Introduction to GIT

We start with some facts about algebraic groups we will need later on.

1 Algebraic groups

- Let $X$ be a variety over $K = \mathbb{K}$ with $\text{char}(K) = 0$.
- Let $G$ be an affine algebraic group acting on $X$:
  \[ \sigma : G \times X \rightarrow X \quad \text{with} \quad g.x := \sigma(g, x) \]
- Denote by $X^G := \{ x \in X \mid g.x = x \forall g \in G \}$ the $G$-invariant points and by $G_x := \{ g \in G \mid g.x = x \}$ the stabiliser of $x$.
- The set $G.x := \{ g.x \mid \forall g \in G \}$ is the orbit of $x$.
- Let $\Psi : G \times X \rightarrow X \times X$ be defined as $\Psi(g, x) = (\sigma(g, x), x)$.
- A map $f : X \rightarrow Y$ is called $G$-invariant if $f(g.x) = f(x) \forall g \in G, x \in X$.

**Definition 1.1.** An affine algebraic group $G$ over $K$ is an affine algebraic variety over $K$ with a group structure such that the inversion map $\beta : G \rightarrow G$ and the group operation $\gamma : G \times G \rightarrow G$ are regular maps.

**Example.** A torus $T = (\mathbb{K}^\ast)^n$ is an affine algebraic group. The general linear group $GL(n, \mathbb{K})$ is an affine algebraic group over $\mathbb{K}$ with matrix multiplication.

**Remark.** Affine algebraic groups can be embedded into $GL(n, \mathbb{K})$ and are therefore linear algebraic groups.

**Definition 1.2.** A linear algebraic group $G$ is called linearly reductive if for any rational representation $\rho : G \rightarrow GL(V)$ and any nonzero $G$-invariant vector $v \in V$ there exists a linear $G$-invariant function $f$ on $V$ such that $f(v) \neq 0$.

**Definition 1.3.** A linear algebraic group $G$ is called geometrically reductive if for any rational representation $\rho : G \rightarrow GL(V)$ and any nonzero $G$-invariant vector $v \in V$ there exists a homogeneous $G$-invariant polynomial $f$ on $V$ such that $f(v) \neq 0$.

**Remark.** Linearly reductive implies geometrically reductive.
Examples of geometrically reductive groups are \((\mathbb{C}^*)^k\) and \(\text{GL}(n)\) for all \(k\) and \(n\). Products of geometrically reductive groups are geometrically reductive. For further details see [Hum75, ch. VII] and [MFK65, p. 191-193]. The reductiveness of a group gives a useful property for a group action which we will need later.

**Lemma 1.1.** Let \(G\) be a geometrically reductive group acting on some affine variety \(X\). Let \(Z_1\) and \(Z_2\) be two closed \(G\)-invariant \((g.Z_i \subseteq Z_i\) for all \(g \in G\)) subsets of \(X\) with \(Z_1 \cap Z_2 = \emptyset\). Then there exists a \(G\)-invariant function \(\Psi \in \mathcal{O}(X)^G\) such that \(\Psi(Z_1) = 1\) and \(\Psi(Z_2) = 0\).

**Proof.**

\[
Z_1 \cap Z_2 = \emptyset \Rightarrow (1) = I(Z_1 \cap Z_2) = I(Z_1) + I(Z_2) \Rightarrow \exists f_1 \in I(Z_1), f_2 \in I(Z_2) : f_1 + f_2 = 1
\]

Let \(W \subseteq \mathcal{O}(X)\) be the linear subspace spanned by \(\{g.f_2 | g \in G\}\) which is finite dimensional with some basis \(\{\varphi_1, ..., \varphi_n\}\).

Set \(f : X \to \mathbb{A}^n, x \mapsto (\varphi_1(x), ..., \varphi_n(x))\). Since the \(Z_i\) are \(G\)-invariant, we have \((g.f_2)(Z_i) = f_2(g^{-1}.Z_i) = f_2(Z_i)\).

\[
\Rightarrow \varphi_j(Z_1) = f_2(Z_1) = 1 \text{ and } \varphi_j(Z_2) = f_2(Z_2) = 0
\]

\[
\Rightarrow f(Z_1) = (1, ..., 1) \text{ and } f(Z_2) = (0, ..., 0)
\]

\(G\) is geometrically reductive \(\Rightarrow \exists F \in \mathbb{K}[x_1, ..., x_n]\) homogeneous such that \(F(1, ..., 1) = 1 \Rightarrow \Psi := f^*(F)\) has the desired properties. \(\Box\)

**Just identifying points of the same orbit gives only a set-theoretical quotient. If we want to construct the quotient of a variety by the group action as a variety and the map to this quotient such that it be a morphism, we need to make the notion of quotients more precise. For this we follow [Dol03].**

### 2 Quotients

**Definition 2.1.** A \(G\)-invariant map \(p : X \to Y\) is called a categorical quotient (of the \(G\) action), if for every \(G\)-invariant \(f : X \to Z\) there exists a unique \(\overline{f} : Y \to Z\), such that \(\overline{f} \circ p = f\).

**Definition 2.2.**

- A \(G\)-equivariant map \(p : X \to Y\) is called a good categorical quotient (of the \(G\)-action), if \(p\) satisfies:

i) For all $U \subseteq Y$ open, $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}(p^{-1}(U))^G$ of $G$-invariant functions.

ii) If $W \subseteq X$ is closed and $G$-invariant, then $p(W) \subseteq Y$ is closed.

iii) If $V_1, V_2 \subseteq X$ are closed, $G$-invariant, and $V_1 \cap V_2 = \emptyset$, then $p(V_1) \cap p(V_2) = \emptyset$.

• A (good) categorical quotient is called a (good) geometric quotient, if $\text{Im}(\Psi) = X \times_Y X$.

Remark. Satisfying the property for being a geometric quotient means that the preimage under $p$ of one point in $Y$ is exactly one $G$-orbit in $X$.

Example. Let $G = \mathbb{C}^*$ act on $X = \mathbb{C}^2 \setminus \{0\}$ by $g.(x_1, x_2) = (gx_1, gx_2)$. Then the map $p : X \to \mathbb{P}_1^2$, $p(x_1, x_2) = (x_1 : x_2)$ is a good geometric quotient, because the smallest closed and $G$-invariant subsets are exactly the orbits and two different orbits are mapped to two different points.

Theorem 2.1. Let $X$ be affine, $G$ geometrically reductive and acting on $X$. Then $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$ is a finitely generated $\mathbb{K}$-algebra and the canonical map $p : X \to Y := \text{Spec}(\mathcal{O}(X)^G)$ is a good categorical quotient.

Proof. Let $R := \mathcal{O}(X)$. Then $R^G$ is finitely generated because of Nagata’s theorem [Dol03, p. 41-45].

i) We show it for basic open sets $D(f) \subseteq Y$ for $f \in R^G$.

$$\frac{g}{f^n} \in (R^G)_f \iff g \in R^G$$

$\implies \frac{g}{f^n} G$-invariant

$\implies \frac{g}{f^n} \in (R_f)^G$

$\implies (R^G)_f = (R_f)^G$

ii) Let $W \subseteq X$ be closed and $G$-invariant and assume that $p(W)$ is not closed.

$\exists y \in p(W) \setminus p(W)$ with $Z := p^{-1}(y) \subseteq X$ closed, $G$-invariant, and $Z \cap W = \emptyset$.

Lemma 1.1 $\Rightarrow \exists \Phi \in \mathcal{O}(X)^G : \Phi(W') = 1$ and $\Phi(Z) = 0$

$p^*$ isomorphism $\Rightarrow \Phi = p^*(\phi)$ for some $\phi \in \mathcal{O}(Y)$

$\Rightarrow \phi(y) = 0$ and $\phi(p(W)) = 1$, which is a contradiction.

iii) Let $V_1, V_2 \subseteq X$ closed, $G$-invariant, and $V_1 \cap V_2 = \emptyset$.

Lemma 1.1 $\Rightarrow \exists \phi \in \mathcal{O}(Y) : \phi(p(V_1)) = 1$ and $\phi(p(V_2)) = 0$

$\Rightarrow p(V_1) \cap p(V_2) = \emptyset$
Remark. If $G$ is not reductive but $\mathcal{O}(X)^G$ is finitely generated, then theorem 2.1 still gives a categorical quotient.

Example. Let the additive group $\mathbb{C}$ act on $\mathbb{C}^2$ by $g.(c_1, c_2) = (c_1, c_2 + gc_1)$ with $g \in \mathbb{C}$ and $(c_1, c_2) \in \mathbb{C}^2$. This corresponds to an action of $\mathbb{C}$ on $\mathbb{C}[x_1, x_2]$ by $g.p(x_1, x_2) = p(x_1, x_2 + gx_1)$. Thus the invariant functions are $\mathbb{C}[x_1, x_2]^\mathbb{C} = \mathbb{C}[x_1]$ and therefore the projection onto the first coordinate is a categorical quotient, but not a good one, because $(0, 0)$ and $(0, 1)$ are two closed $\mathbb{C}$-invariant points but are projected onto the same point.

We are now able to construct good categorical quotients of affine varieties, so we will approach projective varieties. For projective $X$ we will embed $X \hookrightarrow \mathbb{P}^n$. This is done by choosing a (very ample) line bundle $L$ over $X$. The idea is then to cover the image of $X$ with open, affine, $G$-invariant subsets. To do so, we need an extension of the $G$-action on $X$ to a $G$-action on $\mathbb{P}^n$.

3 Linearisation and stability

Definition 3.1. Let $L$ be a line bundle on $X$ with projection $\pi : L \to X$. A $G$-linearisation of $L$ is an extension of the action $\sigma$ on $X$ to an action $\tilde{\sigma}$ on $L$, such that

\[ G \times L \xrightarrow{\sigma} L \]

\[ \text{id} \times \pi \quad \quad \pi \quad \quad \text{commutes.} \]

\[ G \times X \xrightarrow{\sigma} X \]

ii) The zero-section is $G$-invariant.

Remark. Every line bundle $L$ on $X$ is associated to an invertible sheaf $\mathcal{F}$ on $X$ (see [Har77, exercise 5.18]). So in terms of invertible sheafs the $G$-linearisation $\tilde{\sigma}$ on $L$ is the same as an isomorphism

$\Phi : \sigma^*\mathcal{F} \sim \text{pr}_2^*\mathcal{F}$

satisfying the co-cycle condition (see [MFK65, definition 1.6]).

Example 3.1. Let $L := X \times \mathbb{C}$ be the trivial line bundle. Then a linearisation can be given by choosing a character $\chi \in \text{Hom}(G, \mathbb{C}^*)$. An action $\tilde{\sigma}$ is then given by $\tilde{\sigma}(g, (x, z)) := (\sigma(g, x), \chi(g)z)$ for $g \in G$ and $(x, z) \in X \times \mathbb{C}$. Therefore a $G$-linearisation is not unique.
Remark. Let $V := \Gamma(X, L)$ be the global sections of $L$. Then the linearisation induces an action on $V$:

$$(g.s)(x) = \sigma(g, s(\sigma(g^{-1}, x))) = g.(s(g^{-1}.x))$$

for $s \in V, g \in G$, and $x \in X$.

The next problem which occurs is that a cover of open, affine, and $G$-invariant subsets of $X \hookrightarrow \mathbb{P}^n$ where we can find good categorical quotients need not exist, or we do not get a good categorical quotient by gluing them together.

Example 3.2. Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by

$$t.(z_1 : z_2) = (z_1 : tz_2) \quad \text{with} \quad t \in \mathbb{C}^* \quad \text{and} \quad (z_1 : z_2) \in \mathbb{P}^1.$$

This action has three orbits $\{(1 : t), (1 : 0), (0 : 1)\}$. To be a good categorical quotient a morphism from $\mathbb{P}^1$ to some $Y$ has to take the orbit $(1 : t)$ to one point. As the other two points are in the closure of this orbit, they have to go to the same point. Therefore the only possible quotient is $p : \mathbb{P}^1 \mapsto pt$, which does not satisfy property iii) of Def 2.2. This means that no good categorical quotient exists.

The solution is to take an open $G$-invariant subset which has a good categorical quotient.

Definition 3.2. Let $L$ be a $G$-linearised line bundle on $X$. We define

i) $X^{ss}(L) := \{x \in X \mid \exists d > 0, \exists f \in \Gamma(X, L^{\otimes d})^G : f(x) \neq 0 \wedge X_f \text{ is affine}\}$

the set of semistable points with respect to $L$;

ii) $X^s(L) := \{x \in X \mid \exists f \text{ as in i) : } G.y \subseteq X_f \text{ closed } \forall y \in X_f \wedge G.y \text{ is finite}\}$

the set of stable points with respect to $L$;

iii) $X^{us}(L) := X \setminus X^{ss}(L)$

the set of unstable points with respect to $L$;

where $X_f = \{x \in X \mid f(x) \neq 0\}$.

Remark.

- Taking $L' := L^{\otimes n}$ instead of $L$ does not change the set of semistable points and the set of stable points. If $f \in \Gamma(X, L^{\otimes d})^G$ has an affine $X_f$, then $f' := f^n \in \Gamma(X, L'^{\otimes d})^G$ and $X_{f'} = X_f$.

- If $G$ is irreducible and $X$ quasi-projective and normal, then there exists a $G$-linearised line bundle.

In example 3.2 we could take $O(1)$ with the linearisation $t.x_1 = x_1$ and $t.x_2 = tx_2$. Then $X^{ss}(O(1)) = X_{x_1}$, which means that we take out the point $(0 : 1)$. Now property iii) of definition 2.2 is satisfied and hence $p : X^{ss}(O(1)) \mapsto pt$ is a good categorical quotient.
4 Construction of the GIT quotient

With the definition of the open subset of semistable points we can now construct the so-called GIT (geometric invariant theory) quotient.

**Theorem 4.1** (Mumford). Let $X$ be irreducible and $L$ a $G$-linearised line bundle. There exists a good categorical quotient

$$
\pi : X^{ss}(L) \to X^{ss}(L)/G.
$$

There is an open subset $U \subseteq X^{ss}(L)/G$ such that $\pi^{-1}(U) = X^s(L)$ and $\pi|_{\pi^{-1}(U)}$ is a good geometric quotient. Furthermore $X^{ss}(L)/G$ is a quasi-projective variety.

**Proof.** The idea of the proof is to cover the semistable points with a finite number of open sets $X_s$ with $s_i \in \Gamma(X, L^\otimes d_i)^G$ for $i = 1, ..., r$ and $d_i > 0$ (open subsets of $X$ are quasi-compact). The $X_s$ are affine and we can apply theorem 2.1 to get good categorical quotients

$$
\pi : X_s \mapsto \text{Spec}(\mathcal{O}(X_s)^G), \quad i = 1, ..., r.
$$

These can be glued via open subsets defined by $s_i/s_j$ to get a good categorical quotient.

For the complete proof see [Dol03, p. 118-120] or [MFK65, p. 38-39].

**Proposition 4.1** (Mumford). Let $X$ be projective and $L$ be ample. Let

$$
R = \bigoplus_{d \geq 0} \Gamma(X, L^\otimes d).
$$

Then $X^{ss}(L)/G \cong \text{Proj}(R^G)$.

**Proof.** As in theorem 2.1, by Nagata’s theorem [Dol03, p. 41-45] $R^G$ is finitely generated. Take a set of generators $s_0, ..., s_n$. Without loss of generality $s_i \in \Gamma(X, L)$ otherwise we take $L^\otimes n$ instead of $L$. Then the $X_s$ are an affine cover of $X^{ss}(L)$. As in the proof of theorem 4.1 we can take the $Y_i := \text{Spec}(\mathcal{O}(X_s)^G)$ and glue them together to get the good categorical quotient $Y = X^{ss}(L)/G$. On the other hand take the surjective map

$$
\mathbb{K}[T_0, ..., T_n] \to R^G, \quad T_i \mapsto s_i
$$

with the kernel $I \subseteq \mathbb{K}[T_0, ..., T_n]$. Then $Z := \text{Proj}(R^G) \cong V(I) \subseteq \mathbb{P}^n$ and the open subsets $Z_i := Z \cap D(T_i)$ cover $Z$. From the proof of theorem 2.1 and equation (2) it follows that

$$
\mathcal{O}(Z_i) \cong (\mathbb{K}[T_0, ..., T_n]/I)_{(T_i)} \cong (R^G)_{(s_i)} \cong (R_{(s_i)})^G \cong \mathcal{O}(X_s)^G.
$$

This gives maps $f_i : X_s \to Z_i$ which are categorical quotients as well. By the uniqueness of the good categorical quotient we have that $Z_i \cong Y_i$ for all $i$ and as the $Z_i$ glue in the same way as the $Y_i$ it follows that

$$
\text{Proj}(R^G) = Z \cong Y = X^{ss}(L)/G.
$$
5 Hilbert-Mumford-criterion

Let $X \subseteq \mathbb{P}^n$ via a very ample $G$-linearised line bundle $L$. We denote by $x^* \in \mathbb{P}^{n+1}$ a representative of $x \in X$. Then $x$ is unstable iff $0 \in G.x^*$. A one-parameter subgroup of $G$, $\lambda : \mathbb{G}_m \to G$, acts in appropriate coordinates by the formula

$$\lambda(t).x^* = (t^{m_0}x_0, ..., t^{m_n}x_n).$$

If all $m_i$ for which $x_i \neq 0$ are strictly positive then $0 \in G.x^*$ and if all such $m_i$ are strictly negative, take $\lambda^{-1}$ to see that $0 \in G.x^*$. Define $\mu(x, \lambda) := \min(m_i \mid x_i \neq 0)$. Thus we get

$$x \in X^{ss}(L) \Rightarrow \mu(x, \lambda) \leq 0 \quad \forall \lambda \in \chi(G)^*.$$

If $x \in X^{ss}(L)$ and $\exists \lambda : \mu(x, \lambda) = 0$ set $I := \{i \mid x_i \neq 0, m_i > 0\}$ and define $y = (y_0, ..., y_n)$ by $y_i = \begin{cases} x_i & i \notin I \\ 0 & i \in I \end{cases}$. Then $y \in \lambda(G_m).x \subseteq G.x$ and $\lambda(G_m) \subseteq G_y$. Thus either $y \notin G.x$ and hence $G.x$ is not closed and therefore $x$ is not stable. Or $y \in G.x$ and thus in every $X_f$ that contains $x$ and has a non-finite stabiliser and hence $x$ is again not stable. Therefore we get

$$x \in X^{s}(L) \Rightarrow \mu(x, \lambda) < 0 \quad \forall \lambda \in \chi(G)^*.$$

Remark. One can show that $\mu(x, \lambda)$ is independent of the choice of coordinates.

Theorem 5.1. Let $G$ be a reductive group acting on a projective variety $X$. Let $L$ be an ample $G$-linearised line bundle on $X$ and let $x \in X$. Then

$$x \in X^{ss}(L) \iff \mu(x, \lambda) \leq 0 \quad \text{for all } \lambda \in \chi(G)^*,$$

$$x \in X^{s}(L) \iff \mu(x, \lambda) < 0 \quad \text{for all } \lambda \in \chi(G)^*.$$

Proof. See [Dol03, 9.1 - 9.3]. \qed

References


