Probabilistic Finite Automaton (PFA) Emptiness Is Undecidable for a Fixed Automaton

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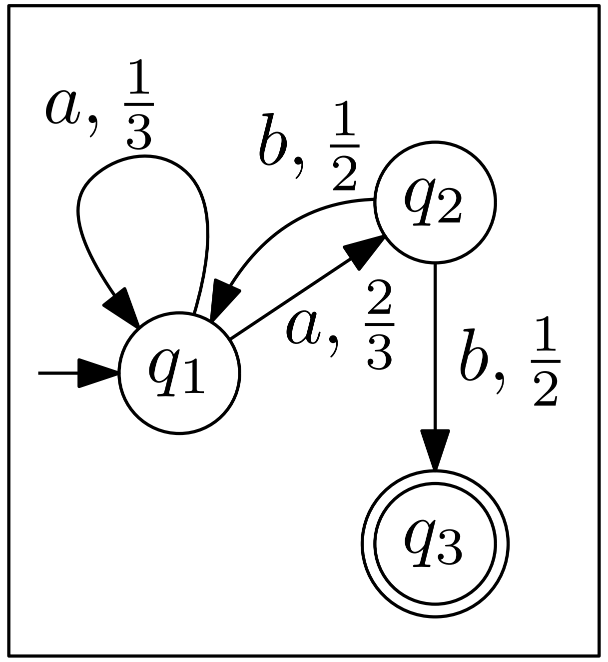
PFA A



Input
$u = abaaca$

$L(A) =$ the language recognized by A [M. Rabin 1963]
$:= \{ u \in \Sigma^* \mid \Pr[A \text{ accepts } u] > \lambda \}$

$\lambda =$ the cutpoint

- stochastic $d \times d$ transition matrices $M_a = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$, $M_b = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$, $M_c \ldots$
- a starting vector $\pi = (1\ 0\ 0)$: a probability distribution
- a final vector $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \{0,1\}^d$ (can also be in $[0,1]^d$ or even in $\mathbb{R}^d$)

$d =$ the number of states of the automaton $=$ the dimension of π, f and M_i

$u = abaaca \in L(A) \iff \pi M_a M_b M_a M_a M_c M_a f > \lambda$

PFA Emptiness

Given A, is $L(A) = \emptyset$ or $L(A) \neq \emptyset$?

Given π, f and a set $\mathcal{M}$ of stochastic matrices, is there a sequence $M^{(1)}, M^{(2)}, \ldots, M^{(n)} \in \mathcal{M}$ such that

$$\pi^T M^{(1)} M^{(2)} \ldots M^{(n)} f > \lambda \ ?$$

with arbitrary matrices in $\mathbb{R}^{d \times d}$, and arbitrary $\pi, f \in \mathbb{R}^d$: *weighted automata* or *generalized probabilistic automata* or *pseudo-stochastic automata*

Undecidability results about the growth of matrix products depend on this:

- Joint spectral radius of several matrices [V. Blondel and N. Tsitsiklis 2000]
- Growth of bilinear systems [G. Rote 2019, M. Rosenfeld 2022, V. Bui 2023]
- Probabilistic planning [A. Condon and R. Lipton 1989]

PFA Emptiness

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$$\pi^T M^{(1)} M^{(2)} \ldots M^{(n)} f > \lambda \text{ ?}$$

UNDECIDABLE [Nasu & Honda 1969, Claus 1981, Condon & Lipton 1989]

Reductions from (a) the Post Correspondence Problem (PCP)
(b) the halting problem for 2-counter machines [with ideas of R. Freivalds, MFCS 1981]

[see my survey at arXiv:2405.03035]

- UNDECIDABLE even for small $k = |\Sigma| = |\mathcal{M}|$ and d.

For example, $k = 5, d = 9$, or $k = 2, d = 20$. [Claus 1981, Blondel & Canterini 2003, Hirvensalo 2007]

New results:

- UNDECIDABLE even for $k = |\mathcal{M}| = 6$ FIXED matrices of size 7×7, and $f = (1, 0, 0, 0, 0, 0, 0)^T$.

The only input is the starting distribution $\pi \in \mathbb{Q}^7$.

- UNDECIDABLE even for $k = |\mathcal{M}| = 5$ FIXED matrices of size 6×6, and fixed distribution $\pi \in \mathbb{Q}^6$.

The only input is the final vector $f \in [0, 1]^6$.

Reductions from the PCP

Overview

- Introduction
  - Problem statement
  - Statement of results
- Techniques and ideas of the proof
- Conclusion
  - How small can d be made?
  - Gap amplification

Modeling words by matrix multiplication

From $a_1 = 0.321$, $u_1 = 0.001$ ("unit factor")
and $a_2 = 0.56789$, $u_2 = 0.00001$
make $a_1 + u_1 a_2 = 0.32\overset{\wedge}{1}56789$ and $u_1 u_2 = 0.00000001$

$a_1 \circ a_2$

$$\begin{pmatrix} 1 & 0 \\ a_1 & u_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 \circ a_2 & u_1 u_2 \end{pmatrix}$$

Modeling words by matrix multiplication

From $a_1 = 0.321$, $u_1 = 0.001$ ("unit factor")
and $a_2 = 0.56789$, $u_2 = 0.00001$
make $a_1 + u_1 a_2 = 0.32156789$ and $u_1 u_2 = 0.00000001$

$0.32156789 \hat{=} a_1 \circ a_2$

$$\begin{pmatrix} 1 & 0 \\ a_1 & u_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 \circ a_2 & u_1 u_2 \end{pmatrix}$$

Word *pairs*: (working towards the Post Correspondence Problem)

$$\begin{pmatrix} 1 & 0 & 0 \\ a_1 & u_1 & 0 \\ b_1 & 0 & v_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a_2 & u_2 & 0 \\ b_2 & 0 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a_1 \circ a_2 & u_1 u_2 & 0 \\ b_1 \circ b_2 & 0 & v_1 v_2 \end{pmatrix}$$

[A. Markov 1947], [M. Paterson 1970]
with *integer* matrices

Modeling words by matrix multiplication

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$0.32156789 = a_1 \circ a_2$

$$\begin{pmatrix} 1 & 0 \\ a_1 & u_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 \circ a_2 & u_1 u_2 \end{pmatrix}$$

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[A. Markov 1947], [M. Paterson 1970] with *integer* matrices

Include the quadratic terms a^2, ab, b^2:

$$M(a,b) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & u & 0 & 0 & 0 & 0 \\ a^2 & 2au & u^2 & 0 & 0 & 0 \\ b & 0 & 0 & v & 0 & 0 \\ b^2 & 0 & 0 & 2bv & v^2 & 0 \\ ab & ub & 0 & va & 0 & uv \end{pmatrix}$$

$$M(a_1, b_1) M(a_2, b_2) = M(a_1 \circ a_2, b_1 \circ b_2)$$

[Claus 1981], [Blondel & Canterini 2003] for *integer* matrices

With $\pi = (0,0,-1,0,-1,2)$ and $f = (1,0,0,0,0,0)^T$: $\pi M(a,b) f = -a^2 - b^2 + 2ab = -(a-b)^2$

$$\pi M(a,b) f \ge 0 \iff a = b$$

The Post Correspondence Problem (PCP)

Given a list of word pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$,
decide if there is a nonempty sequence $i_1 i_2 \ldots i_n$ of indices $i_j \in \{1, 2, \ldots, k\}$ such that

$$a_{i_1} a_{i_2} \ldots a_{i_n} = b_{i_1} b_{i_2} \ldots b_{i_n}.$$

UNDECIDABLE already for $k = 5$ five word pairs. [T. Neary 2015]

Form the matrices $M_i = M(a_i, b_i)$.

$$\pi M(a,b) f \geq 0 \iff a = b$$

$$\pi M_{i_1} M_{i_2} \ldots M_{i_n} f \geq 0 \iff a_{i_1} a_{i_2} \ldots a_{i_n} = b_{i_1} b_{i_2} \ldots b_{i_n}.$$

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Given a list of word pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$,
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make matrices M_i stochastic!

make π positive!

change ≥ 0 to $> \lambda$!

$\Downarrow$

general method of P. Turakainen [(1969) 1975]
would add 2 states: $d \to d + 2$

M_i has small nonnegative entries $\to$ can do it with 1 extra state (fixed f) or 0 extra states (fixed π). $\square$

Make column sums 1

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$$\pi M_{i_1} M_{i_2} \dots M_{i_m} f = (\pi V)\underbrace{(V^{-1} M_{i_1} V)}\underbrace{(V^{-1} M_{i_2} V)}\dots(V^{-1} M_{i_m} V)(V^{-1} f) \text{ with } V = \begin{pmatrix} 1&1&1&1&1&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

The column sums of the new matrices $V^{-1} M_i V$ are 1:

$$\begin{aligned} &(1,1,1,1,1,1)V^{-1} M_i V \\ &= (1,0,0,0,0,0) M_i V \\ &= (1,0,0,0,0,0) V \\ &= (1,1,1,1,1,1) \end{aligned}$$

$$M_i = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & u & 0 & 0 & 0 & 0 \\ a^2 & 2au & u^2 & 0 & 0 & 0 \\ b & 0 & 0 & v & 0 & 0 \\ b^2 & 0 & 0 & 2bv & v^2 & 0 \\ ab & ub & 0 & va & 0 & uv \end{pmatrix}, \qquad V^{-1} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Make row sums 1 with an extra state

$$M = \begin{pmatrix} * & * & * & * & * & * & \\ * & * & * & * & * & * & \\ * & * & * & * & * & * & \\ * & * & * & * & * & * & \\ * & * & * & * & * & * & \\ * & * & * & * & * & * & \\ & & & & & & \end{pmatrix}$$

Make row sums 1 with an extra state

$$M = \begin{pmatrix} * & * & * & * & * & * & r_1 \\ * & * & * & * & * & * & r_2 \\ * & * & * & * & * & * & r_3 \\ * & * & * & * & * & * & r_4 \\ * & * & * & * & * & * & r_5 \\ * & * & * & * & * & * & r_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} =1 \\ =1 \\ =1 \\ =1 \\ =1 \\ =1 \\ =1 \end{matrix}$$

r_j's can be negative.

Make row sums 1 with an extra state

$$M = \begin{pmatrix} * & * & * & * & * & * & r_1 \\ * & * & * & * & * & * & r_2 \\ * & * & * & * & * & * & r_3 \\ * & * & * & * & * & * & r_4 \\ * & * & * & * & * & * & r_5 \\ * & * & * & * & * & * & r_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} =1 \\ =1 \\ =1 \\ =1 \\ =1 \\ =1 \\ =1 \end{matrix}$$

r_j's can be negative.

- Make the matrices doubly-stochastic:

$$J = \begin{pmatrix} 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \end{pmatrix},$$

form the matrices $M' := 0.99J + 0.01M > 0$

$JJ = JM = MJ = J$

$\pi J = 0$, because the entries of π sum to 0.

- Turn π into a probability distribution $\pi' := (\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}) + 0.01\pi$

Fixed matrices

Undecidable PCP instances of Yu. Matiyasevich & G. Sénizergues [2005] with 7 word pairs

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5), (a_6, b_6), (a_7, b_7).$$

Two special pairs (a_1, b_1) and (a_2, b_2) must be used at the beginning and at the end (and can be used nowhere else).

$$\pi M_1 M_{i_2} M_{i_3} \dots M_{i_{n-1}} M_2 f > \lambda$$

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$$\underbrace{\pi M_1}_{\pi'} M_{i_2} M_{i_3} \dots M_{i_{n-1}} \underbrace{M_2 f}_{f'} > \lambda$$

$$\pi' M_{i_2} M_{i_3} \dots M_{i_{n-1}} f' > \lambda$$

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$$\underbrace{\pi M_1}_{\pi'} M_{i_2} M_{i_3} \dots M_{i_{n-1}} \underbrace{M_2 f}_{f'} > \lambda$$

$$\pi' M_{i_2} M_{i_3} \dots M_{i_{n-1}} f' > \lambda$$

All words a_i, b_i except a_1 can be held fixed. [observed by Halava, Harju, & Hirvensalo 2007]

$\Longrightarrow$ Everything except π' is fixed.

Fixed matrices

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$$\pi M_1 M_{i_2} M_{i_3} \ldots M_{i_{n-1}} M_2 f > \lambda$$

$$\pi' \qquad\qquad f'$$

$$\pi' M_{i_2} M_{i_3} \ldots M_{i_{n-1}} f' > \lambda$$

All words a_i, b_i except a_1 can be held fixed. [observed by Halava, Harju, & Hirvensalo 2007]

$\Longrightarrow$ Everything except π' is fixed.

WHAT ARE THOSE 5 FIXED MATRICES M_3, M_4, M_5, M_6, M_7?

Can we explicitly write down the fixed matrices?

Undecidability reduction chain

Halting problem for some universal Turing machine

↓ (also: ? ↓)

Individual word problem (or *individual accessibility problem*) for some particular (fixed) semi-Thue system:
Can the fixed target word w_∞ be derived from a given $w \xrightarrow{*} w_\infty$?

⇓ [Yu. Matiyasevich & G. Sénizergues 2005]

Individual word problem for a semi-Thue system with **3** fixed rules.

↓ [V. Claus 1980]

PCP with 7 word pairs (6 of which are fixed) ⟶ 5 fixed matrices for the PFA

Semi-Thue system:

$$\begin{array}{l} a_1 \longrightarrow b_1 \\ a_2 \longrightarrow b_2 \\ \quad\vdots \\ a_k \longrightarrow b_k \end{array}$$

rewriting rules

Can we explicitly write down the fixed matrices?

Undecidability reduction chain

Halting problem for some universal Turing machine

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Individual word problem (or *individual accessibility problem*) for some particular (fixed) semi-Thue system:
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PCP with 7 word pairs (6 of which are fixed)

5 fixed matrices for the PFA

Semi-Thue system:

$$\begin{array}{l} a_1 \longrightarrow b_1 \\ a_2 \longrightarrow b_2 \\ \vdots \\ a_k \longrightarrow b_k \end{array}$$

rewriting rules

Can we explicitly write down the fixed matrices?

$$M_{\mathtt{a},\mathtt{a}} = \frac{1}{3}\begin{pmatrix} 2.2592 & 0.33 & 0.006 & 0.33 & 0.0385 & 0.0363 \\ 2.2193 & 0.36 & 0.0126 & 0.33 & 0.0385 & 0.0396 \\ 2.2589 & 0.33 & 0.0063 & 0.33 & 0.0385 & 0.0363 \\ 2.2189 & 0.33 & 0.006 & 0.36 & 0.0455 & 0.0396 \\ 2.2589 & 0.33 & 0.006 & 0.33 & 0.0388 & 0.0363 \\ 2.2589 & 0.33 & 0.006 & 0.33 & 0.0385 & 0.0366 \end{pmatrix} \quad M_{\mathtt{b},\mathtt{b}} = \frac{1}{11}\begin{pmatrix} 7.9863 & 1.32 & 0.0473 & 1.32 & 0.168 & 0.1584 \\ 7.8367 & 1.43 & 0.0737 & 1.32 & 0.168 & 0.1716 \\ 7.9852 & 1.32 & 0.0484 & 1.32 & 0.168 & 0.1584 \\ 7.8351 & 1.32 & 0.0473 & 1.43 & 0.196 & 0.1716 \\ 7.9852 & 1.32 & 0.0473 & 1.32 & 0.1691 & 0.1584 \\ 7.9852 & 1.32 & 0.0473 & 1.32 & 0.168 & 0.1595 \end{pmatrix}$$

Another matrix from a direct construction with $k = 50$ fixed matrices, using the universal Turing machine $U_{15,2}$ of T. Neary & D. Woods [2009]:

$$M_{\mathtt{baaaabbaab},\mathtt{baabbaaaaab}} =$$

$$\frac{1}{33}\begin{pmatrix} 23.858541515971309615329133136143029195355357 6 & 3.99666666669999666696 & 0.15070740741548067414515958922651 \\ 23.8585415159713096148792331328093625953494616 & 3.99666666669999666729 & 0.15070740741548067422509292256051 \\ 23.8585415159713096153291331361430291953520576 & 3.99666666669999666696 & 0.15070740741548067414515958922651 \\ 23.8585415159713096153245856916278814179108832 & 3.99666666669999666696 & 0.15070740741548067414515958922651 \\ 23.85854151597130961532913313614302919535535727 & 3.99666666669999666696 & 0.15070740741548067414515958922651 \\ 23.8585415159713096153291331361430291953553246 & 3.99666666669999666696 & 0.15070740741548067414515958922651 \end{pmatrix}$$

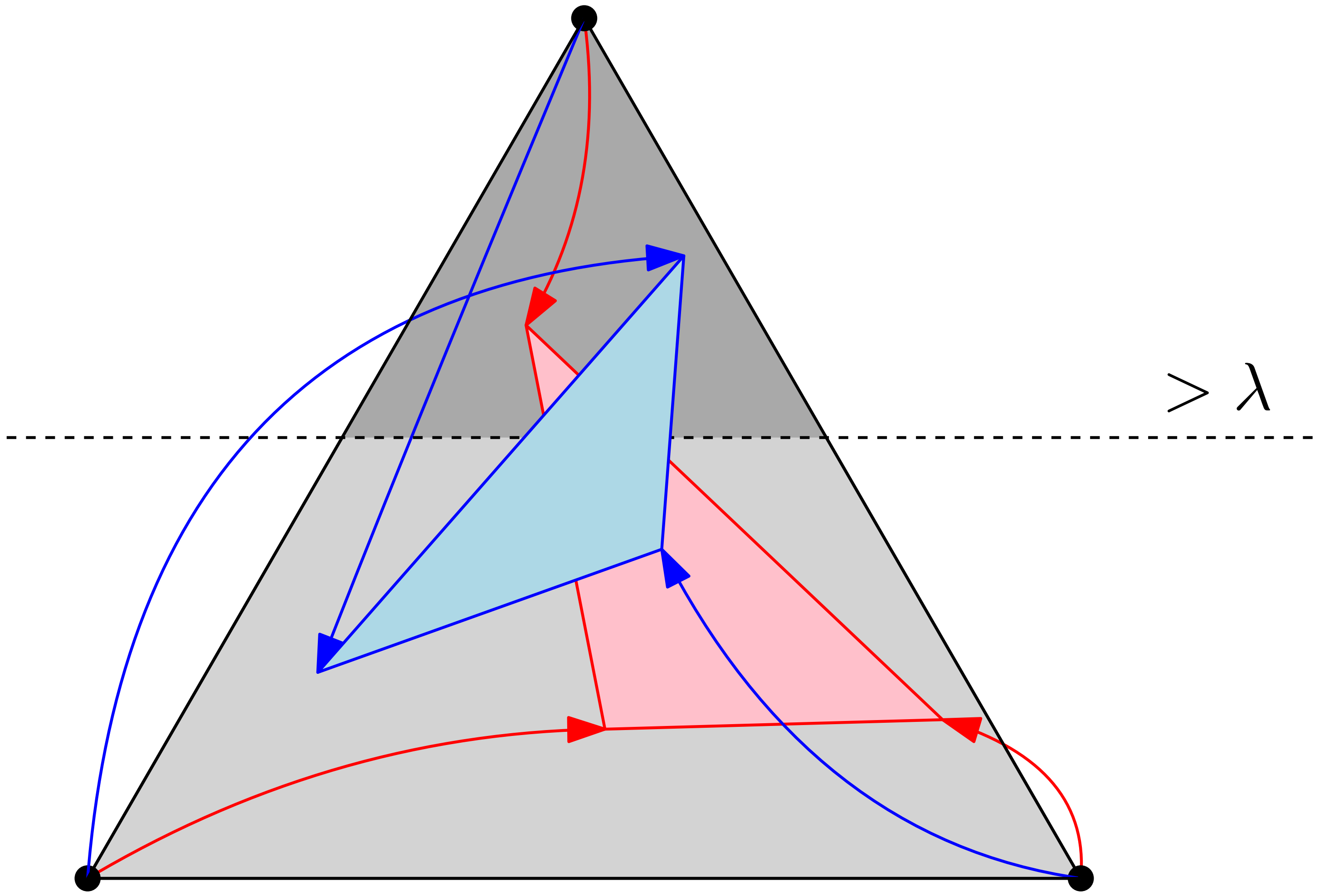
(40-digit numbers)

How small can the dimension d be made?

$d = 6$: UNDECIDABLE

$d = 2$: decidable: linear mappings on the interval $[0, 1]$ [V. Claus 1981]

$d = 3$: linear mappings on the triangle. Geometric techniques?



Gap Amplification

PFA A_6:
four 6×6 matrices

$$\sup_{u \in \Sigma^*} \Pr[A_6 \text{ accepts } u] > \lambda = \tfrac{1}{2}$$

$$\sup_{u \in \Sigma^*} \Pr[A_6 \text{ accepts } u] \le \lambda = \tfrac{1}{2}$$

$\Downarrow$

[H. Gimbert & Y. Oualhadj 2010]
[N. Fijalkow 2017]

PFA B_{15}:
six 15×15 matrices (not fixed)

$$\sup_{u \in \Sigma^*} \Pr[B_{15} \text{ accepts } u] = 1$$

$$\sup_{u \in \Sigma^*} \Pr[B_{15} \text{ accepts } u] \le \varepsilon$$

any chosen $\varepsilon > 0$