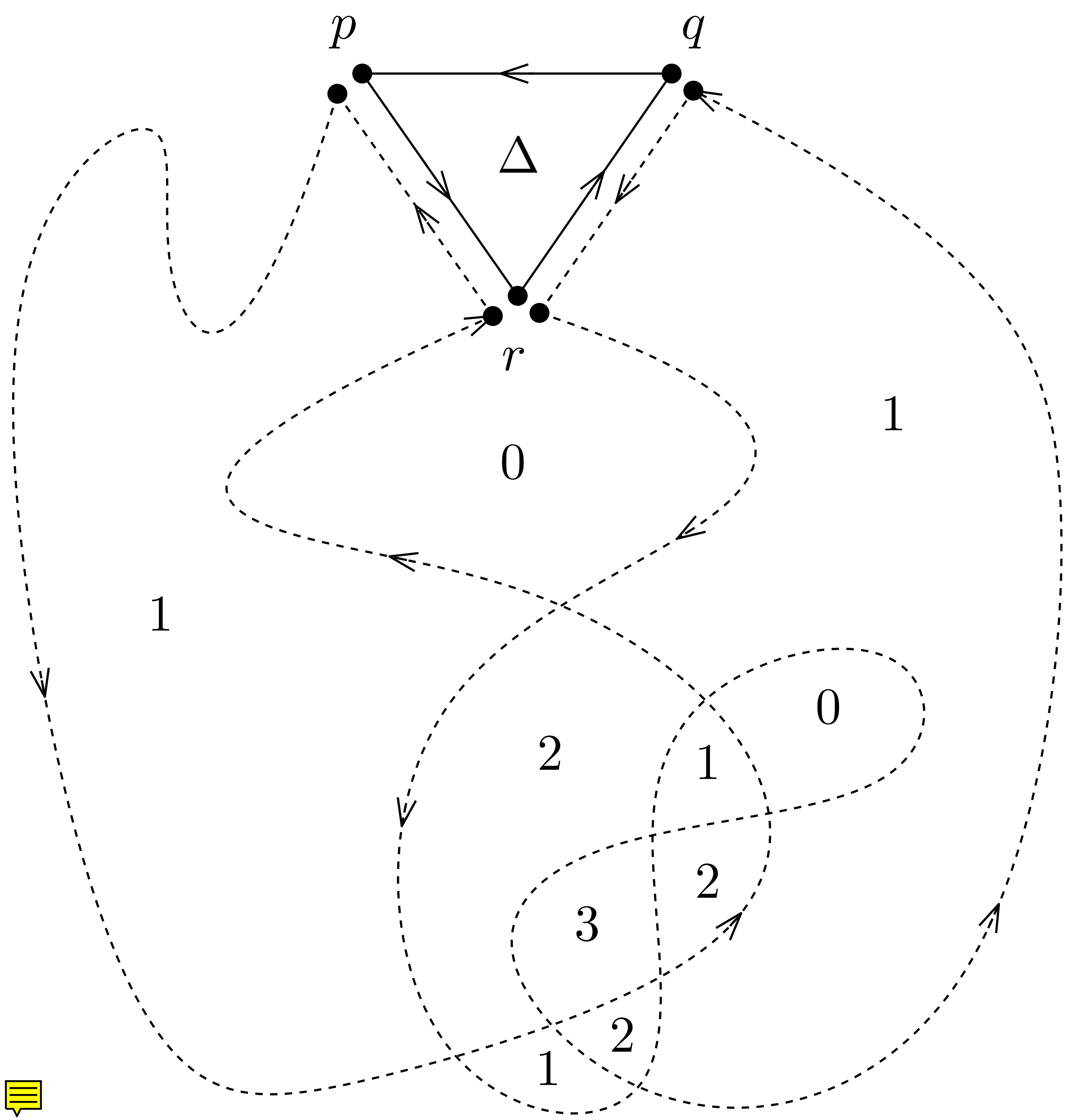


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UNCROSSING ALGORITHM:
[Kotzig 1968] [Akitaya, Cs. Tóth 2018]

- ignore directions
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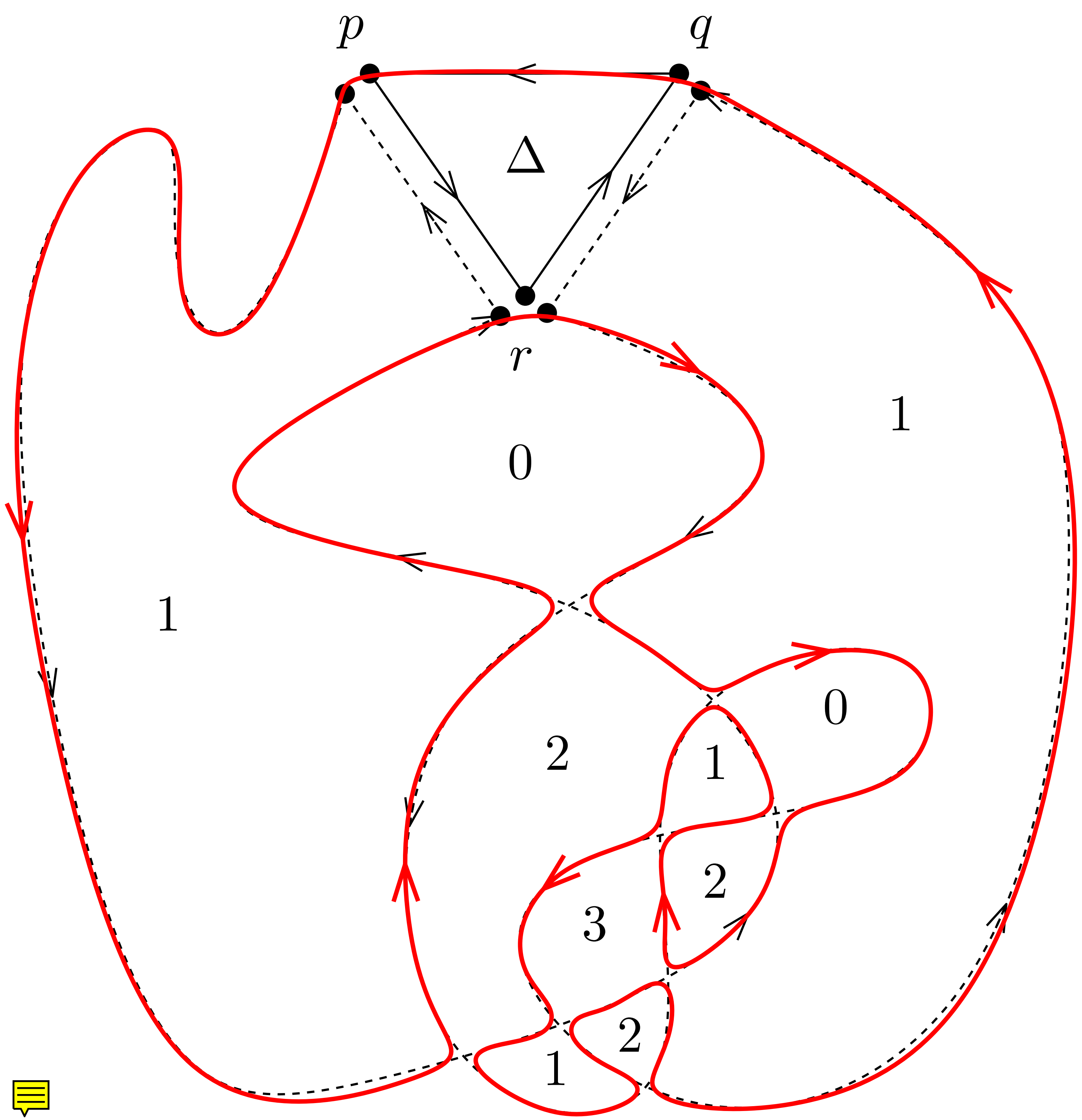


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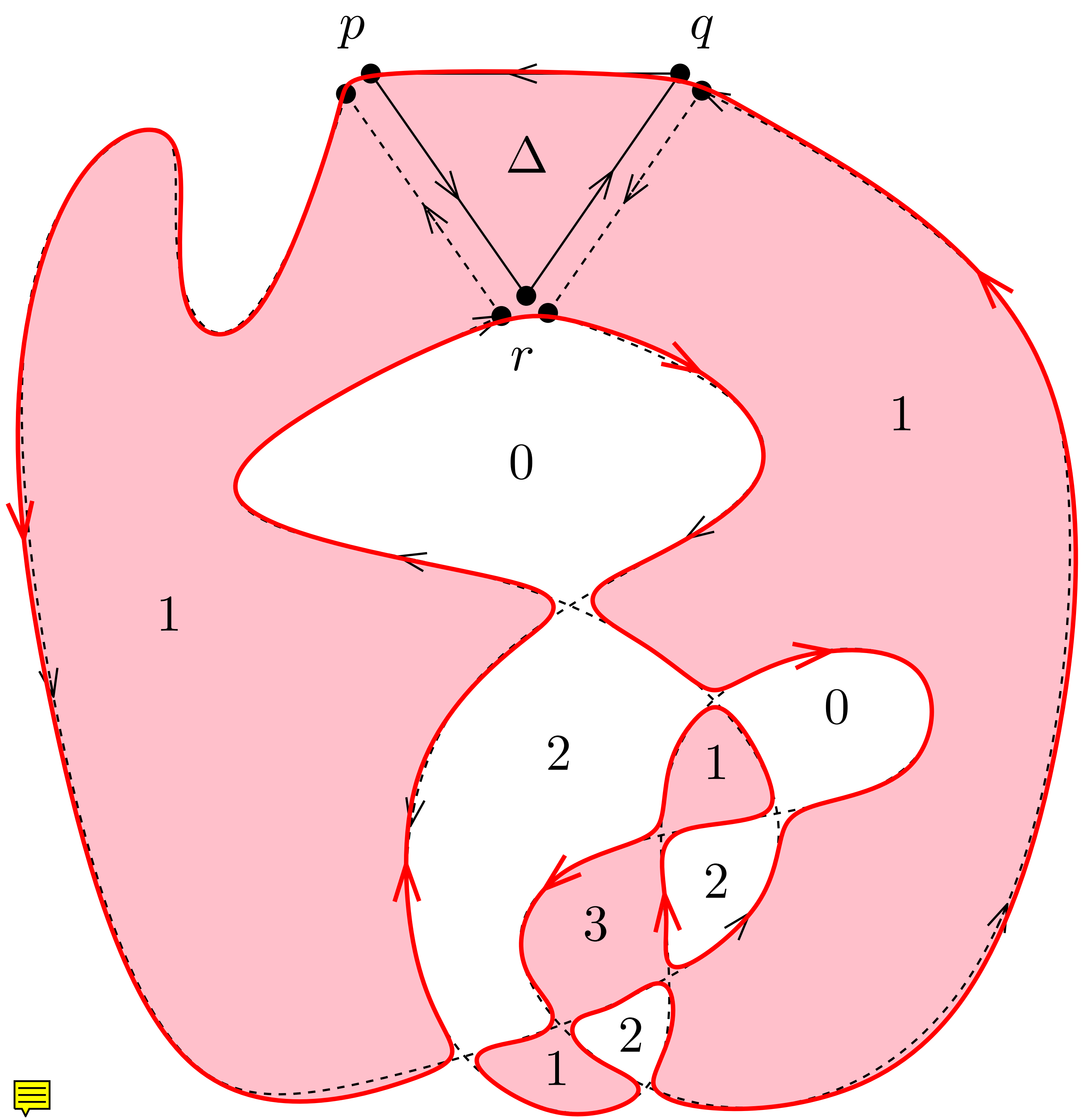


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Stays feasible. Cost can only go down.

# Correctness



$$\mathrm{cost}_{\mathrm{OPT}} \le \mathrm{cost}_{\mathrm{ALG}} \le \mathrm{cost}_{\mathrm{DP}} \le \mathrm{cost}_{\mathrm{OPT}}$$

↑ after uncrossing

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DP + UNCROSS produces a feasible solution.

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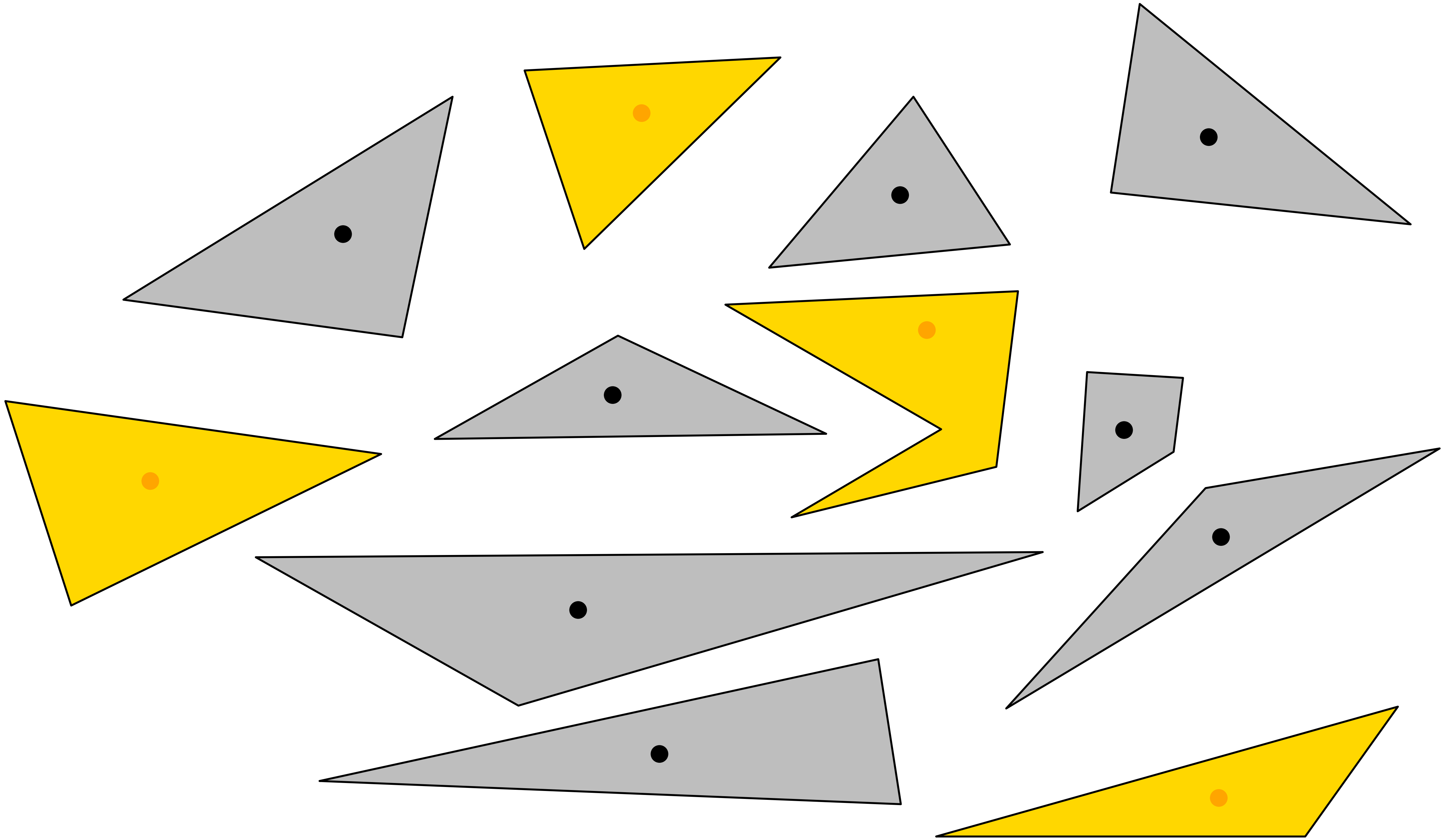

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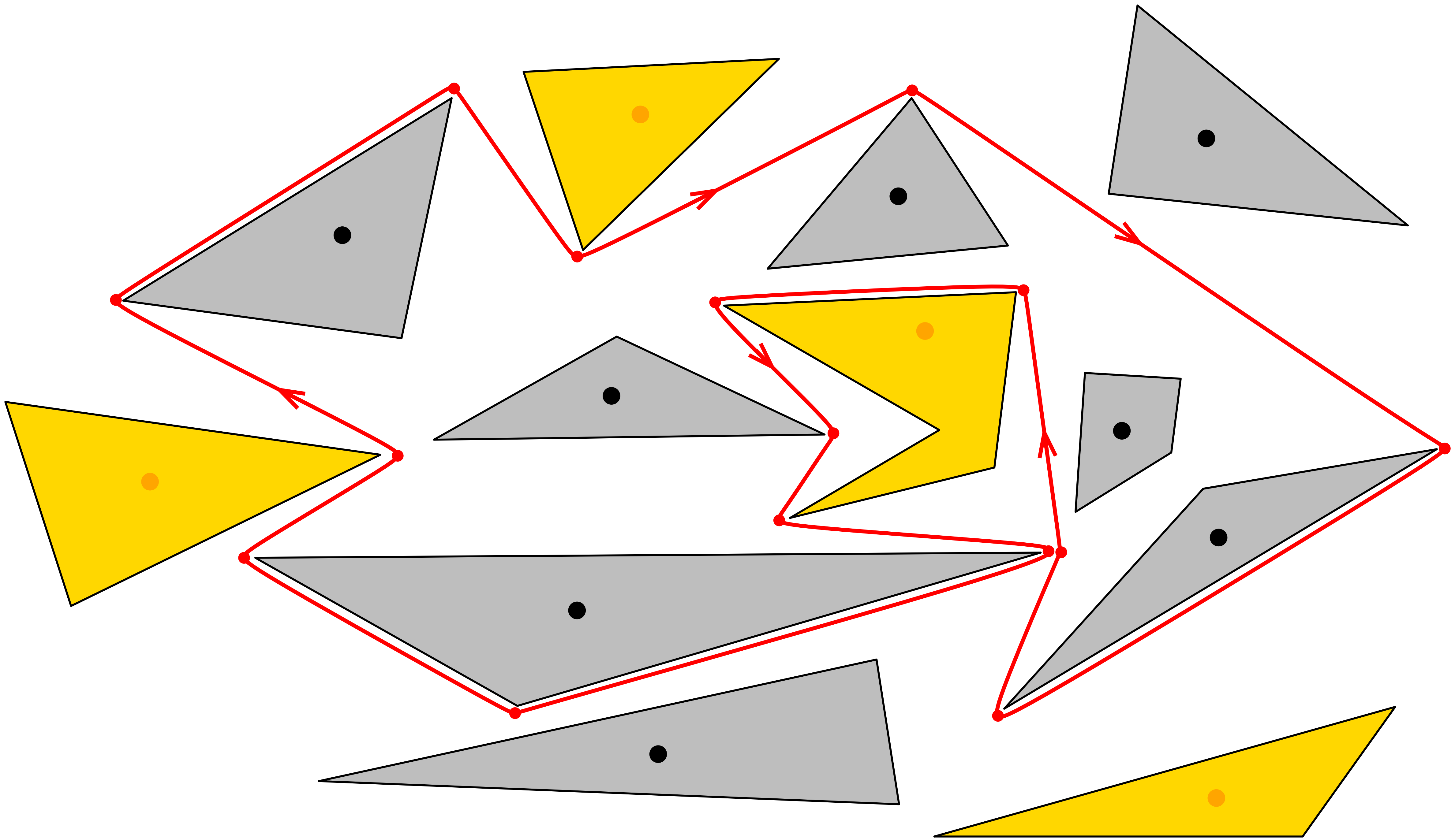

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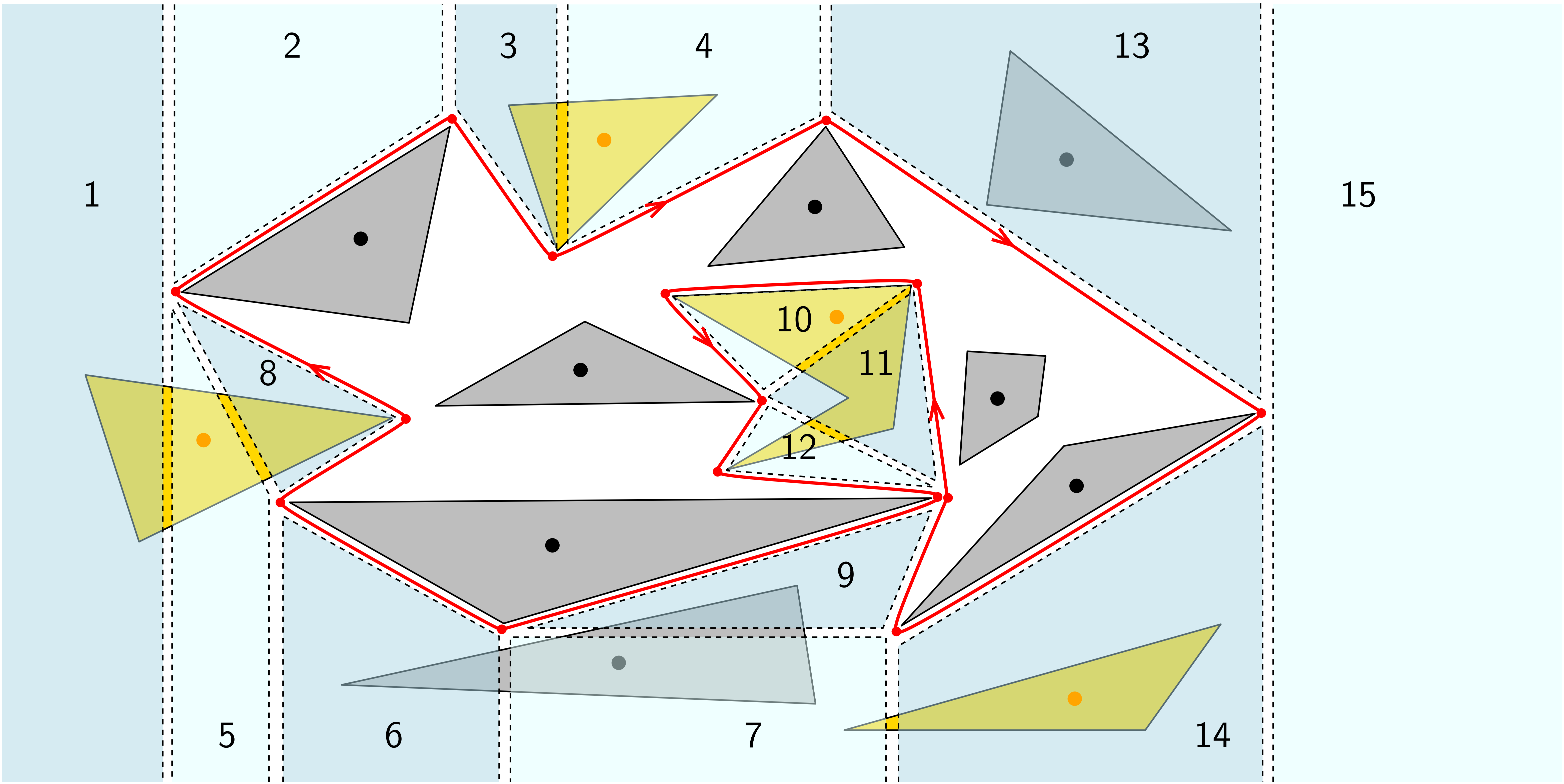

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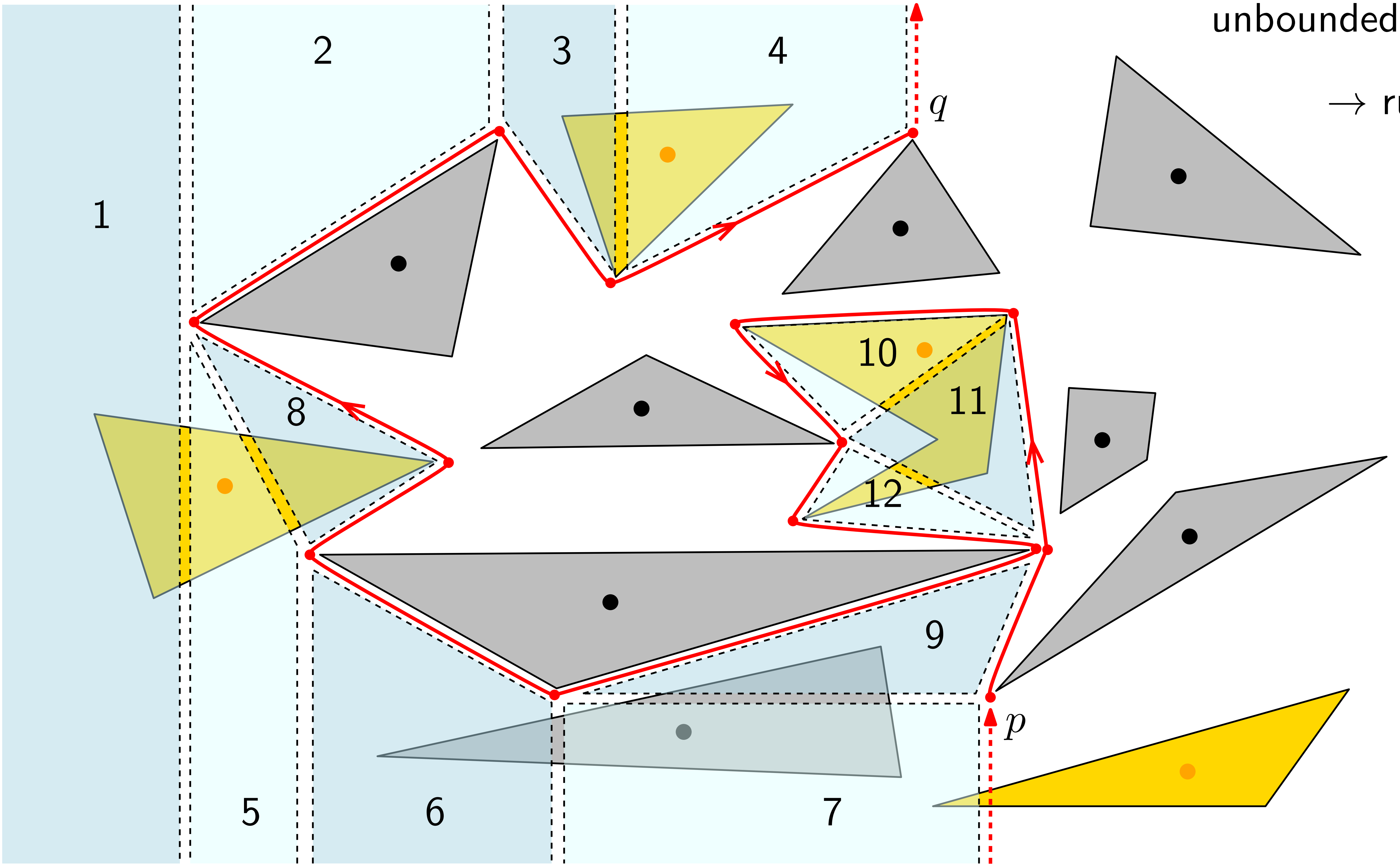



unbounded subproblems $U(p, q)$

$\rightarrow$ runtime is unchanged.

# Speedup by Dijkstra-style algorithm



From $O(3^k n^5)$ to $O(3^k n^3)$:

$$M(pq, \xcancel{t,} B) := \min \begin{cases} C(p, \xcancel{t-1,} B) + w_{pq}\ , & \text{if } pq \text{ is a free-space edge} \\ \min\{\, M(pr, \xcancel{t_1,} B_1) + M(rq, \xcancel{t_2,} B_2) + \pi(\Delta) \mid r, \xcancel{t = t_1 + t_2,}\ B = B_1 \sqcup B_2 \sqcup R(\Delta) \,\} \end{cases}$$

$$C(p, \xcancel{t,} B) := \dots$$

- Maintain *tentative* values $M(pq, B)$ and $C(p, B)$
- The smallest of the tentative values is made *permanent*.
- For all equations where this value appears on the *right-hand side*, the tentative values on the left-hand side are updated.

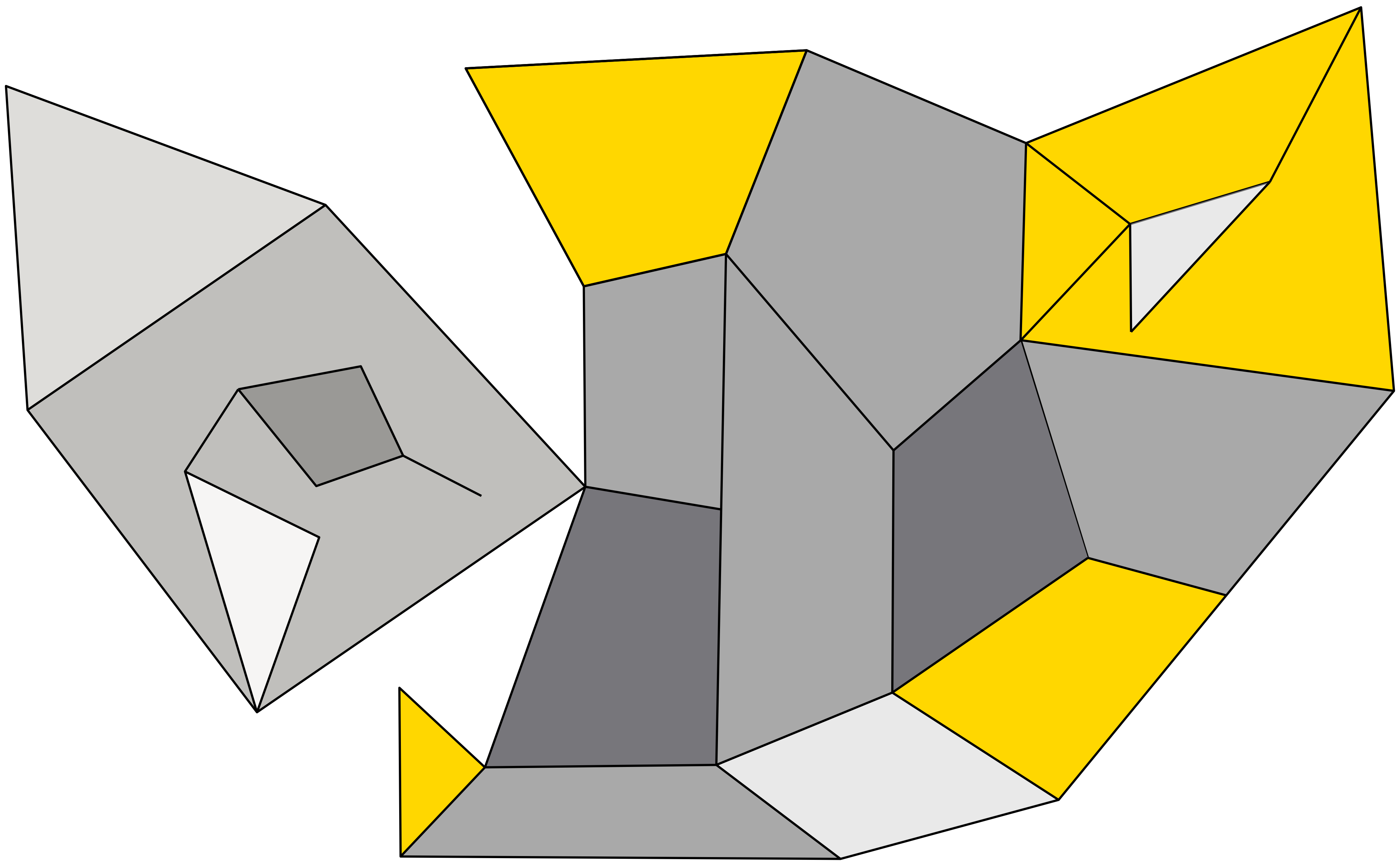
Cyclic dependencies? Why does this work:
The left-hand side is larger than the values appearing on the right-hand side.

[Knuth 1977]: "superior context-free grammar"

# GRAPH-Enclosure with penalties

INPUT: a plane graph

- required and optional *faces*
- *arbitrary* edge costs $> 0$



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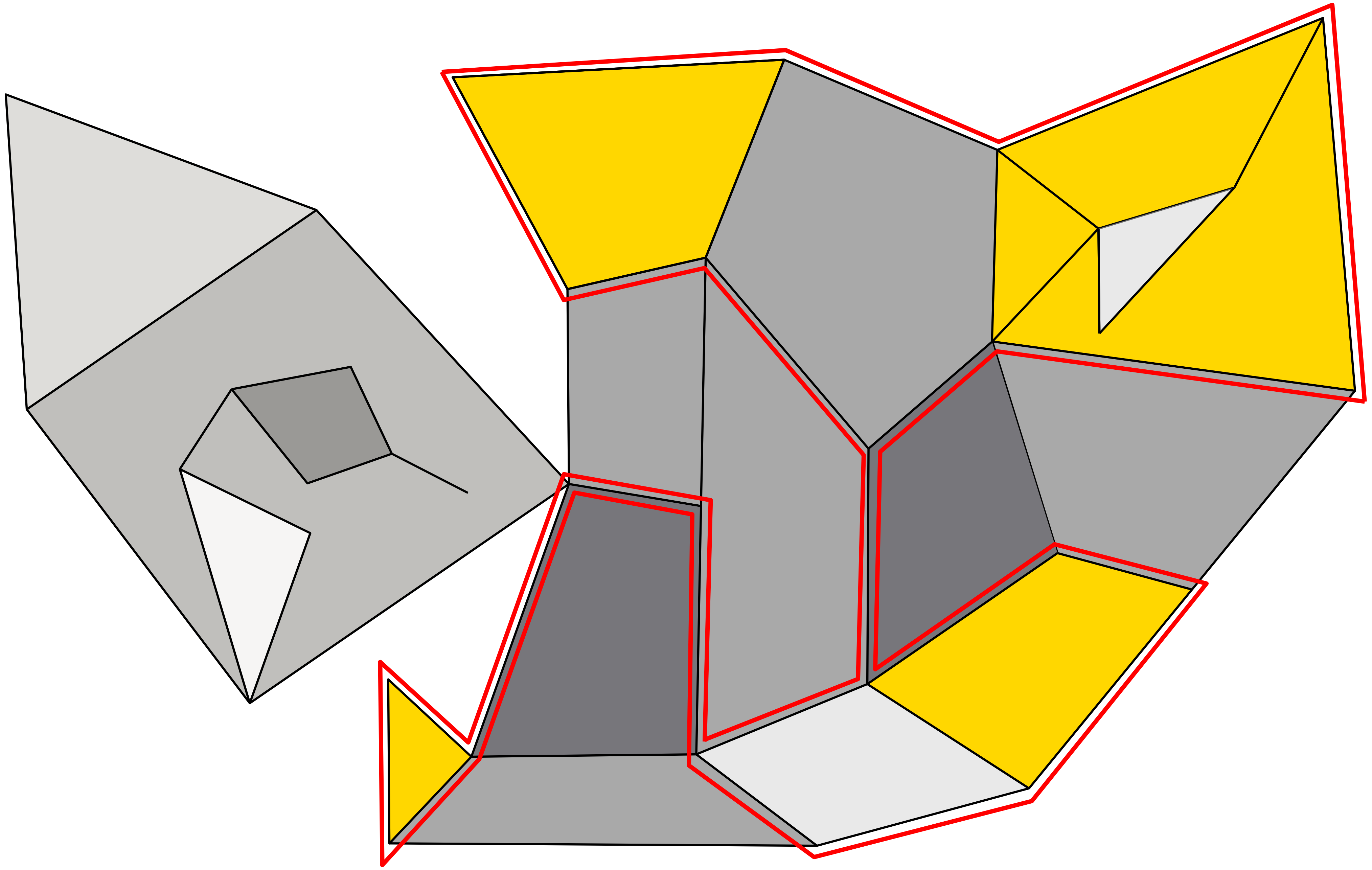

Reduction to (slighly generalized) GEOMETRIC-enclosure by an (arbitrary) straight-line embedding

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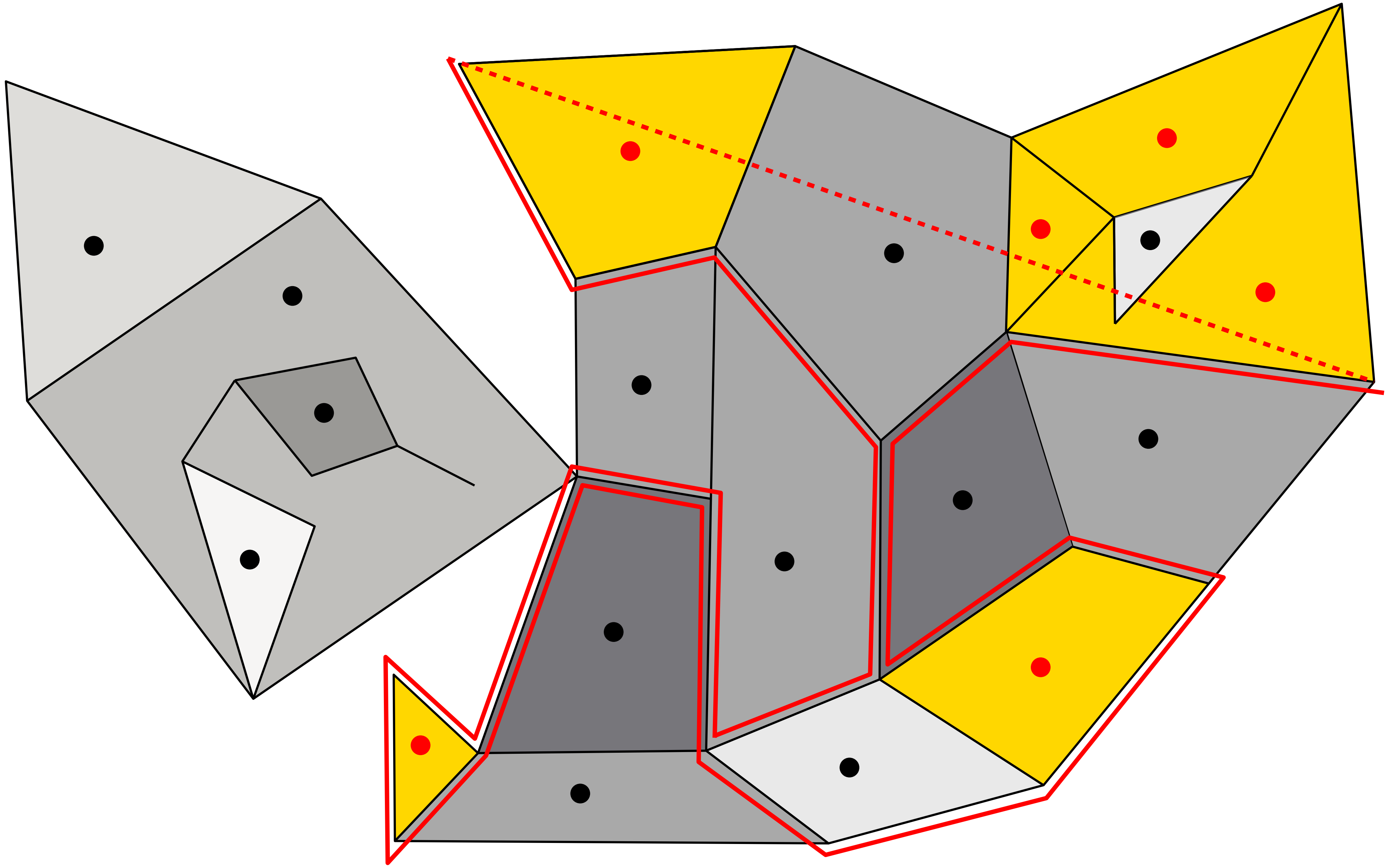

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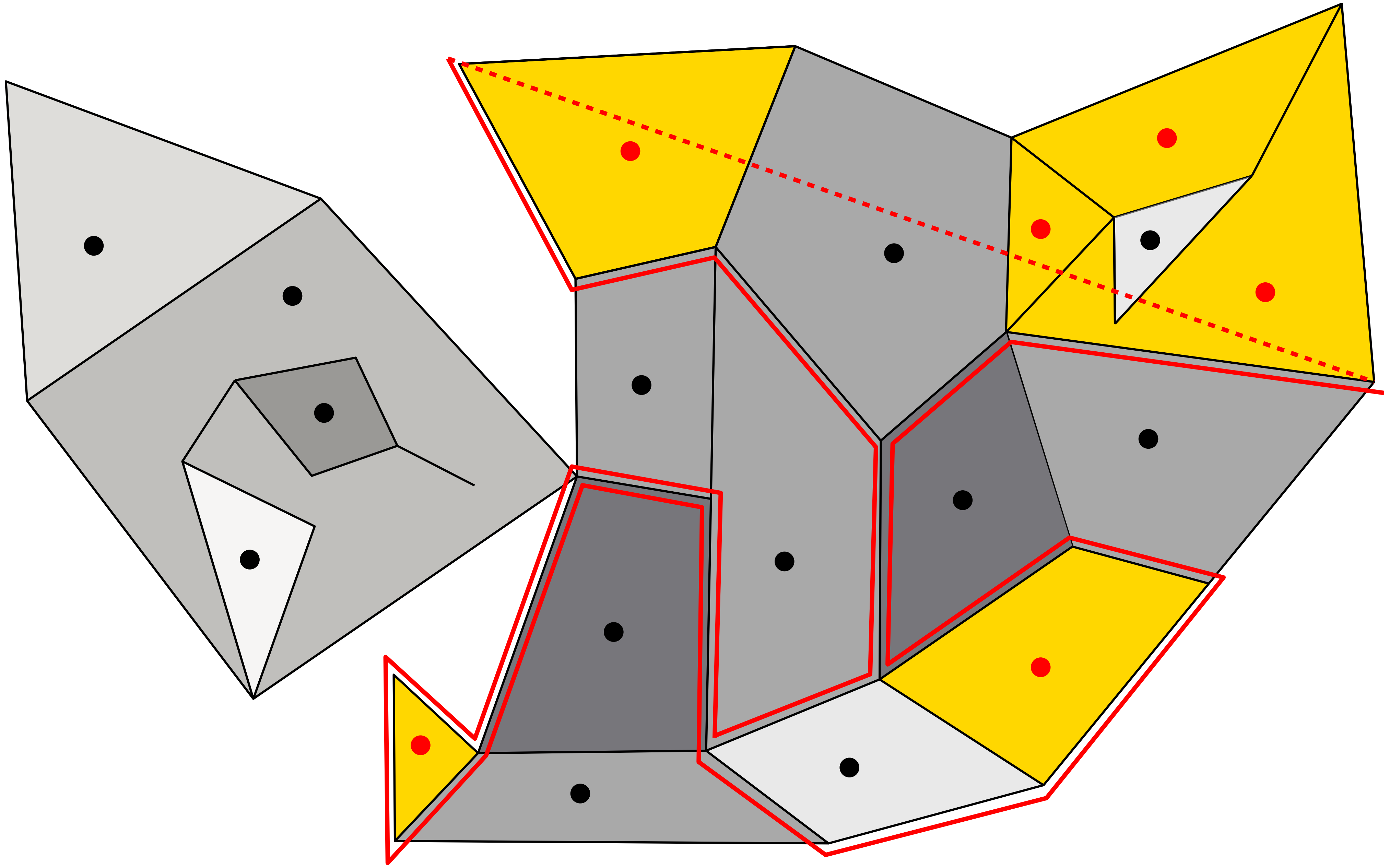
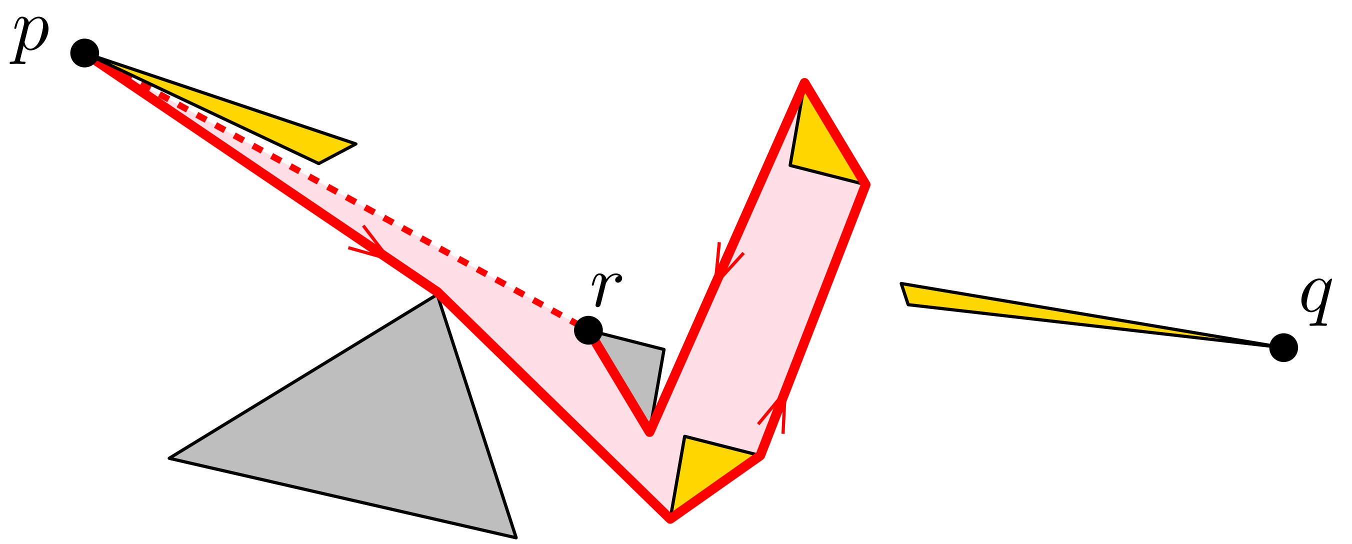


Reduction to (slighly generalized) GEOMETRIC-enclosure by an (arbitrary) straight-line embedding

Point objects can be handled. Extension to weakly simple objects is open.

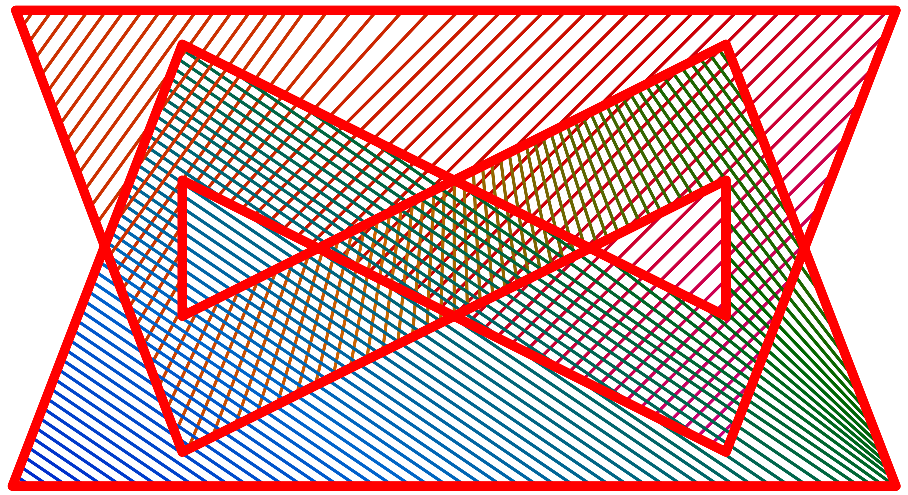
# What does $C(p, t, B)$ and $M(pq, t, B)$ REALLY represent?

- For subproblems $M(qp)$, self-intersections *do* occur:

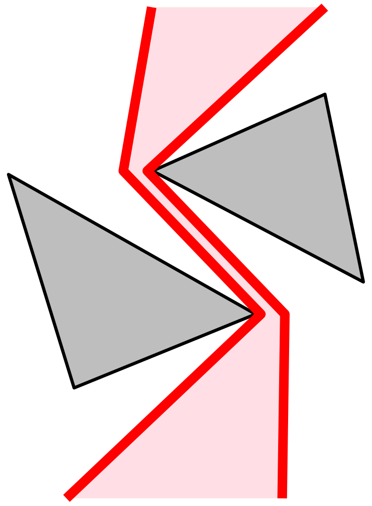



- weakly simple immersed polygons

self-overlapping + weakly simple



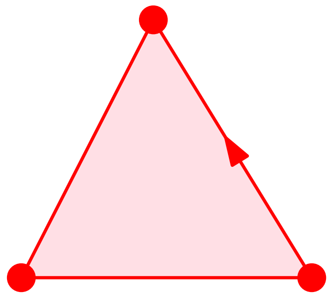
Milnor's doodle



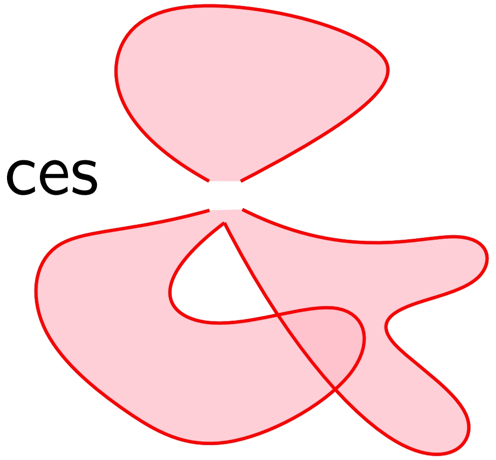
- Recognition algorithm?

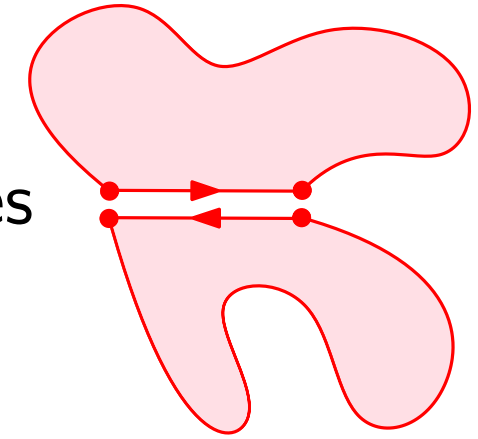
- basic building blocks

digons 

triangles 

- glue together at

vertices 

edges 

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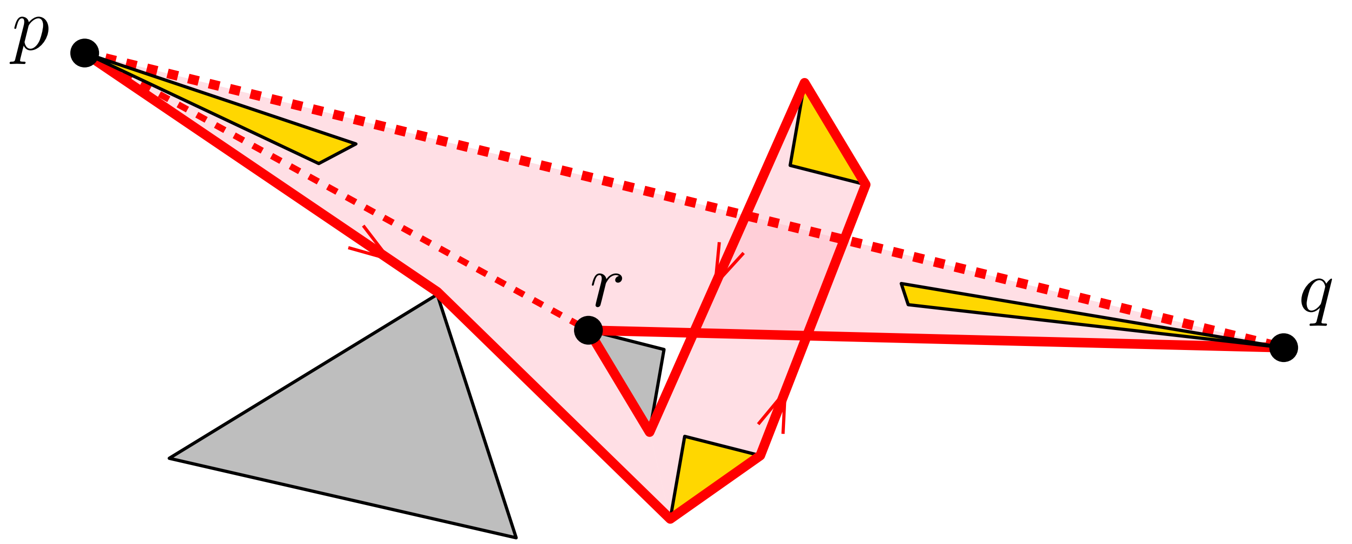


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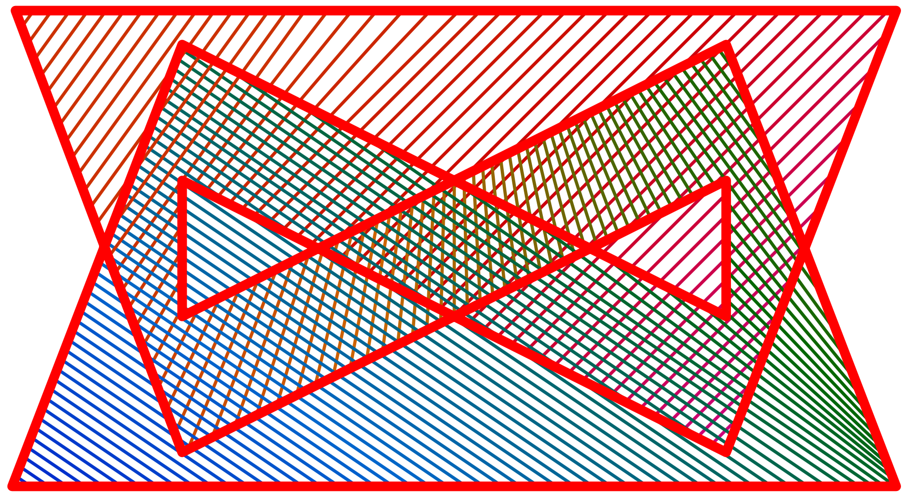

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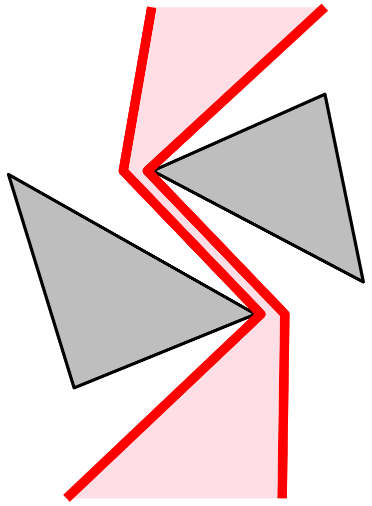

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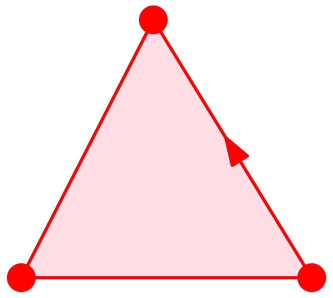
triangles 

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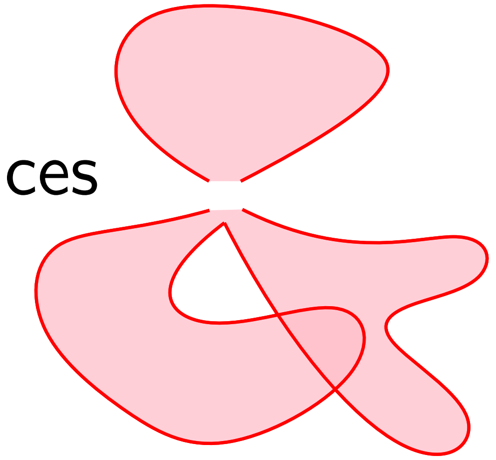
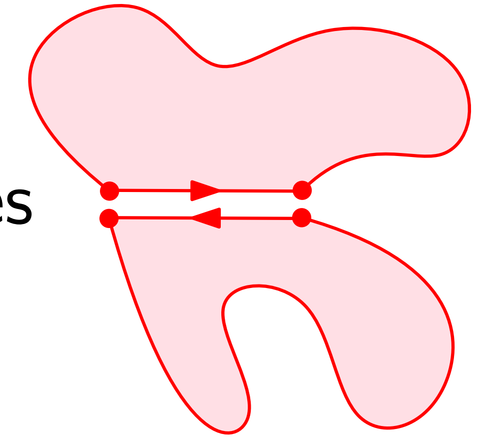
vertices 

edges 

# Lower bound under ETH



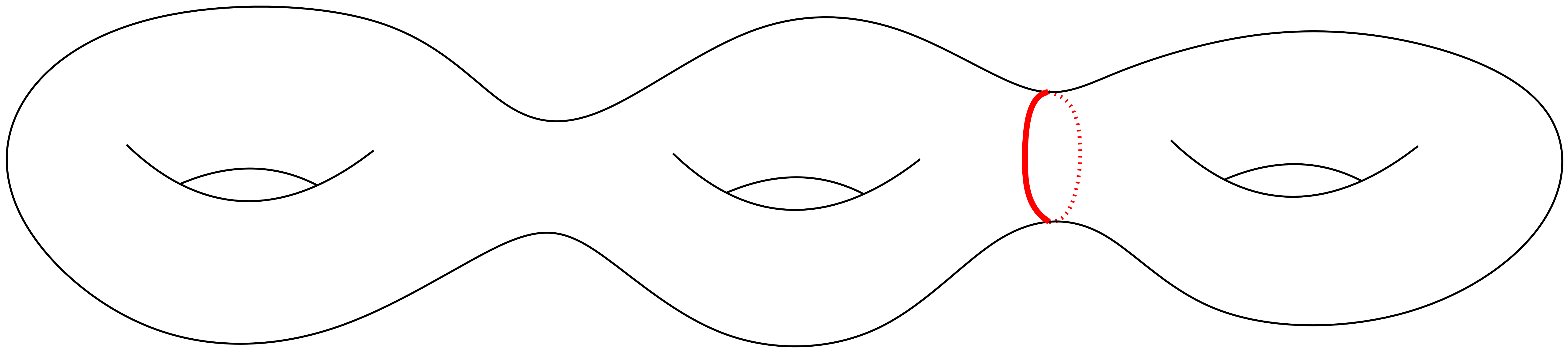
Exponential Time Hypothesis $\rightarrow$

GEOMETRIC/GRAPH-ENCLOSURE WITH PENALTIES cannot be solved in $2^{o(k)} \cdot n^{O(1)}$ time, even when all weights are $1$ and all penalties are $\infty$.

(Reduction from unweighted PLANAR STEINER TREE)

# What we originally set out to find

Find a shortest cycle that cuts off a *single handle* on a triangulated surface of high genus $g$.
(or a part of specified genus $g' < g$)



- FPT in $g$. [ Chambers, Colin de Verdière, Erickson, Lazarus, Whittlesey 2008 ]
- FPT in $g'$ ?