How many compositions of two polyominoes?

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joint work with Andrei Asinowski, Gill Barequet, Gil Ben-Shachar, Martha Carolina Osegueda
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\[ P_1, \text{ size } n_1 = 14 \quad \text{and} \quad P_2, \text{ size } n_2 = 4 \]
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Background

(Wrong) LEMMA. Two polyominoes of total size $n_1 + n_2 = n$ have at most $2n$ compositions.

[ G. Barequet and R. Barequet 2015 ]
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PROPOSITION. Every polyomino of size $n$ can be composed from two polyominoes of size $n_1$ and $n_2$ with $n_1, n_2 \geq \frac{n-1}{4}$.

$A_n =$ the number of polyominoes of size $n$

$$A_n \leq \sum_{n_1 = n/4}^{3n/4} A_{n_1} A_{n-n_1} 2n$$
Background

(Wrong) LEMMA. Two polyominoes of total size $n_1 + n_2 = n$ have at most $2n$ compositions.

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\begin{align*}
A_n &\leq \sum_{n_1=n/4}^{3n/4} A_{n_1} A_{n-n_1} 2n
\end{align*}

[ G. Barequet, G. Rote, Mira Shalah 2019 ]: Improved bounds on the number of polyiamonds
Almost tight bounds

OBSERVATION. Two polyominoes of size $n_1$ and $n_2$ have at most $4n_1n_2$ compositions.
Almost tight bounds

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Two polycubes in $d$ dimensions of size $n_1$ and $n_2$ have at most $2dn_1 n_2$ compositions.

In $d \geq 3$ dimensions, this is tight up to a constant factor.
Almost tight bounds

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In $d \geq 3$ dimensions, this is tight up to a constant factor.

THEOREM. Two polyominoes of size $n$ can have as many as

$$\frac{n^2}{2 \cdot 8 \cdot \sqrt{\log_2 n}}$$

compositions.
Compositions & Minkowski difference

Represent polyomino $P$ by the set $A$ of square centers

$P_1 \quad \begin{array}{ccc} \times & \times & \times \\ \times & \times & \quad \end{array} \quad A_1 \quad \begin{array}{ccc} \times & \times & \times \\ \times & \times & \quad \end{array}$
Compositions & Minkowski difference

Represent polyomino $P$ by the set $A$ of square centers

\[
P_1 \begin{array}{cccc}
\times & \times & \times \\
\times & \times 
\end{array} \quad A_1 \begin{array}{ccc}
\times & \times & \times \\
\times & \times
\end{array}
\]

OBSERVATION. $P_1$ and $P_2 + t$ overlap $\iff t \in M := A_1 \oplus (-A_2) := \{ x_1 - x_2 \mid x_1 \in A_1, \ x_2 \in A_2 \}$
Compositions & Minkowski difference

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\[
P_1 \quad × \quad × \quad ×
\]
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Proof: $t = x_1 - x_2 \iff x_1 = x_2 + t$
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$P_1$ $A_1$ $P_2$ $A_2$

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Proof: $t = x_1 - x_2 \iff x_1 = x_2 + t$
Represent polyomino $P$ by the set $A$ of square centers

$P_1$  
\[
\begin{array}{cccc}
\times & \times & \times & \times \\
\times & \times & & \\
\end{array}
\]

$A_1$  
\[
\begin{array}{ccc}
\times & \times & \times \\
\times & & \\
\end{array}
\]

$P_2$  
\[
\begin{array}{ccc}
\circ & \circ & \\
\circ & & \\
\circ & & \\
\end{array}
\]

$A_2$  
\[
\begin{array}{c}
\circ \\
\circ \\
\end{array}
\]

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Proof: $t = x_1 - x_2 \iff x_1 = x_2 + t$
Compositions & Minkowski difference

Represent polyomino $P$ by the set $A$ of square centers

$P_1$  

$A_1$  

$P_2$  

$A_2$  

OBSERVATION. $P_1$ and $P_2 + t$ overlap $\iff$ 
$t \in M := A_1 \oplus (-A_2) := \{ x_1 - x_2 \mid x_1 \in A_1, \ x_2 \in A_2 \}$

Proof: $t = x_1 - x_2 \iff x_1 = x_2 + t$
Compositions & Minkowski difference

Represent polyomino \( P \) by the set \( A \) of square centers

\[
P_1 \begin{array}{c}
\times \\
\times \\
\times \\
\times
\end{array} \quad A_1 \begin{array}{c}
\times \\
\times \\
\times
\end{array} \quad P_2 \begin{array}{c}
ocircle
\ocircle
\ocircle
\ocircle
\end{array} \quad A_2 \begin{array}{c}
\ocircle
\ocircle
\ocircle
\ocircle
\end{array}
\]

OBSERVATION. \( P_1 \) and \( P_2 + t \) overlap \iff \( t \in M := A_1 \oplus (-A_2) := \{ x_1 - x_2 \mid x_1 \in A_1, \ x_2 \in A_2 \} \)

Proof: \( t = x_1 - x_2 \iff x_1 = x_2 + t \)
Compositions & Minkowski difference

Represent polyomino $P$ by the set $A$ of square centers

$P_1$  $A_1$  $P_2$  $A_2$

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Proof: $t = x_1 - x_2 \iff x_1 = x_2 + t$

$P_1$ and $P_2 + t$ valid $\iff t \notin M$ and $t$ is adjacent to $M$
Compositions & Minkowski difference

Represent polyomino $P$ by the set $A$ of square centers

$P_1$ $x$ $x$ $x$ $x$ $A_1$ $x$ $x$ $x$ $P_2$ $x$ $x$ $A_2$ $o$ $o$ $o$ $o$

OBSERVATION. $P_1$ and $P_2 + t$ overlap $\iff$ $t \in M := A_1 \oplus (-A_2) := \{ x_1 - x_2 \mid x_1 \in A_1, x_2 \in A_2 \}$

Proof: $t = x_1 - x_2 \iff x_1 = x_2 + t$

$P_1$ and $P_2 + t$ valid $\iff$ $t \not\in M$ and $t$ is adjacent to $M$
Motion Planning
Motion Planning

\[ \Theta(n^2) \]
Motion Planning

$\Theta(n^4)$
Many Compositions

\[ \sqrt{n} \times \sqrt{n} \times \sqrt{n} \times 2\sqrt{n} \times n \]
Many Compositions

\[
\begin{align*}
\sqrt{n} & \quad \times \quad \sqrt{n} \\
3\sqrt{n} & \quad \times \quad \sqrt{n} \\
\end{align*}
\]

\[
\begin{align*}
\sqrt{n} & \quad \times \quad \sqrt{n} \\
2\sqrt{n} & \quad \times \quad n \\
\end{align*}
\]

\[
\text{size} = \Theta(n) \quad \Theta(n^{3/2}) \text{ compositions}
\]
Even more compositions

$\Theta(n^{5/3})$

sparse toothbrush $S_2$

dense toothbrush $D_2$
Even more compositions

\[ \Theta\left(n^{5/3}\right) \]

sparse toothbrush $S_2$

dense toothbrush $D_2$
Even more compositions

\[ \Theta(n^{5/3}) \]

sparse toothbrush \( S_2 \)
dense toothbrush \( D_2 \)
Numerical experiments

\[ \text{normalized number of compositions } n + n \]
Numerical experiments

normalized number of compositions $n + n$

$4n$
Compute the (number of) compositions

Find $M := A_1 \oplus (-A_2)$ and find all its neighbors.
Compute the (number of) compositions

Find $M := A_1 \oplus (-A_2)$ and find all its neighbors
Compute the (number of) compositions

Find $M := A_1 \oplus (-A_2)$ and find all its neighbors

In $d$ dimensions: $O(n^2d)$ space and $O(n^2d^2)$ time. (Radix sort)