A Curious Identity on Self-Stresses

Günter Rote, Freie Universität Berlin

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We have a straight-line drawing of the graph K_4 in the plane on 4 vertices p_1, p_2, p_3, p_4. We write $[p_i p_j \dots p_n]$ for the signed area of the polygon $p_i p_j \dots p_n$. We assign to each edge ij a stress

$$\omega_{ij} := \frac{1}{[p_i p_j p_k][p_i p_j p_l]},$$

where the triangles in the denominator are the two triangles incident to ij, i. e., $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then there will be equilibrium at every vertex p_i, see Figure 1a:

$$\sum_{j:ij \in E} \omega_{ij}(p_j - p_i) = 0, \text{ for every } i, \tag{1}$$

where E denotes the edge set of the graph. (This equilibrium stress is determined by (1) up to a constant factor.) Moreover, if we pick two arbitrary points a and b and define

$$f_{ij} := [p_i p_j a] \cdot [p_i p_j b],$$

then the following identity can be shown [2]:

$$\sum_{1 \le i < j \le 4} \omega_{ij} f_{ij} = 1 \tag{2}$$

Regarding this fact, a reviewer of [2] wrote: "I believe there is some underlying homology in this situation. Given the fact that motions and stresses also fit into a setting of cohomology and homology as well, the authors might, at least, mention possible homology descriptions." These connections are still unexplored.

However, by experimenting, I found that the formula (2) generalizes to the graph of a k-sided pyramid (or the k-wheel) with tip p_0 and vertices $p_1, \dots, p_k$

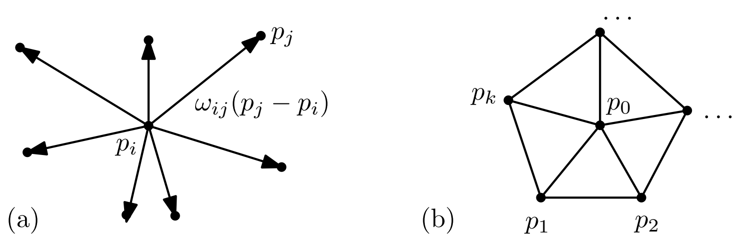

Figure 1: (a) equilibrium at the vertex p_i, (b) a 5-wheel ($k = 5$)

around the base, see Figure 1b. Again, a stress satisfying the equilibrium conditions (1) is unique up to a scalar multiple. We can compute such an equilibrium stress by the following formulas:

$$\omega_{i,i+1} := \frac{1}{[p_i p_{i+1} p_0][p_1 p_2 \dots p_k]}, \tag{3}$$

$$\omega_{0i} := \frac{1}{[p_0 p_i p_{i-1}][p_0 p_i p_{i+1}]} \cdot \frac{[p_{i-1} p_i p_{i+1}]}{[p_1 p_2 \dots p_k]} \tag{4}$$

for $i = 1, \dots, k$, where indices are taken modulo k. These stresses satisfy the generalization of (2):

$$\sum_{ij \in E} \omega_{ij} f_{ij} = 1 \tag{5}$$

Are there other graphs for which stresses can be defined by an analogous formula such that (5) *holds?* Of course, one could just overlay two pyramids, for example, and get a double-pyramid, but then the stresses are no longer such simple products and quotients of face areas.

Formulas (2) and (5) hold for a whole class of different functions f_{ij}. For example, we may replace f_{ij} by the path integral of the function $\|x\|^2$ over the segment $p_i p_j$.

Incidentally, the stress (3–4) lies behind the so-called *Wachspress coordinates*, which are used for barycentric interpolation and which go back to E. Wachspress [3], see also [1]. The equilibrium condition in the central vertex p_0 says that p_0 is the weighted barycenter of the surrounding vertices if we take the weights ω_{0i} and normalizes them to have sum 1.

References

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[2] Günter Rote, Francisco Santos, and Ileana Streinu. Expansive motions and the polytope of pointed pseudo-triangulations. In Boris Aronov, Saugata Basu, János Pach, and Micha Sharir, editors, *Discrete and Computational Geometry—The Goodman–Pollack Festschrift*, volume 25, pages 699–736. Springer Verlag, 2003. doi:10.1007/978-3-642-55566-4_33.

[3] Eugene Wachspress. *A Rational Finite Element Basis*. Academic Press, 1975.