Hints to the exercises can as a rule easily be supplemented to a complete solution. Exercises that are essential for the text are solved completely. The reader may mail improved solutions to the author whose website is www.math.fu-berlin.de/~raut.
Section 1.1

1. (a): \( x_k \) is fictional in \( f \) iff \( a_k = 0 \). (b): Because of the uniqueness, 
\( 2^{n+1} (= \text{number of subsets of} \{0, \ldots, n\} \) is the number sought for. (c): induction on formulas in \( \rightarrow, + \) and \( p_1, \ldots, p_n \).

2. Consider on \( \mathcal{F} \) the property \( \mathcal{E} \varphi \): ‘\( \varphi \) is prime or there are \( \alpha, \beta \in \mathcal{F} \) with \( \varphi = \neg \alpha \) or \( \varphi = (\alpha \circ \beta) \) where \( \circ = \wedge \) or \( \circ = \vee \).’ Formula induction shows \( \mathcal{E} \varphi \) for all \( \varphi \in \mathcal{F} \).

3. Verify by induction on \( \varphi \) the property \( \mathcal{E} \varphi \): ‘no proper initial segment of \( \varphi \) is a formula nor can \( \varphi \) be a proper initial segment of a formula’.
Induction step: Case \( \varphi = \neg \alpha \). Then a proper initial segment of \( \neg \alpha \) either equals \( \neg \) (hence is not a formula), or has the form \( \neg \xi \) where \( \xi \) is a proper initial segment of \( \alpha \). Thus \( \xi \notin \mathcal{F} \) by the induction hypotheses, hence also \( \neg \xi \notin \mathcal{F} \) (since a formula starting with \( \neg \) must have the form \( \neg \beta \) for some formula \( \beta \) by Exercise 2). Case \( \varphi = (\alpha \circ \beta) \). Let \( \xi \) be a proper initial segment of \( \varphi \) or conversely. Assume that \( \xi \) is a formula so that \( \xi = (\alpha' \circ' \beta') \), some \( \alpha', \beta' \in \mathcal{F} \) (Exercise 2). Then \( \alpha \neq \alpha' \), for otherwise necessarily \( \xi = \varphi \). Hence \( \alpha' \) is a proper initial segment of \( \alpha \) or conversely, a contradiction to the induction hypothesis \( \mathcal{E} \alpha \).

4. Assume that \( (\alpha \circ \beta) = (\alpha' \circ' \beta') \), hence \( \alpha \circ \beta = \alpha' \circ' \beta' \). If \( \alpha \neq \alpha' \) then \( \alpha \) is a proper initial segment of \( \alpha' \) or conversely. This is impossible by Exercise 3. Consequently \( \alpha = \alpha' \), hence \( \circ = \circ' \) and \( \beta = \beta' \).

Section 1.2

1. \( w((p \rightarrow q_1) \wedge (\neg p \rightarrow q_2)) = 0 \) iff \( wp = 1, wq_1 = 0 \) or \( wp = 0, wq_2 = 0 \), and the same condition holds for \( w(p \wedge q_1 \vee \neg p \wedge q_2) = 0 \). In a similar way the second equivalence is treated.

2. \( \neg p \equiv p + 1, 1 \equiv p + \neg p, p \leftrightarrow q \equiv p + \neg q, p + q \equiv p \leftrightarrow \neg q \equiv \neg (p \leftrightarrow q) \).

3. Induction on the \( \alpha \in \mathcal{F}_n \{0, 1, \wedge, \vee \} \) (= set of formulas in \( 0, 1, \wedge, \vee \) and \( p_1, \ldots, p_n \)). If \( f, g \in B_n \) are monotonic then so is \( \vec{a} \mapsto f \vec{a} \circ g \vec{a} \), where \( \circ \) is \( \wedge \) or \( \vee \). For simplicity, treat first the case \( n = 1 \). Converse: Induction on the arity \( n \). Clear for \( n = 0 \), with the formulas 0 and 1 representing the two constants. With \( f \in B_{n+1} \) also \( f_k : \vec{x} \mapsto f(\vec{x}, k) \) is monotonic \( (k = 0, 1) \). Let \( \alpha_k \in \mathcal{F}_n \{0, 1, \wedge, \vee \} \) represent \( f_k \) (induction hypothesis). Then \( \alpha_0 \vee (\alpha_1 \wedge p_{n+1}) \) represents \( f \). Note that \( w \alpha_0 \preceq w \alpha_1 \) for all \( w \).
4. By Exercise 3, a not representable \( f \in B_{n+1} \) is not monotonic in the last argument, say. Then \( f(\vec{a},1) = 0 \) and \( f(\vec{a},0) = 1 \) for some \( \vec{a} \in \{0,1\}^n \), hence \( g: x \mapsto f(\vec{a},x) \) is negation. This proves the claim.

**Section 1.3**

1. (a): MP easily yields \( p \rightarrow q \rightarrow r, p \rightarrow q, p \models r \). Apply (D) three times.

2. The deduction theorem yields \( \models (\alpha \rightarrow \beta) \rightarrow (\gamma \rightarrow \beta) \).

3. Assume that \( w \models X, \alpha \lor \beta \). Then clearly \( w \models X, \alpha \lor \beta \) or \( w \models X, \beta \).

5. Let \( X \vdash \alpha \notin X \). Then \( X, \alpha \models \beta \) for each \( \beta \). Thus, \( X \vdash \beta \) by (T).

**Section 1.4**

1. \( X \cup \{ \neg \alpha \mid \alpha \in Y \} \vdash \bot \Rightarrow X \cup \{ \neg \alpha_0, \ldots, \neg \alpha_n \} \vdash \bot \), some \( \alpha_0, \ldots, \alpha_n \in Y \). Hence \( X \vdash (\bigwedge_{i \leq n} \neg \alpha_i) \rightarrow \bot \), or equivalently, \( X \vdash \bigvee_{i \leq n} \alpha_i \). This all is easily verified if \( \vdash \) is replaced by \( \models \).

2. Supplement Lemma 4.4 by proving \( X \vdash \alpha \lor \beta \iff X \vdash \alpha \lor X \vdash \beta \).

3. Choose \( X, \varphi \) such that \( X \not\models \varphi \) and \( X \vdash' \varphi \). Let \( Y \supseteq X \cup \{ \neg \varphi \} \) be maximally consistent in \( \vdash \). Define \( \sigma \) by \( p^\sigma = \top \) for \( p \in Y \) and \( p^\sigma = \bot \) otherwise. Induction on \( \alpha \) yields the aid of \( (\land) \) and \( (\neg) \) page 28

\[
(\ast) \quad \alpha \in Y \Rightarrow \vdash \alpha^\sigma \quad ; \quad \alpha \notin Y \Rightarrow \vdash \neg \alpha^\sigma.
\]

In proving \((\ast)\), \( \vdash \top, \vdash \alpha \Rightarrow \vdash \neg \alpha, \neg \alpha \vdash \neg (\alpha \land \beta), \) and \( \neg \beta \vdash \neg (\alpha \land \beta) \) are needed which easily follow from the \( \neg \neg \) rules. By \((\ast)\), \( \vdash \neg \varphi^\sigma \), hence \( \vdash' \neg \varphi^\sigma \). Clearly \( \vdash Y^\sigma \) (i.e., \( \vdash \alpha^\sigma \) for all \( \alpha \in Y \)), and so \( \vdash Y^\sigma \). But \( Y^\sigma \vdash Y^\sigma \) (substitution invariance). Thus, \( \vdash' \varphi^\sigma \). Therefore \( \vdash' \alpha \) for all \( \alpha \) by \((\neg)\), so that \( \vdash \) is maximal by definition.

4. There is a smallest consequence relation with the properties \((\land)1)-(\neg)2\), namely the calculus \( \vdash \) of this section. Since \( \vdash \subseteq \vdash \) and \( \vdash \) is already maximal according to Exercise 3, \( \vdash \) must coincide.

**Section 1.5**

1. For finite \( M \) easily shown by induction on the number of elements of \( M \). Note that \( M \) has a maximal element. General case: Add to the formulas in Example 1 the set of formulas \( \{ p_{ab} \mid a \leq_0 b \} \).
2. \( \Rightarrow \): Assume \( M, N \notin F \). Then \( (M \cup N) = \setminus M \cap \setminus N \in F \), because \( \neg M, \neg N \in F \). Therefore \( M \cup N \notin F \). \( \Leftarrow \): \( M \in F \) implies \( M \cup N \in F \) by condition (b). For proving \( (\neg) \) from \( (\cap) \) observe that \( M \cup \setminus M \in F \).

3. \( \Rightarrow \): Let \( U \) be trivial, i.e., \( E \in U \) for some finite \( E \subseteq I \). Induction on the number of elements in \( E \) and Exercise 2 easily show that \( \{i_0\} \in U \) for some \( i_0 \in E \). The converse is obvious.

Section 1.6

1. First verify the deduction theorem, which holds for each calculus with MP as the only rule and A1, A2 among the axioms; cf. Lemma 6.3. X is consistent iff \( X \not\vdash \bot \), for \( X \vdash \bot \Rightarrow X \vdash (\alpha \rightarrow \bot) \rightarrow \bot = \neg \neg \alpha \) by A1, hence \( X \vdash \alpha \) by A3. Now prove \( X \vdash \alpha \rightarrow \beta \iff X \vdash \alpha \Rightarrow X \vdash \beta \), provided \( X \) is maximally consistent. This allows one to proceed along the lines of Lemma 4.5 and Theorem 4.6.

2. Apply Zorn’s lemma to \( H := \{Y \supseteq X \mid Y \not\vDash \alpha\} \). Note that if \( K \subseteq H \) is a chain then \( \bigcup K \in H \) due to the finitarity of \( \vdash \).

3. (a): Such a set \( X \) satisfies \( (\ast) \): \( X \vdash \varphi \rightarrow \alpha \) for all \( \alpha \). For otherwise \( X, \varphi \rightarrow \alpha \vdash \varphi \), hence \( X \vdash (\varphi \rightarrow \alpha) \rightarrow \varphi \), and so \( X \vdash \varphi \) by Peirce’s axiom. Suppose \( \alpha \notin X \). Then \( X, \alpha \vdash \varphi, \varphi \rightarrow \beta \) by \( (\ast) \), and so \( X, \alpha \vdash \beta \).

(b): With (a) easily follows \( X \vdash \alpha \rightarrow \beta \iff X \vdash \alpha \Rightarrow X \vdash \beta \) as in Exercise 1. Proceed with an adaptation of Lemma 4.5.

4. Crucial for completeness is the proof of \( (m) \): \( \alpha \vdash \beta \Rightarrow \alpha \gamma \vdash \beta \gamma \) by induction on the rules of \( \vdash \). \( (m) \) implies \( (M) \): \( X, \alpha \vdash \beta \Rightarrow X, \alpha \gamma \vdash \beta \gamma \), proving first that a calculus \( \vdash \) based solely on unary rules satisfies \( X \vdash \beta \Rightarrow \alpha \vdash \beta \) for some \( \alpha \in X \). E.g., \( \alpha \vdash \alpha \beta \Rightarrow \alpha \gamma \vdash \gamma \alpha \beta \vdash \alpha \beta \gamma \). Although \( \alpha (\beta \gamma) \vdash (\alpha \beta) \gamma \) and conversely, it is still tricky to show that \( \alpha (\beta \gamma) \delta \vdash (\alpha \beta) \gamma \delta \). \( (M) \) implies \( X, \alpha \vdash \gamma \& X, \beta \vdash \gamma \Rightarrow X, \alpha \beta \vdash \gamma \), because \( X, \alpha \vdash \gamma \Rightarrow X, \alpha \beta \vdash \gamma \beta \vdash \beta \gamma \) and \( X, \beta \gamma \vdash \gamma \gamma \vdash \gamma \), therefore \( X, \alpha \beta \vdash \gamma \). From this it follows \( [\vee] \): \( X \vdash \alpha \beta \iff X \vdash \alpha \) or \( X \vdash \beta \), provided \( X \) is \( \varphi \)-maximal, for note that

\[
X \not\vDash \alpha \& X \not\vDash \beta \Rightarrow X, \alpha \vdash \varphi \& X, \beta \vdash \varphi \Rightarrow X, \alpha \beta \vdash \varphi \Rightarrow X \not\vDash \alpha \beta.
\]

Having \( [\vee] \) one may proceed with a slight modification of Lemma 4.5.
Section 2.1
1. There are 10 essentially binary Boolean functions $f$. The corresponding algebras $\langle \{0, 1\}, f \rangle$ split into 5 pairs of isomorphic ones. For example, $\langle \{0, 1\}, \wedge \rangle \cong \langle \{1, 0\}, \lor \rangle$.

2. Choose $c = a$ in $a \approx b \land a \approx c \Rightarrow b \approx c$ to get $a \approx b \Rightarrow b \approx a$.

3. For simplicity, treat first the case $n = 2$ using transitivity.

5. For simplicity, let the signature contain only the symbols $r, f$, both unary. Then $ra \Rightarrow ra \jmath \Rightarrow rha$ and $hfa = h(fa)_{i\in I} = fa = fha$.

Section 2.2
1. Trivial if $t$ is a prime term. A terminal segment of $f\vec{t}$ either equals $f\vec{t}$ or has the form $t'_{k+1}\cdots t_n$ for some $k \leq n$ ($t'_k t_{k+1} \cdots t_n$ means $t'_n$ in case $k = n$), where $t'_k$ a terminal segment of $t_k$. By the induction hypotheses, $t'_k$ is a term concatenation, hence so is $t'_k t_{k+1} \cdots t_n$.

2. It suffices to prove (a') $t\xi = t'\xi' \Rightarrow t = t'$, for all $t, t' \in T$, all $\xi, \xi' \in S_L$ by induction on $t$. This is obvious for prime $t$. Let $t = ft_1\cdots t_n$ and $t\xi = t'\xi'$ with $t' = f't'_1\cdots t'_m$. Then clearly $f = f'$ and $m = n$, hence $t_1\cdots t_n\xi = t'_1\cdots t'_m\xi'$. Thus $t_1 = t'_1$ and $t_2\cdots t_n\xi = t'_2\cdots t'_m\xi'$ by the induction hypothesis for $t_1$. Similarly, $t_2 = t'_2\ldots, t_n = t'_n$ and also $\xi = \xi'$. This proves (a').

3. (a): Similar to Exercise 3 in 1.1. (b) follows readily from (a). (c): If $\neg\xi \in L$ then by (b), $\neg\xi = \neg\alpha$ for some $\alpha \in L$. Hence $\xi = \alpha$. Similarly, $\alpha, \alpha \land \xi \in L \Rightarrow \alpha \land \xi = \beta \land \gamma$, some $\beta, \gamma \in L$, hence $\alpha = \beta$ and $\xi = \gamma$.

5. Can completely be reduced to Corollary 1.2.2 by some bijection from $X$ onto a set $V$ of propositional variables.

Section 2.3
1. If $\mathcal{M} \vdash X$ and $x \notin \text{free } X$ then $\mathcal{M}_a \vdash X$ for each $a$ (Theorem 2.3.1).

2. $\forall x (\alpha \rightarrow \beta), \forall x \alpha \vdash \alpha \rightarrow \beta, \alpha \vdash \beta$ and Exercise 1.

3. The Theorems 3.1 and 3.5 yield $\mathcal{A} \vdash \alpha [a] \iff \mathcal{A'} \vdash \alpha [a] \iff \mathcal{A} \vdash \alpha_1 (a)$.

4. (a): $\exists_n \land \exists_m \equiv \exists_m$ for $n \leq m$, and $\exists_n \land \neg \exists_m \equiv \exists_0 (\equiv \bot)$ for $n > m$.
   (b): Exercise 5 in 2.2, and $\exists_n \land \neg \exists_m \equiv \bigvee_{n \leq k < m} \exists_k$ for $n < m$. 
Section 2.4

1. \( \alpha \equiv \beta \Rightarrow \exists \forall \vec{x} (\alpha \leftrightarrow \beta) \Rightarrow \exists (\alpha \leftrightarrow \beta)^\vec{x} \left( = \alpha^\vec{x} \leftrightarrow \beta^\vec{x} \right) \).

3. W.l.o.g. \( \alpha \equiv \forall \vec{y} \alpha'(\vec{x}, \vec{y}) \) and \( \beta \equiv \forall \vec{z} \beta'(\vec{x}, \vec{z}) \) with disjoint tuples \( \vec{x}, \vec{y}, \vec{z} \).

4. Simultaneous induction on \( \phi \) and \( \neg \phi \). Clear if \( \phi \) is prime. If the claim holds for \( \alpha, \beta \) then also for \( (\alpha \land \beta) \) and \( \neg (\alpha \land \beta) \) (\( \equiv \neg \alpha \lor \neg \beta \)). The step for \( \lor \) is similar. Step for \( \neg \): Simply observe that \( \neg \neg \alpha \equiv \alpha \).

5. \( \exists x (Px \rightarrow \forall y Py) \equiv \forall x Px \rightarrow \forall y Py \) according to (10) in 2.4.

Section 2.5

1. Proof very similar to that of Exercise 6 in 2.4

2. \( \Rightarrow \): \( S \models \alpha^\vec{y} \rightarrow \beta \Rightarrow S, \alpha \models \beta \) and (e) page 79. \( \Leftarrow \): (9) in 2.4.

3. \( \beta \in T + \alpha \iff T, \alpha \models \beta \iff T \models \alpha \rightarrow \beta \) by the deduction theorem.

Section 2.6

1. The "if" part follows as Theorem 6.1 because \( y \models f^\vec{t} \equiv \exists T_f \delta_f(\vec{t}, y) \). The "only if" part: \( y \models f^\vec{t} \equiv T_f \delta_f(\vec{t}, y) \) and \( T_f \models \exists \vec{x} \exists! y y = f^\vec{x} \). Hence also \( T_f \models \forall \vec{x} \exists! y \delta(\vec{x}, y) \).

2. \( N \models x = 0 \iff \forall y x \neq Sy \). Careful calculation confirms the definition \( x + y = z \iff x = y = z = 0 \lor z \neq 0 \land S(x \cdot z) \cdot S(y \cdot z) = S(z^2 \cdot S(x \cdot y)) \). Therein \( z^2 \) denotes the term \( z \cdot z \).

3. Let \( xy = xz = e \) (\( \circ \) not written). Choose some \( y' \) with \( yy' = e \). Then \( yx = (yx)(yy') = y(xy)y' = ye = e \) and so \( ex = (yx)x = x(yx) = xe = x \) for all \( x \). In other words, \( e \) is a left and right unit element. We hence obtain \( y = ye = y(xz) = (yx)z = ez = z \). For the additional claim derive the axioms of \( T^e_G \) from those of \( T_G \) and conversely.

4. If \( < \) were definable then \( < \) would be invariant under automorphisms of \( (\mathbb{Z}, 0, +) \). This is not the case for the automorphism \( n \mapsto -n \). This approach to the problem is also called Padoa’s method.
Section 3.1
1. Let $X \vdash \alpha \frac{4}{x}$. Then $X, \forall x \neg \alpha \vdash \alpha \frac{4}{x}, \neg \alpha \frac{4}{x}$. Hence $X, \forall x \neg \alpha \vdash \exists x \alpha$. Also $X, \neg \forall x \neg \alpha \vdash \exists x \alpha (= \neg \forall x \neg \alpha)$. Thus $X \vdash \exists x \alpha$ according to (−2).

2. Let $\alpha' := \alpha \frac{y}{y}, u \not\approx \text{var} \alpha, u \neq y$. Then $\forall x \alpha \vdash \alpha' \frac{y}{y} (= \alpha \frac{u}{y})$ by (forall).

Hence $\forall x \alpha \vdash \forall y \alpha'$ by (forall), with $X = \{\forall x \alpha\}, \alpha'$ for $\alpha$, and $y$ for $x$.

3. $\forall y (\alpha \frac{y}{y}) \vdash \forall x \alpha \vdash \forall z (\alpha \frac{z}{z})$ according to Exercise 2.

4. $\Rightarrow$: $X \not\models \varphi \Rightarrow X, \varphi \vdash \bot \Rightarrow X \vdash \neg \varphi$. $\Leftarrow$: $X \not\models \alpha \Rightarrow X \vdash \neg \alpha \Rightarrow X, \alpha \vdash \bot$.

Section 3.2
1. First prove ($*$): $\exists \varphi$ iff $\exists \varphi \in T^n_0$ (\varphi \in \mathcal{L} open); use Theorem 2.3.5. Next prove ($^\ast$) $X \vdash \alpha \Leftrightarrow \exists \varphi \in \mathcal{L}$ (open) by induction on $\times, \neg$; observe that $\mathcal{L}$ is $\vdash$-free. Let $X \vdash \forall \varphi$ (\varphi open) and $\bar{t} \in T^n_0$. Then also $X \vdash \alpha := \varphi \frac{\bar{t}}{\bar{t}}$, hence $\exists \varphi \models \alpha$ by ($^\ast$). Thus, $\exists \varphi \models \forall \varphi$ by ($*$), and so $\exists \varphi \models U$.

2. $K \vdash \alpha \Rightarrow T \vdash \alpha$ for some $T \in K$ (finiteness theorem)

4. (i)$\Rightarrow$(ii): (12) in 2.4. Observe also $(x = t \rightarrow \alpha) \frac{t}{t} \equiv \alpha \frac{t}{t}$.

Section 3.3
1. Prove $\vdash_{PA} \forall z (x + y) + z = x + (y + z)$ by induction on $z$. Obvious for $z = 0$. The induction step follows easily from $\vdash_{PA} x + Sy = S(x + y)$.

Most proofs of the arithmetical laws in PA need much patience.

2. $x + x = x \rightarrow z = 0$ (induction on $x$) readily yields $x \leq y \leq x \rightarrow x = y$.

3. Informally: $x < y \Rightarrow \exists z Sx + x = y \Rightarrow \exists z + Sx = y \Rightarrow Sx \leq y$.

The converse $Sx \leq y \rightarrow x < y$ follows from $\vdash_{PA} x < Sx$. The induction hypothesis of $x \leq y \lor y \leq x$ may be written as $x < y \lor y \leq x$. If $x < y$ then $Sx \leq y$, hence $x \leq y \lor y \leq Sx$ (induction claim). We get the same in the case $y \leq x$, since then $y \leq Sx$ ($\leq$ is transitive).

4. (a): Put $\varphi := (\forall y < x) \alpha \frac{y}{y}$. It suffices to prove (i) $\forall x (\varphi \rightarrow \alpha) \vdash_{PA} \varphi \frac{0}{x}$ (which is trivial) and (ii) $\forall x (\varphi \rightarrow \alpha) \vdash_{PA} \varphi \rightarrow \varphi \frac{x}{x}$ since by IS then $\forall x (\varphi \rightarrow \alpha) \vdash_{PA} \forall x \varphi \vdash_{PA} \forall x \alpha$. Now, $\varphi, \varphi \rightarrow \alpha \vdash_{PA} \varphi \rightarrow \varphi \frac{x}{x}$, hence $\forall x (\varphi \rightarrow \alpha) \vdash_{PA} \varphi \rightarrow \varphi \frac{x}{x}$ which confirms (ii). (b): Follows from
(a) by contraposition. (c): For $\phi := (\forall x<v)\exists y\gamma \rightarrow \exists z(\forall x<v)(\exists y<z)\gamma$ holds $\vdash_{PA} \phi \not\in \mathbb{P}$, and $\phi \vdash_{PA} \phi \not\in \mathbb{P}$. This yields the claim by IS.

Section 3.4
1. $T \cup \{v_i \neq v_j \mid i \neq j\}$ is satisfiable because each finite subset is.
2. $ThA \cup \{v_{n+1} < v_n \mid n \in \mathbb{N}\}$ has a model with a descending $\omega$-chain.
3. If $\alpha \not\in T$ then $T$ has a completion $T'$ with $-\alpha \in T'$, hence $\alpha \not\in T'$.
4. Consider the identical operator on the universe $V$ and restrict it to a given set $u$ in $AS$.
5. Informally: Suppose $\text{fin}(a)$, $\varphi_x(\emptyset)$, and $\forall u \forall e (\varphi_x(u) \rightarrow \varphi_x(u \cup \{e\}))$. Then holds also $\emptyset \in s \land (\forall u \in s)(a \setminus u \neq \emptyset \rightarrow (\exists e \in a \setminus u) u \cup \{e\} \in s)$ for the set $s := \{u \in \Psi a \mid \varphi_x(u)\}$. Hence $a \in s$, i.e. $\varphi_x(a)$.

Section 3.5
2. Let $T + \{\alpha_i \mid i \in \mathbb{N}\}$ be an infinite extension of $T$. We may assume $\bigwedge_{i \leq n} \alpha_i \not\in T \alpha_{n+1}$. Hence, $T + \bigwedge_{i \leq n} \alpha_i \land \neg \alpha_{n+1}$ is consistent. Let $T_n$ be a completion of $T + \bigwedge_{i \leq n} \alpha_i \land \neg \alpha_{n+1}$. Then $T_n \neq T_m$. Thus, a theory with finitely many completions cannot have an infinite extension and, in particular, no infinite completion.
3. Let $T_0, \ldots, T_n$ be the completions of $T$. According to Exercise 3 in 3.4, $\alpha \in T$ iff $\alpha \in T_i$ for all $i \leq n$. Thus, $T$ is decidable provided each $T_i$ is, and this follows from Theorem 5.2, for each $T_i$ is a finite extension of $T$ according to Exercise 2, hence is axiomatizable as well.
4. Starting with a effective enumeration $(\alpha_n)_{n \in \mathbb{N}}$ of $L^0$, a Lindenbaum completion of $T$ as constructed in 1.4 is effectively enumerable.
5. According to Exercise 3 in 3.4, there is a bijection between the set of consistent extensions of $T$ (including $T$) and the set of nonempty subsets of the collection $\{T_1, \ldots, T_n\}$ of all completions of $T$.
Section 3.6

1. \( x = y \not\equiv \forall x x = y \). The same holds for \( \vdash \), since \( \vdash \subseteq \models \).

2. (a): Let \((\varphi_n)_{n\in\mathbb{N}}\) and \((A_n)_{n\in\mathbb{N}}\) be effective enumerations of all sentences and of all finite \( T \)-models (up to isomorphism). In step \( n \) write down all \( \varphi_i \) for \( i \leq n \) with \( A_n \not\models \varphi_i \). (b): Let \((\alpha_n)_{n\in\mathbb{N}}\) and \((\beta_n)_{n\in\mathbb{N}}\) be effective enumerations of sentences provable or refutable in \( T \), respectively. Each \( \alpha \in \mathcal{L}^\alpha \) occurs in one of these sequences. In the first case is \( \alpha \in T \).

3. Condition (ii) from Exercise 2 is then granted because the validity of only finitely many axioms has to be tested in a finite structure.

Section 3.7

1. For \( H \): Let \( \mathcal{B} = hA \) be a homomorphic image of \( A \), \( w: \text{Var} \to A \), and define \( hw: \text{Var} \to \mathcal{B} \) by \( x^{hw} := h x^w \). Then \( ht^{A, w} = t^{\mathcal{B}, hw} \) for all terms \( t \) and there is some \( w: \text{Var} \to A \) with \( hw = w' \) for any given \( w': \text{Var} \to \mathcal{B} \). For \( S \): (3) in 2.3 page 66. For \( P \): Set \( \mathcal{B} = \prod_{i \in I} A_i \). Then \( t^{\mathcal{B}, w} = (t^{A_i, w_i})_{i \in I} \) with \( x^w = (x^{w_i})_{i \in I} \).

Section 3.8

1. (a): Let \( \alpha_{\text{unc}} \) in \( \mathcal{L}_{II} \) formalize the sentence ‘there is a continuous order’. \( \alpha_{\text{unc}} \) has no countable model. In \( \mathcal{L}_{\mathbb{Q}}^1 \) one may take \( \forall x x = x \) for \( \alpha_{\text{unc}} \). (b): \( X = \{ i \neq j \mid i, j \in I, i \neq j \} \cup \{ \neg \forall x x = x \} \) has no model if \( I \) is uncountable, although each finite subset of \( X \) has a model.

2. Define \( \mathbb{R} \) as a continuously ordered set with a countable dense subset.

4. Let \( x \) be a variable not in \( P, \mathbb{Q} \). A possible definition is provided by

\[ x := 0; \text{ WHILE } \alpha \vee x = 0 \text{ DO } P; \text{ } x := S0 \text{ END.} \]

Section 4.1

1. Note that \( \bar{t} \) in case \( k = 0 \) is defined for ground terms only.

2. The most important case is \( k = 0 \). It deals with ground terms only.
Section 4.2
1. First prove (a) \((\forall i \in I) A_i \models \pi [w_i] \iff B \models \pi [w] (x^w = (x^{w_i})_{i \in I})\), \(\pi\) prime and \(B = \prod_{i \in I} A_i\). Then prove (b) \((\forall i \in I) A_i \models \alpha [w_i] \Rightarrow B \models \alpha [w]\) by induction over basic Horn formulas \(\alpha\) as in Theorem 2.1. (b) yields the induction steps over \(\land, \lor, \exists\). Observe \(t^{B, w} = (t^{A_i, w_i})_{i \in I}\). For a universal Horn theory apply (ii) \(\Rightarrow\) (i) of Theorem 2.3.2.

2. A set of positive Horn formulas has the trivial (one-element) model.

Section 4.4
1. With \(w \models p_1, p_3, \neg p_2\) and \(w \models p_2, p_3, \neg p_1\) we have \(w_1, w_2 \models \mathcal{P}\). Since \(w \models \mathcal{P}\) implies \(w \models p_3\) and either \(w \models p_1\) or \(w \models p_2\), there is no valuation \(w \leq w_1, w_2\) such that \(w \models \mathcal{P}\).

2. For arbitrary \(w \models \mathcal{P}\), \(w \models p_{m,n,m+n}\) follows inductively on \(n\). Hence \(w_s \leq w_r\), and consequently \(w_s = w_r\).

3. (a): Theorem 4.2. (b): \(w_r \not\models p_{n,m,k}\) if \(k \neq n + m\), so \(\mathcal{P}, \neg p_{n,m,k} \not\models_{HR} \). 

Section 4.5
2. \(\Rightarrow\): \(x_i \in \text{var} t_j \Rightarrow x^\sigma_j = t_j \neq t^\sigma_j = x^{\sigma^2}_j\), hence \(\sigma \neq \sigma^2\). \(\Leftarrow\): \(t^\sigma_i = t_i\) since necessarily \(x^\sigma = x\) for all \(x \in \text{var} t_i\).

3. Let \(\omega\) be a unifier of \(K_0 \cup K_1\). Then \(K^\omega_0 = K^\omega_1\) is a singleton. Put \(x^\omega = x^{\rho \omega}\) for \(x \in \text{var} K^\rho_0\) and \(x^\omega = x^\omega\) else. Then \(K^{\rho \omega}_0 = K^{\rho \omega}_1 = K^\omega\) since \(\rho^2 = \iota\), and \(K^\omega_0 = K^\omega_1\). Thus, \(K^\rho_0 \cup K_1\) is unified by \(\omega\). The converse need not hold. Let \(r_2\) be a binary relation symbol, \(f\) a unary operation symbol, and \(0\) a constant. \(K_0 = \{r_2 f v f x\}\) and \(K_1 = \{r_2 f 0 v\}\) are not unifiable, but \(K^\rho_0\) and \(K_1\) are, with \(\rho = (\iota^0)\). Indeed, for \(\omega = 0 \frac{v}{f} \frac{f}{x}\) we get \(K^{\rho \omega}_0 = \{r_2 f u f x\}^\omega = \{r_2 f 0 f x\} = K^\omega_0\).

Section 4.6
1. Join \(\mathcal{P}_g\) and \(\mathcal{P}_h\) and add to the resulting program the rules
\[ r_f(\bar{x}, 0, u) \leftarrow r_g(\bar{x}, u) \text{ and } r_f(\bar{x}, Sy, u) \leftarrow r_f(\bar{x}, y, v), r_h(\bar{x}, y, v, u). \]

2. Add to the programs the rule \(r_f \bar{x} u \leftarrow r_{g_1} \bar{x} y_1, \ldots, r_{g_m} \bar{x} y_m, r_h \bar{y} u\).
Section 5.1
1. Let $\alpha = \alpha(x), \overline{a} \in A^n$, and $A \vDash \alpha(\overline{a})$. Then $C \vDash \alpha(\overline{a})$ as well, and since $B \preceq C$, also $B \vDash \alpha(\overline{a})$.

3. Prove first the following simple lemma: Let $0 < b < c < 1$. Then there is a strictly monotonic bijection $f : [0, 1] \rightarrow [0, 1]$ (an automorphism of the closed interval $[0,1]$) such that $fb = c$. W.l.o.g. $a_1 < \cdots < a_n$, $n \geq 2$, and $b \in [a_1, a_n]$ irrational. Let $a_k < b < a_{k+1}$. W.l.o.g. we may assume $a_k = 0$ and $a_{k+1} = 1$. Choose some $c \in \mathbb{Q}$ with $b < c < 1$ and an automorphism $f : [0, 1] \rightarrow [0, 1]$ with $fb = c$ according to the above lemma. $f$ can be extended in a trivial way to an automorphism of the whole of $(\mathbb{R}, <)$ by setting $fx = x$ outside $[0, 1]$.

4. W.l.o.g. $A \cap B = \emptyset$. It suffices to show that $D_{el}A \cup D_{el}B$ is consistent. Assume the contrary. Then there is some conjunction $\gamma(\overline{b})$ of members of $D_{el}B$ and some $\overline{b} \in B^n$ such that $D_{el}A, \gamma(\overline{b}) \vDash \bot$. Thus, $D_{el}A \vDash \neg \gamma(\overline{b})$. Since $A \cap B = \emptyset$, the $b_1, \ldots, b_n$ do not occur in $A$, hence constant quantification yields $D_{el}A \vDash \forall \overline{x} \neg \gamma$ and so $A \vDash \forall \overline{x} \neg \gamma$. But clearly $B \vDash \exists \overline{x} \gamma$, a contradiction to $A \equiv B$.

5. (a): $\{t^A \mid t \in T_G\}$ is closed with respect to all $f^A$. It is the smallest such set and hence exhausts $A$. (b): By (a), we may choose to each $a \in A \setminus G$ some $t_a \in T_G$ such that $D_A \vdash a \equiv t_a$. Thus, $T + D_A$ can be regarded as a definitorial and hence conservative extension of $T + D_GA$, so that $D_A \vdash_{T} \alpha \iff D_{A^E} \vdash_{T} \alpha$ for all sentences $\alpha \in \mathcal{L}G$.

Section 5.2
2. $T_{suc} \vdash \text{IS}$ because $(\mathbb{N}, 0, S) \vDash \text{IS}$ and $T_{suc}$ is complete. To prove the “no circle” scheme which is equivalent to $(\ast) \forall x S^n\neg x \neq x$ ($n \geq 1$), we start from $(\#) S^{n+1}x = S^n(S^nx)$ for every $n$. $(\#)$ is easily verified by meta-induction on $n$, while the induction schema IS is needed in order to prove $(\ast)$ by induction on $x$. Clearly, $S^00 \neq 0$ by the axiom $\forall x 0 \neq Sx$. From the induction hypothesis $S^n\neg x \neq x$ we get the induction claim $S^n(S^nx) = S(S^n\neg x) \neq Sx$ by applying $(\#)$ and the second axiom of $T_{suc}$.

3. Let $\alpha \in G \vDash T$ and $\overline{a}$ the element with $n \overline{a} = a$, and $\overline{m} : a \mapsto m \overline{a}$ for $m \overline{a} \in \mathbb{Q}$. Then $G$ becomes the vector group of a $\mathbb{Q}$-vector space. This group is easily shown to be $\aleph_1$-categorical.
4. Each consistent $T' \supseteq T$ is the intersection of its completions in $\mathcal{L}$.

5. Each $\mathcal{A} \models T$ has a countable elementary substructure (Theorem 1.5).

Section 5.3

1. For $SO_{00}$: In the first round player II may play arbitrarily, then according to the winning strategies for models of $SO_{01}$ or $SO_{10}$ in the decomposed segments.

2. If player I starts with $a \in A$ and to the right and the left of $a$ remain at least $2^{k-1}$ elements, player II should choose correspondingly. Otherwise he should answer with the elements of the same distance from the left or right edge element, respectively.

3. $SO_{11} \subseteq FO$ is obvious. $FO \subseteq SO_{11}$: If $\mathcal{A} \models SO_{11}$ then for each $k > 0$ there is some finite $B \models SO_{11}$ such that $\mathcal{A} \sim_k B$.

4. Prove first that $SO_{11} \cup \{\exists i \mid i > 0\}$ is complete. Then apply Theorem 2.3.

Section 5.4

1. Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism, $\mathcal{M} = (\mathcal{A}, w)$, $\mathcal{M}' = (\mathcal{B}, w')$ with $x^{w'} = hx^w$. Verify $\mathcal{M} \models \varphi [\vec{a}] \Rightarrow \mathcal{M}' \models \varphi [h\vec{a}]$ by induction on $\varphi$.

2. Let $\mathcal{A} = (\mathcal{A}, <)$ be ordered. Replacing each $a \in A$ by a copy of $(\mathbb{Z}, <)$ or of $(\mathbb{Q}, <)$ results in a discrete or a dense order $B \supseteq A$, respectively.

3. Let $\mathcal{A}_0 \models T_0$. Choose $\mathcal{A}_1$ with $\mathcal{A}_0 \subseteq \mathcal{A}_1 \models T_1$, $\mathcal{A}_2$ with $\mathcal{A}_1 \subseteq \mathcal{A}_2 \models T_0$ etc. This results in a chain $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ such that $\mathcal{A}_{2i} \models T_0$ and $\mathcal{A}_{2i+1} \models T_1$. Then $\mathcal{A}^* := \bigcup_{i \in \mathbb{N}} \mathcal{A}_{2i} = \bigcup_{i \in \mathbb{N}} \mathcal{A}_{2i+1} \models T_0, T_1$ and hence $\mathcal{A}^* \models T := T_0 + T_1$. This shows that $T$ is consistent and model compatible with $T_0$ (hence likewise with $T_1$). Clearly, $T$ is an $\forall \exists$-theory and therefore also inductive.

4. The union $S$ of a chain of inductive theories model compatible with $T$ has again these properties. By Zorn’s lemma there exists a maximal, hence in view of Exercise 3 a largest theory of this kind.
Section 5.5

1. Let $(i, j) \neq (0, 0)$. Then $DO_{ij}$ has models $A \subseteq B$ with $A \not\equiv B$. To show that $DO_{00}$ is the model completion of $DO$ note first that $T := DO_{00} + DA$ is model complete for each $A \models DO$. Moreover, $T$ is complete since $T$ has a prime model: For instance, let $A \models DO_{10}$. Then the ordered sum $\mathbb{Q} + A$ (i.e., $(\forall x \in \mathbb{Q})(\forall y \in A)x < y$) is a prime model of $T$.

2. (a) Lindström’s criterion. $T$ is $\aleph_1$-categorical because a $T$-model can be understood as a $\mathbb{Q}$-vector space. (b) Each $T_0$-model $G$ is embeddable in a $T$-model $H$. One gains such $H$ by defining a suitable equivalence relation on the set of all pairs $\frac{a}{n}$ with $a \in G$ and $n \in \mathbb{Z} \setminus \{0\}$.

3. Uniqueness follows similarly to uniqueness of the model completion. If $A \models T^*$ and $A \subseteq B \models T$ then $B \subseteq C \models T^*$ for some $C$, hence $A \not\leq C$ in view of $A \subseteq C$, and therefore $A \leq_{ee} B$ according to Lemma 4.8.

4. The algebraic closure $\mathbb{F}_p$ of the prime field $\mathbb{F}_p$ is equal to $\bigcup_{n \geq 1} \mathbb{F}_p^n$, where $\mathbb{F}_p^n$ is the finite field of $p^n$ elements. Thus, an $\forall \exists$-sentence valid in all finite fields is valid in all a.c. fields of prime characteristics and hence in all a.c. fields (proof indirectly with (1) in 3.3).

Section 5.6

1. Let $A, B \models \mathbb{Z}G$, $A \subseteq B$. Then also $A' \subseteq B'$ for the $\mathbb{Z}GE$-expansions $A'$ and $B'$ of $A$ and $B$, respectively, because $m1$ has in $\mathbb{Z}G$ both an $\forall$- and an $\exists$-Definition. Thus $A' \not\leq B'$ and hence $A \not\leq B$.

2. Similar to quantifier elimination in $\mathbb{Z}GE$ but somewhat more simple.

3. Inductively over quantifier-free $\varphi = \varphi(x)$ follows: either $\varphi^A$ or $(\neg \varphi)^A$ is finite for each $A \models RCF^0$. This is not the case for $a(x)$.

4. CS holds in the real closed field $\mathbb{R}$, hence in each $A \in RCF$. The proofs from CS of $(\forall x \geq 0)\exists y x = y \cdot y$, and that each polynomial of odd degree has a zero must be carried out without a theory of continuous functions, which is very instructive.
Section 5.7

1. If $F$ is trivial then there is some $i_0 \in I$ with $i_0 \in J$ for each $J \in F$ by Exercise 3 in 1.5. Then $a \cong_F b \iff i_0 \in I_{a=b} \iff a_{i_0} = b_{i_0}$, for all $a, b \in \prod_{i \in I} A_i$. This implies $\prod_{i \in I} A_i \simeq A_{i_0}$.

2. $x \mapsto x^I/F$ $(x \in A)$ is an embedding (to be checked in detail) and moreover an elementary embedding.

3. Let $X \models_K \varphi$ and $I$, $J$, $F$ defined as in the proof of Theorem 7.3 and assume that for each $i \in I$ there is some $\mathcal{A}_i \in K$ and $w_i : \operatorname{Var} \to \mathcal{A}_i$ such that $w_i \alpha \in D\mathcal{A}_i$ for all $\alpha \in i$ but $w_i \varphi \notin D\mathcal{A}_i$. Put $\mathcal{C} := \prod_{i \in I} F_i \mathcal{A}_i \in K$ and $w = (w_i)_{i \in I}$. Then $wX \subseteq D\mathcal{C}$ and $w \varphi \notin D\mathcal{C}$, hence $X \not\models \varphi$, a contradiction to $X \models_K \varphi$.

4. W.l.o.g. $A = 2$ and $2 \subseteq B \subseteq 2^I$ for some set $I$ by Stone’s representation theorem. $2 \models \alpha \Rightarrow 2^I \models \alpha \Rightarrow B \models \alpha$ according to Theorem 7.5.

Section 6.1

1. $b \in \operatorname{ran} f \iff (\exists a \leq b)f a = b$ (this predicate is p.r. iff $f$ is p.r.).

2. Injectivity: Let $\varphi(a, b) = \varphi(c, d)$. In order to prove $a = c$ and $b = d$ assume first that $a + b < c + d$. This leads to a contradiction since $\varphi(a, b) < \varphi(a, b) + b + 1 = t_{a+b} + a + b + 1 = t_{a+b+1} \leq t_{c+d} \leq \varphi(c, d)$. Thus $a + b = c + d$. But then $a = \varphi(a, b) - t_{a+b} = \varphi(c, d) - t_{c+d} = c$, hence also $b = d$. Surjectivity: Since $\varphi(0, 0) = 0 \in \operatorname{ran} \varphi$ it suffices to prove $\varphi(a, b) + 1 \in \operatorname{ran} \varphi$, for all $a, b$. Clear for $b = 0$ because $\varphi(a, 0) + 1 = t_a + a + 1 = t_{a+1} = \varphi(0, a+1)$. In case $b \neq 0$ is $\varphi(a, b) + 1 = t_{a+b} + a + 1 = t_{a+1+b-1} + a + 1 = \varphi(a+1, b-1)$. This proof also confirms the correctness of the diagram for $\varphi$, that is, the arrows truly reflect the successively growing values of $\varphi$.

3. $\exists_1 n = (\mu k \leq n) [(\exists m \leq n) \varphi(k, m) = n]$.

4. $\operatorname{lcm}\{f \nu | \nu \leq n\} = \mu k \leq \prod_{\nu \leq n} f \nu | k \neq 0 \land (\forall \nu \leq n) f \nu | k$.

5. $\Rightarrow$: Let $R$ be recursive, $M = \{a \in \mathbb{N} \mid \exists b R a b\}$, and $c \in M$ fixed. Put $f n = k$ in case $(\exists m \leq n) n = \varphi(m, k) \land R k m$, and $f n = c$ otherwise.
Section 6.2

1. Let $\alpha_0, \alpha_1, \ldots$ be a recursive enumeration of $X$, $\beta_n = \alpha_n \land \ldots \land \alpha_n$. By Exercise 1 in 6.1, $\{\beta_n \mid n \in \mathbb{N}\}$ is recursive and axiomatizes $T$ as well.

2. Follow the proof of the unique term reconstruction property.

3. Similar to Exercise 2 with the unique formula reconstruction.

4. (a): A proof $\Phi = (\varphi_0, \ldots, \varphi_n)$ of $\varphi = \varphi_n$ from an axiom system $X$ in $T + \alpha$ can easily and in a p.r. manner be converted into a somewhat longer proof $\Phi'$ of $\alpha \rightarrow \varphi$ in $T$, following the case distinction in Lemma 1.6.3: $\varphi_i \in \Phi$ should in case $\varphi_i = \alpha$ be replaced by a proof of $\alpha \rightarrow \alpha$ in $T$, and in case $\varphi_i \in X \cup \Lambda$ by a proof of $\varphi_i \rightarrow \alpha \rightarrow \varphi_i$ in $T$ followed by $\varphi_i$ and $\alpha \rightarrow \varphi_i$. If $\varphi_k \in \Phi$ results from $\varphi_i \in \Phi$ and $\varphi_j = \varphi_i \rightarrow \varphi_k \in \Phi$ by applying MP, then the axiom $(\alpha \rightarrow \varphi_i \rightarrow \varphi_k) \rightarrow (\alpha \rightarrow \varphi_i) \rightarrow \alpha \rightarrow \varphi_k$, followed by $(\alpha \rightarrow \varphi_i) \rightarrow \alpha \rightarrow \varphi_k$ and $\alpha \rightarrow \varphi_k$ should replace $\varphi_k$. One may also proceed inductively on the length of $\Phi$ in constructing $\Phi'$.

Section 6.3

1. $\exists x \exists y \alpha \equiv_N \exists z (\exists x \leq z)(\exists y \leq z)(z = \varphi(x, y) \land \alpha)$ where $z \notin \text{var} \alpha$. Similarly for $\forall x \forall y \alpha$. Note also that $\exists x \exists y \alpha \equiv_N \exists z (\exists x \leq z)(\exists y \leq z)\alpha$. In all these equivalences, $\equiv_N$ can be replaced by $\equiv_{PA}$.

2. $(\forall z < y)\exists x \alpha \equiv_{PA} \exists u (\forall z < y)(\exists x < u)\alpha$. Contraposition and renaming of $\alpha$ readily yields $(\exists z < y)\forall x \alpha \equiv_{PA} \forall u (\exists z < y)(\forall x < u)\alpha$.

3. Prove $R^=$ by case distinction.

4. Prove by induction on $\varphi$ that both $\varphi$ and $\neg \varphi$ satisfy the claim.

Section 6.4

1. (a): $p | a \Rightarrow a \bot p \Rightarrow \exists xy xa + 1 = yp$ (Euclid’s lemma)
   $$\Rightarrow \exists xy b = yp - xab \Rightarrow p | b.$$  

   (b): Let $m := \text{lcm}\{a_\nu \mid \nu \leq n\}$, so that $m = a_\nu c_\nu$ for suitable $c_\nu$. Assume that $(\forall \nu \leq n)p | a_\nu$. Then $(\forall \nu \leq n)p | c_\nu$ by (a). Thus $m = pm'$ and $c_\nu = pc'_\nu$ for suitable $m', c'_\nu$. This leads to contradiction to the definition of $m$. (c) easily follows from (b).
2. \( \exists u[\text{beta } u \leq 2 \land (\forall v < x)(\exists w, w' \leq y)(\text{beta } w w \land \text{beta } u S w' \land w < w') \land \text{prim } w \land \text{prim } w' \land (\forall z < w')(\text{prim } z \Rightarrow z \leq w) \land \text{beta } u x y)] \).

3. (a): Prove this first for \( x \) instead of \( \vec{x} \). (b): It suffices to show that \( \text{sb}_{\vec{x}}(\dot{\varphi}, x) = \dot{\varphi} \) for \( x \notin \text{free } \varphi \). \( \text{sb}_{\vec{x}}((\forall x \alpha'), x) = (\forall x \alpha)' \) for closed \( \alpha \).

Section 6.5

2. (ii)\( \Rightarrow \)(i): If \( T \) is complete and \( T' + T \) is consistent then \( T' \subseteq T \) provided \( T \) and \( T' \) belong to the same language.

4. Trivial if \( T + \Delta \) is inconsistent. Otherwise let \( \vDash \) be the conjunction of all sentences \( \forall \vec{x} \exists y \alpha(\vec{x}, y) \), \( \alpha \) running through all defining formulas for operations from \( \Delta \). If \( T \) is decidable then so is \( T + \vDash \). Moreover \( \vdash_{T+\Delta} \alpha \iff \vdash_{T+\vDash} \alpha^{rd} \).

5. Set \( fa = (\dot{\varphi})_{last} \) if there is a proof \( \Phi \) in \( Q \) with \( a = \dot{\Phi} \), and \( fa = 0 \) otherwise. \( \text{ran } f = \{0\} \cup \{\dot{\varphi} \mid \vdash_{Q} \varphi \} \) is not recursive, since otherwise \( \dot{Q} \) would be recursive which is not the case.

Section 6.6

1. Let \( T \supseteq T_1 \) be consistent. \( S = \{\alpha \in L_0 \mid \alpha^p \in T^\Delta + CA\} \) is a theory, see the proof of Theorem 6.2. \( S \) extends \( T_0 \) consistently, hence is undecidable. The same then holds for \( T^\Delta + CA \), hence for \( T^\Delta \) (since \( CA \) is finite), and therefore also for \( T \).

2. Identify \( P \) with \( \omega \) and define for arbitrary \( n, m, k \in \omega \)
\[
n + m = k \iff \exists ab(a \sim n \land b \sim m \land a \cap b = a \cup b).
\]
For an explicit definition of multiplication on \( \omega \) the cross product has to be used. These definitions reflect the naive set-theoretic standard definitions of addition and multiplication in \( \mathbb{N} \).

Section 6.7

2. \( \Delta_0 \) is r.e. but not \( \Delta_1 \) (Remark 2 in 6.4). \( \dot{Q} \) is \( \Sigma_1 \) but not \( \Delta_1 \).

3. \( T \) is \( \omega \)-inconsistent iff \( (\exists \varphi \in L^1)(\forall n \ bwb_T \neg \varphi(n) \ \& \ bwb_T \exists x \varphi) \).
Section 7.1
1. Prove \( \vdash_{\text{PA}} \exists r \delta_{\text{rem}}(a, b, r) \) for \( b \neq 0 \) by induction on \( a \).

2. (a): Follow the proof of Euclid’s lemma in 6.4. (b): Use \(<\)-induction. (c): Let \( p \mid ab. p \mid a \Rightarrow \exists x, y x a + 1 = y p \Rightarrow \exists x, y x a b + b = y b p \Rightarrow p \mid b. \)

3. Similar to part (c) of Exercise 1 in 6.4.

4. Existence: \(<\)-induction. Uniqueness: Prove first \( p \nmid q^k (p, q \text{ prime}) \) by induction on \( k \), applying Exercise 2(c).

5. (a): \( \Box_{T+\alpha} \varphi \vdash_T \Box (\alpha \rightarrow \varphi) \) formalizes part (b) of Exercise 4 in 6.2.

Section 7.3
1. \( \vdash_T \Box \alpha \rightarrow \alpha \Rightarrow \vdash_T \neg \Box \alpha \Rightarrow \vdash_T \neg \Box \alpha \neg \Box \alpha \) by (5). Thus, \( T' \) is inconsistent by (1), hence \( \vdash_T \alpha \).

3. Clear if \( n = 0 \). Let \( T^n = T + \neg \Box^{n-1} \) and \( \neg \Box^{n-1} \) (the induction hypothesis). Now, \( \Box^{n-1} \vdash_T \neg \Box^{n-1} \) by D3. Hence, we obtain \( T^{n+1} = (T + \neg \Box^{n-1}) + \neg \Box^{n-1} = T + \neg \Box^{n+1} \). Further, by (5) page 281, \( \neg \Box^{n+1} \equiv_T \neg \Box^{n+2} \).

4. For arithmetical sentences \( \alpha \) the statement ‘If \( \alpha \) is provable in \( \text{PA} \) then \( \alpha \) is true in \( \mathcal{N} \)’ is provable in \( \text{ZFC} \). Formulated: \( \vdash_{\text{ZFC}} \Box_{\text{PA}} \alpha \rightarrow \alpha. \)

Section 7.4
1. \( \Box p \rightarrow \Box \Box p \) is responsible for transitivity, Löb’s formula for irreflexivity.

2. \( \vdash_G p \rightarrow \Box p \rightarrow p \Rightarrow \vdash_G \Box (p \rightarrow \Box p \rightarrow p) \Rightarrow \vdash_G \Box p \rightarrow \Box (\Box p \rightarrow p). \)

Section 7.5
1. Prove first \((*)\) \( \vdash_{G_n} H \Leftrightarrow \vdash_G \Box^n \neg \rightarrow H \) for all \( H \in \mathcal{F}_n \). The direction \( \Rightarrow \) in \((*)\) follows by induction on \( \vdash_{G_n} H \). Then continue as follows:

\[
\begin{align*}
\vdash_{G_n} H & \Leftrightarrow \vdash_G \Box^n H \\
& \Leftrightarrow \vdash_{\text{PA}} (\Box^n H) \text{ for all } i \quad \text{(by \((*)\))} \\
& \Leftrightarrow \vdash_{\text{PA}} \Box^n H \text{ for all } i \quad \text{(Theorem 5.2)} \\
& \Leftrightarrow \vdash_{\text{PA}_n} H \text{ for all } i \quad \text{(property of } i) \\
& \Leftrightarrow \vdash_{\text{PA}_n} (\Box^n H) \text{ for all } i \quad \text{(PA}_n = \text{PA} + \Box^n). 
\end{align*}
\]
2. The first claim follows immediately from Exercise 3 in 7.3. For determining the provability logic of $\text{PA}_n^\perp$, use (6) in 7.3 and Theorem 5.3.

4. Prove that $\not\vdash \text{GS} \neg[\neg\Box(p \to q) \land \neg\Box(p \to \neg q) \land \neg\Box(q \to p) \land \neg\Box(q \to \neg p)]$ and observe Theorem 5.4.

Section 7.7

1. We show there is some $\pi : g \to n$ with $P < Q \iff \pi P < \pi Q$ for $n := \text{lh } g$ (the length of a longest path in $g$). Trivial for $\text{lh } g = 0$, with $\pi P = 0$ for all $P \in g$. Let $\text{lh } g = n + 1$ and $g' := g \setminus \text{max } g$ where max $g$ denotes the set of all maximal points in $g$. Then $\text{lh } g' = n$ and $g'$ has property (p) as well as is readily checked. Hence $g'$ is a preference order with a mapping $\pi' : g' \to n$ by the induction hypothesis. Extend $\pi'$ to $\pi : g \to n + 1$ by putting $\pi P = n$ for all $P \in \text{max } g$. Obviously, $P < Q \Rightarrow \pi P < \pi Q$. For proving the converse let $\pi P < \pi Q$ with $Q \in \text{max } g$. Then certainly $P' \in \text{max } g$ for some $P' > P$. Hence, by (p), either $P < Q$ or $Q < P'$. The latter is impossible since $Q \in \text{max } g$. Thus $P < Q$.

2. If (i) is falsified in $g$ (that is, if $\Diamond(\Box p \land \Diamond \neg q) \land \Diamond(\Diamond q \land \neg p)$ is satisfiable in some point $O \in g$) then $g$ contains the diagram from page 296 as a subdiagram, with no arrow from $P$ to $Q$ and from $Q$ to $P'$. It easily follows that the finite poset $g$ cannot be a preference order.

3. It is a matter of routine to check that $\Box(\Box p \land p) \lor \Box(\Box q \land p)$ is satisfied in an ordered $\text{G}$-frame. For the converse assume that $g$ is initial but not (totally) ordered. Then $g$ contains the “fork” from page 298 as a subframe, in which the $\text{G}$-axiom can easily be refuted.

4. Soundness of the $\text{G}$-axioms and rules is shown as the soundness part of Theorem 7.3 which was given in the text. Soundness of the $\text{G}$-axiom follows by contraposition. Assume that there are cardinals $\kappa, \lambda$ such that $V_\kappa \models \Box \alpha \land \alpha \land \neg \beta$, and $V_\lambda \models \Box \beta \land \neg \alpha$. Then each of the assumptions $\kappa < \lambda$, $\kappa > \lambda$, or $\kappa = \lambda$ yields a contradiction.