Time-space Trade-offs for Voronoi Diagrams

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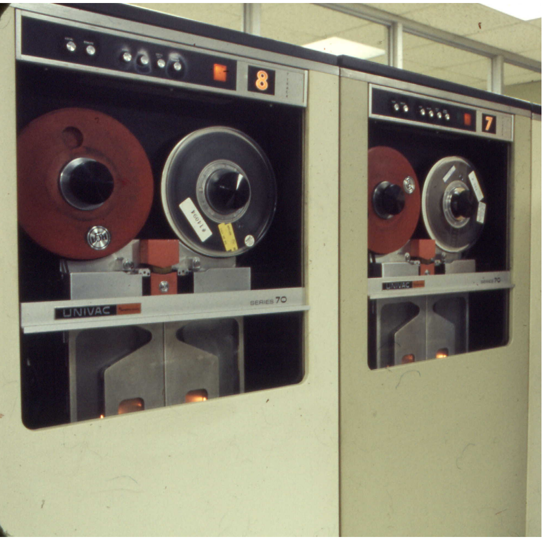
Limited Memory

Started in the 70's



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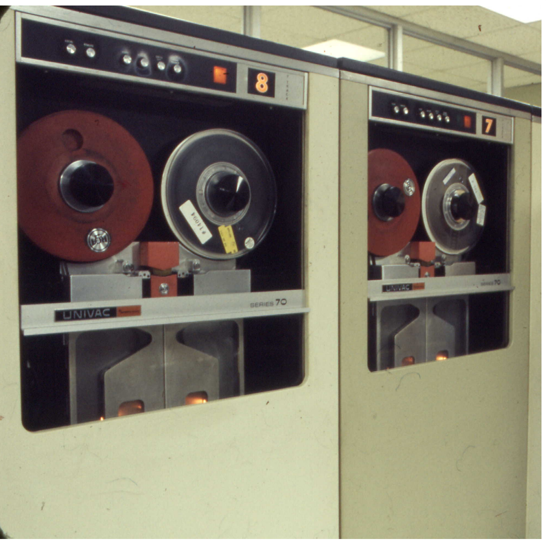

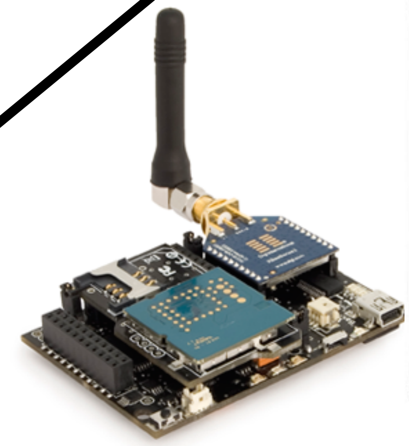

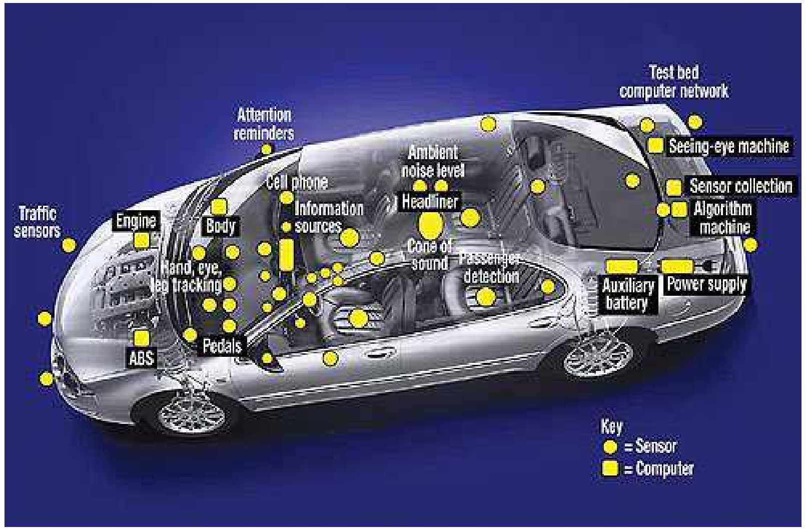

Increased interest recently

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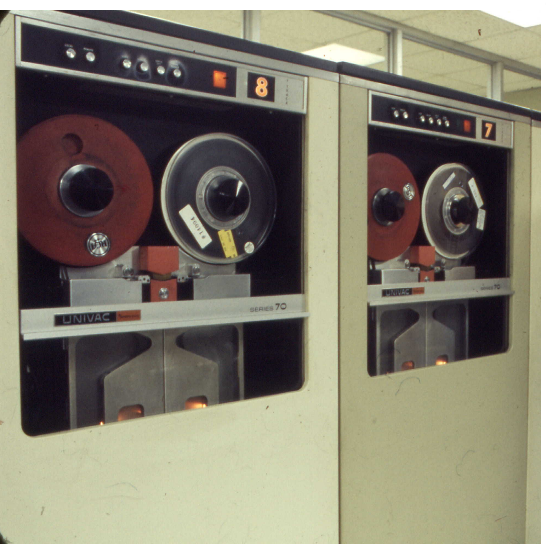
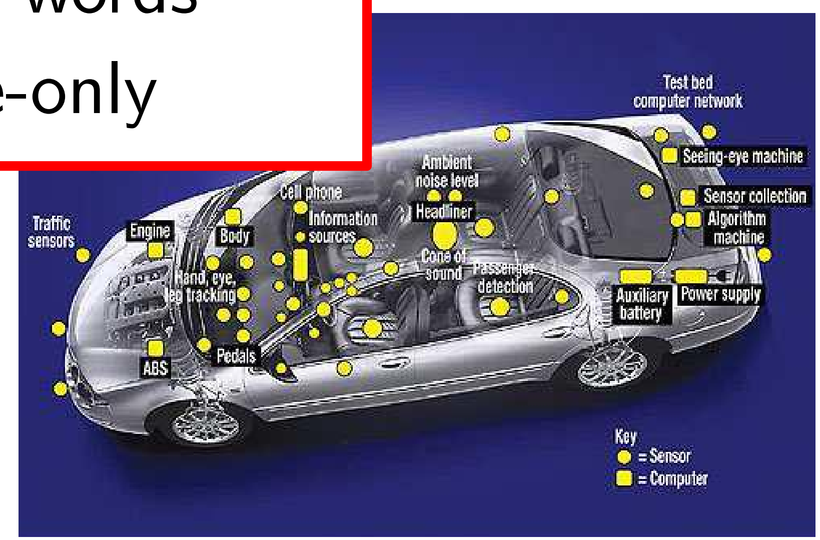


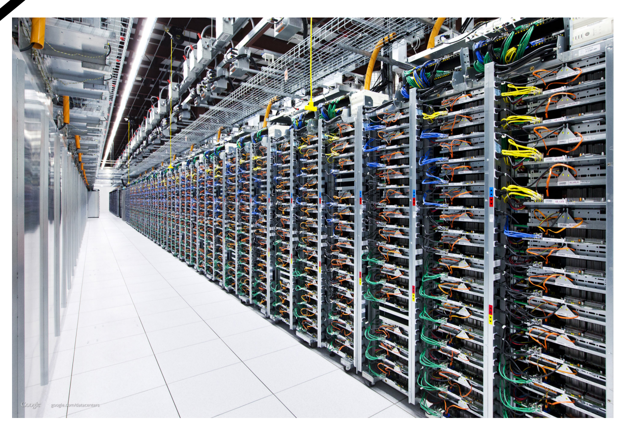
Model

Input: read-only, random-access

Memory: $O(s)$ words

Output: write-only





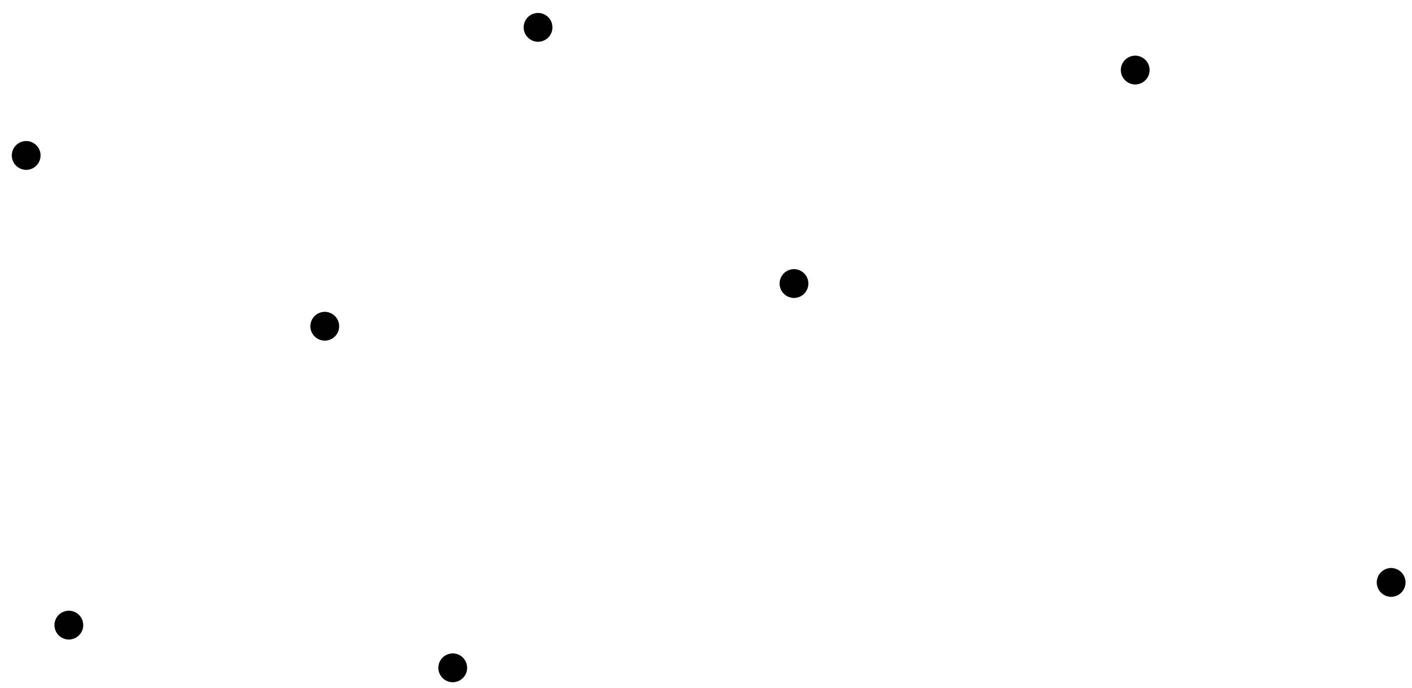
Increased interest recently

Voronoi Diagram

Input: set P of points in $\mathbb{R}^2$

Output: Subdivision of $\mathbb{R}^2$, such that each region has a common nearest neighbor in P.

Output format: vertices of Voronoi diagram in arbitrary order



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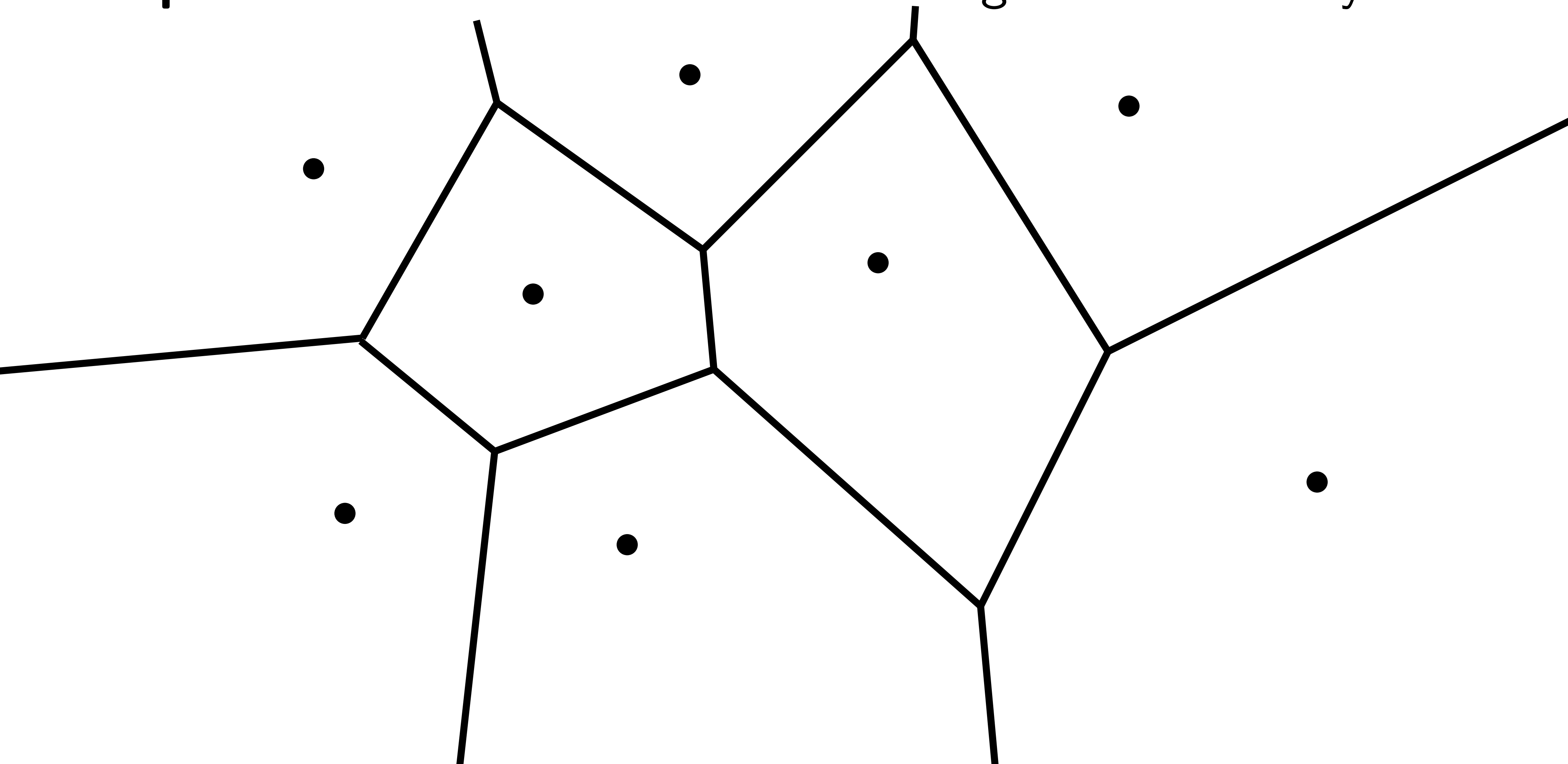

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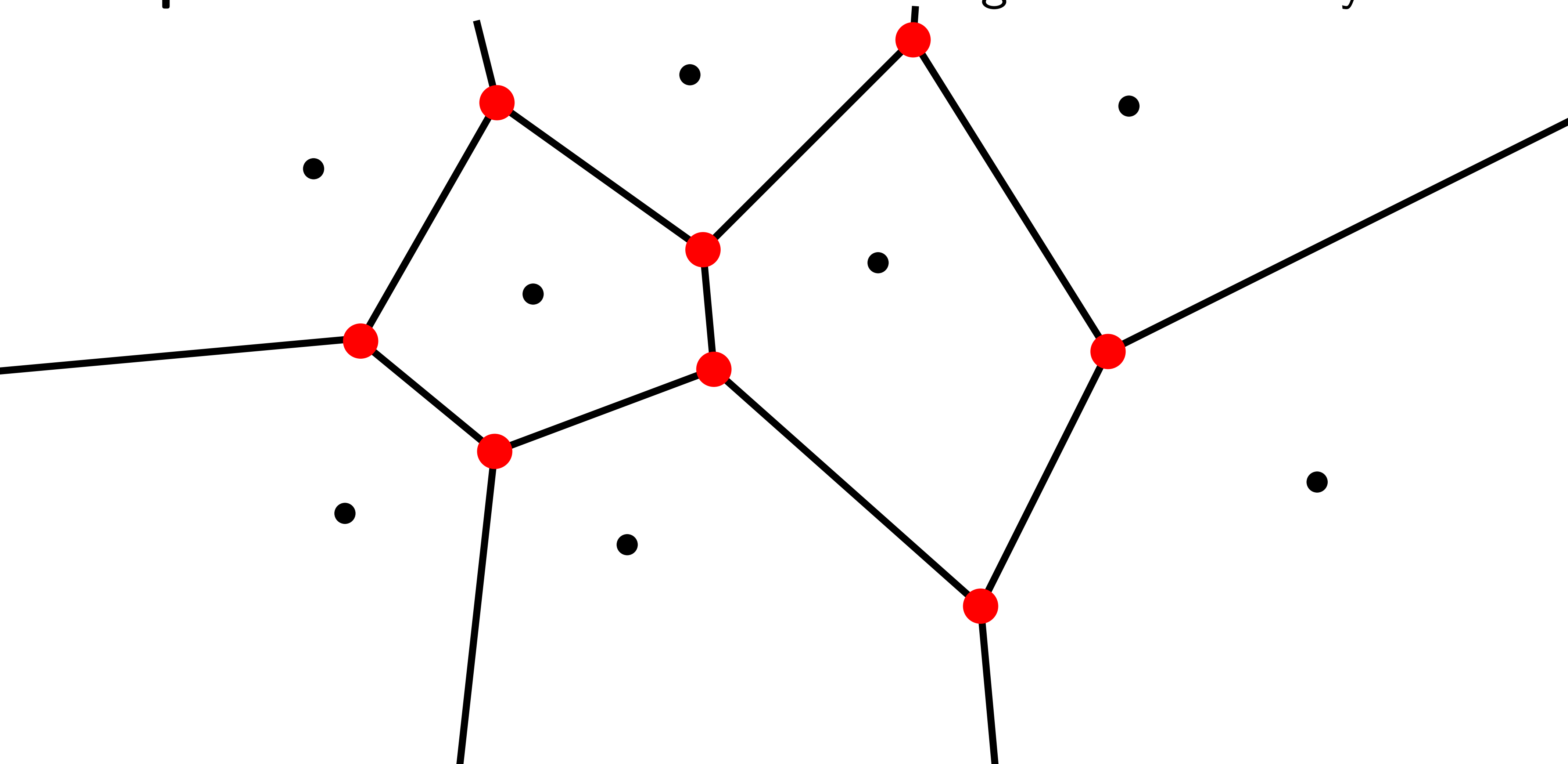

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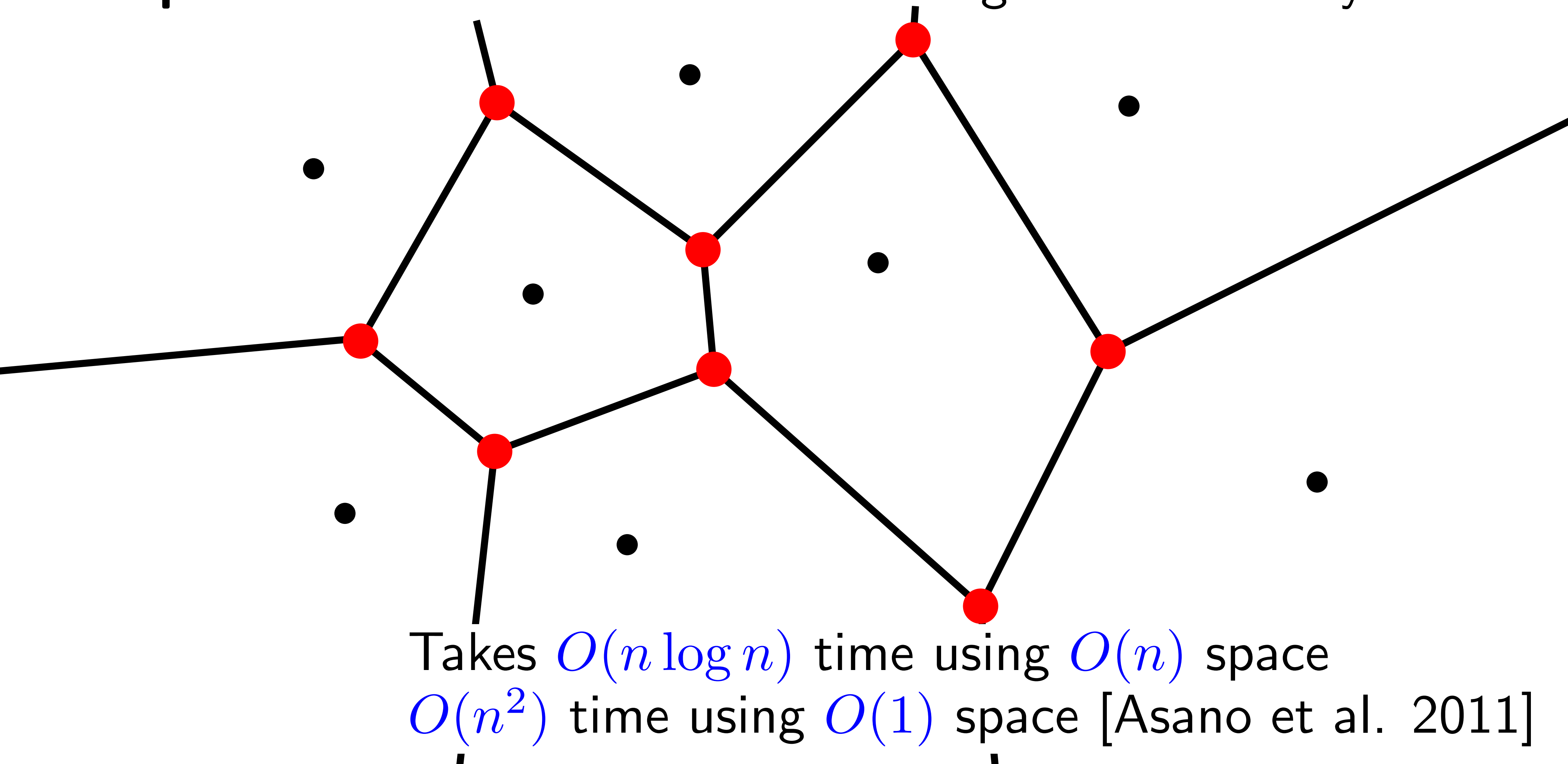
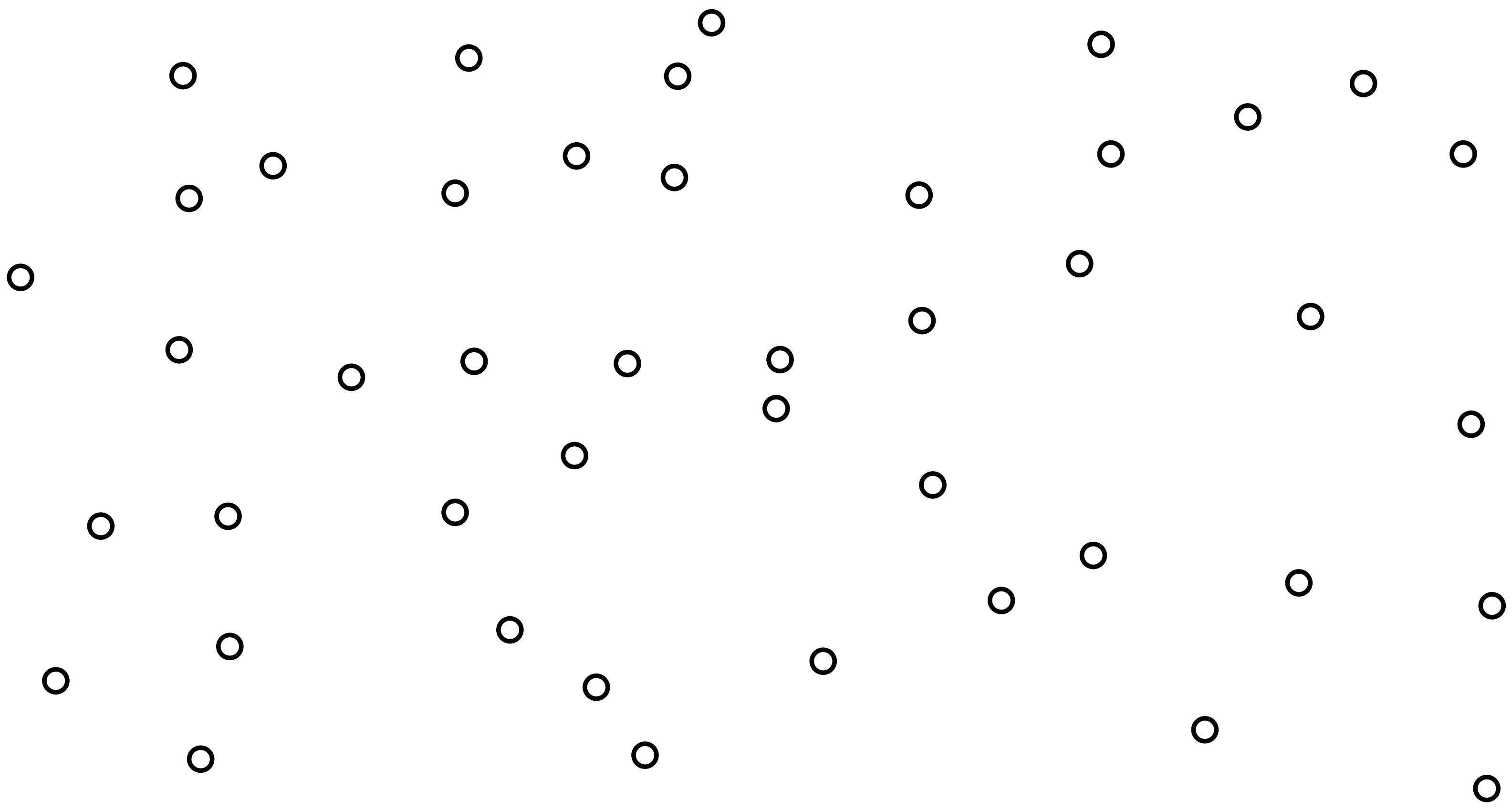


Takes $O(n \log n)$ time using $O(n)$ space

$O(n^2)$ time using $O(1)$ space [Asano et al. 2011]

General approach

- Find $R \subset P$ of size $O(s)$
- Compute $VD(R)$
- Triangulate $VD(R)$
- For each triangle, report the vertices of $VD(P)$ inside it



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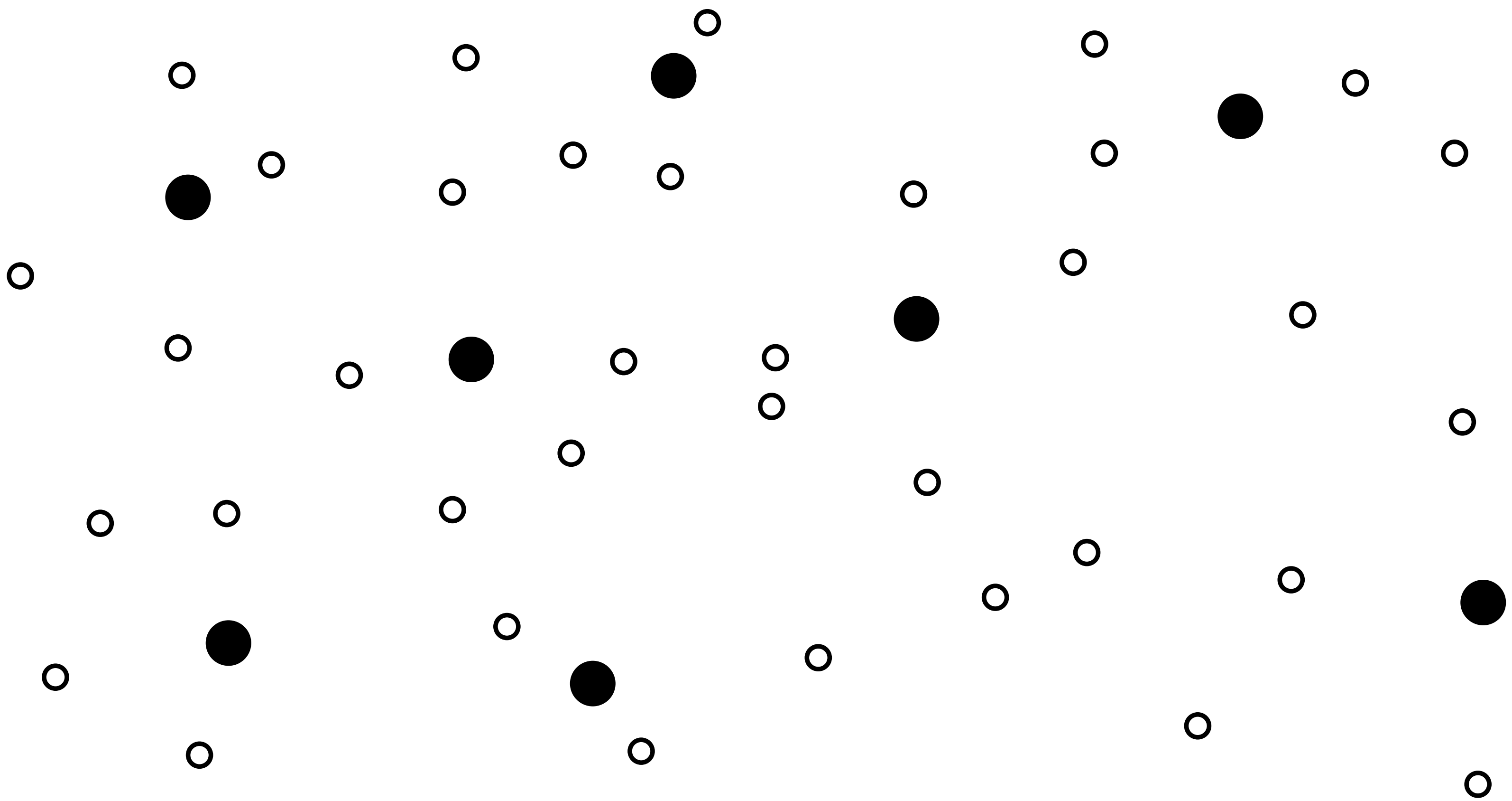

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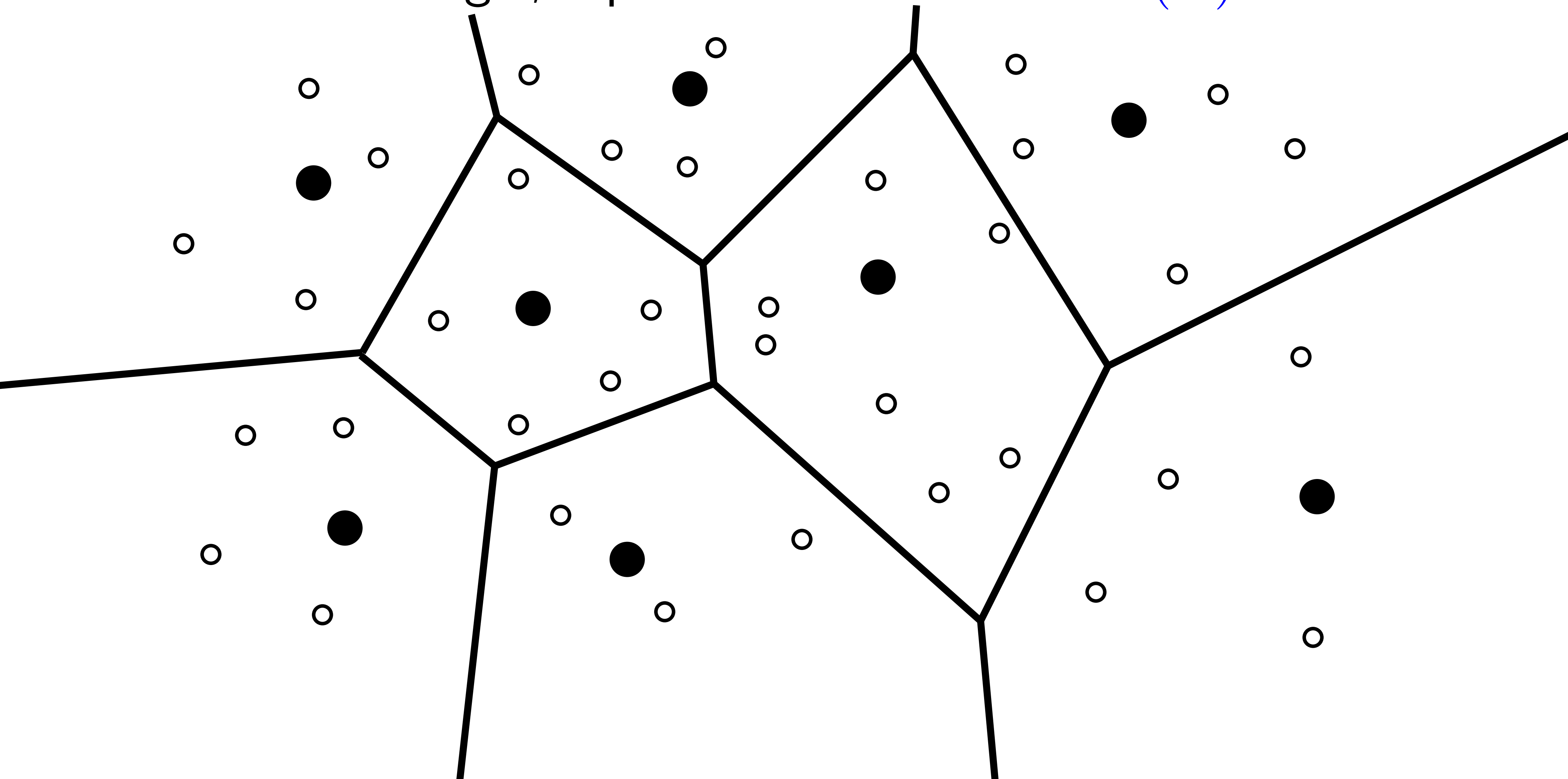

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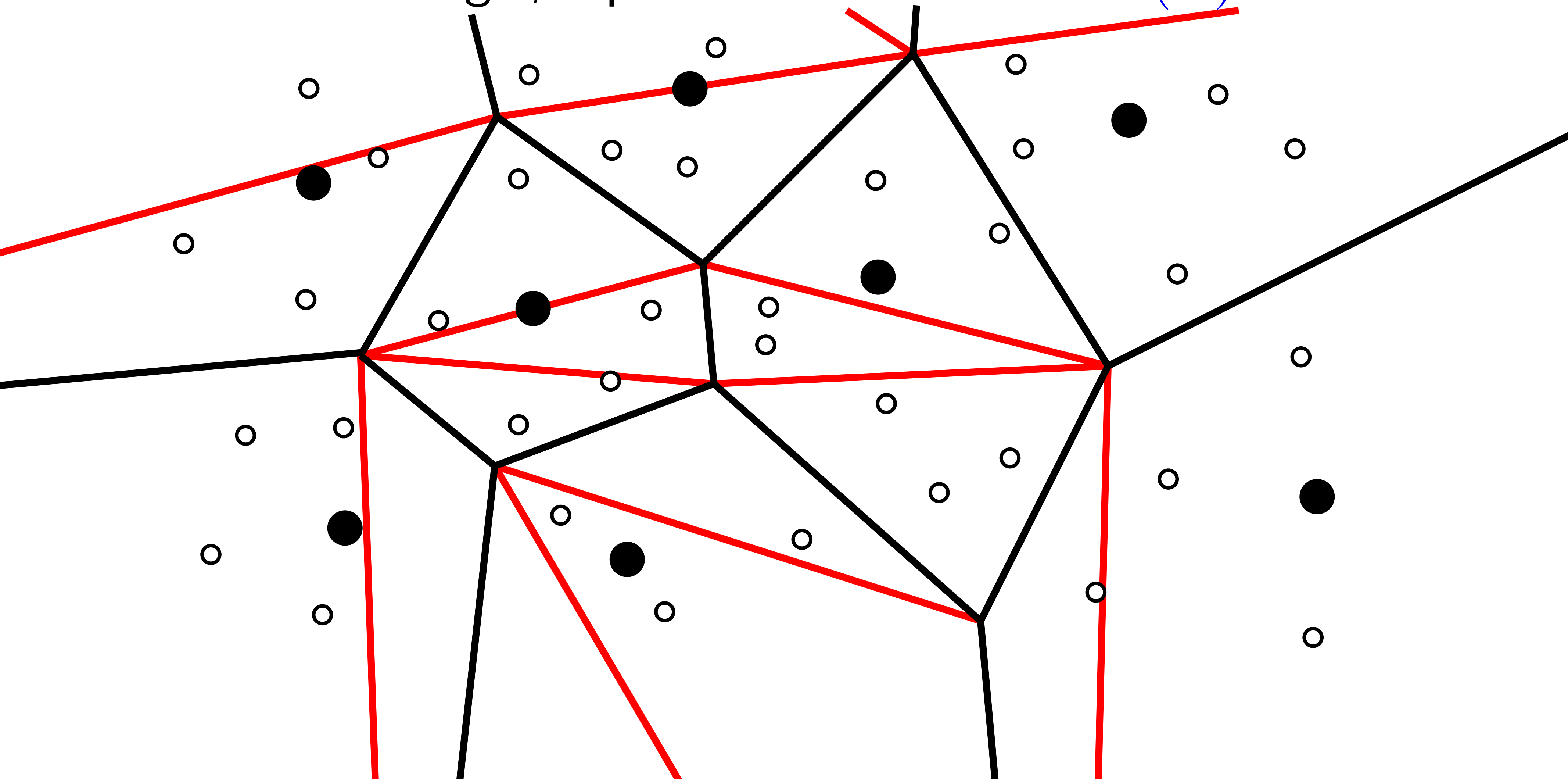

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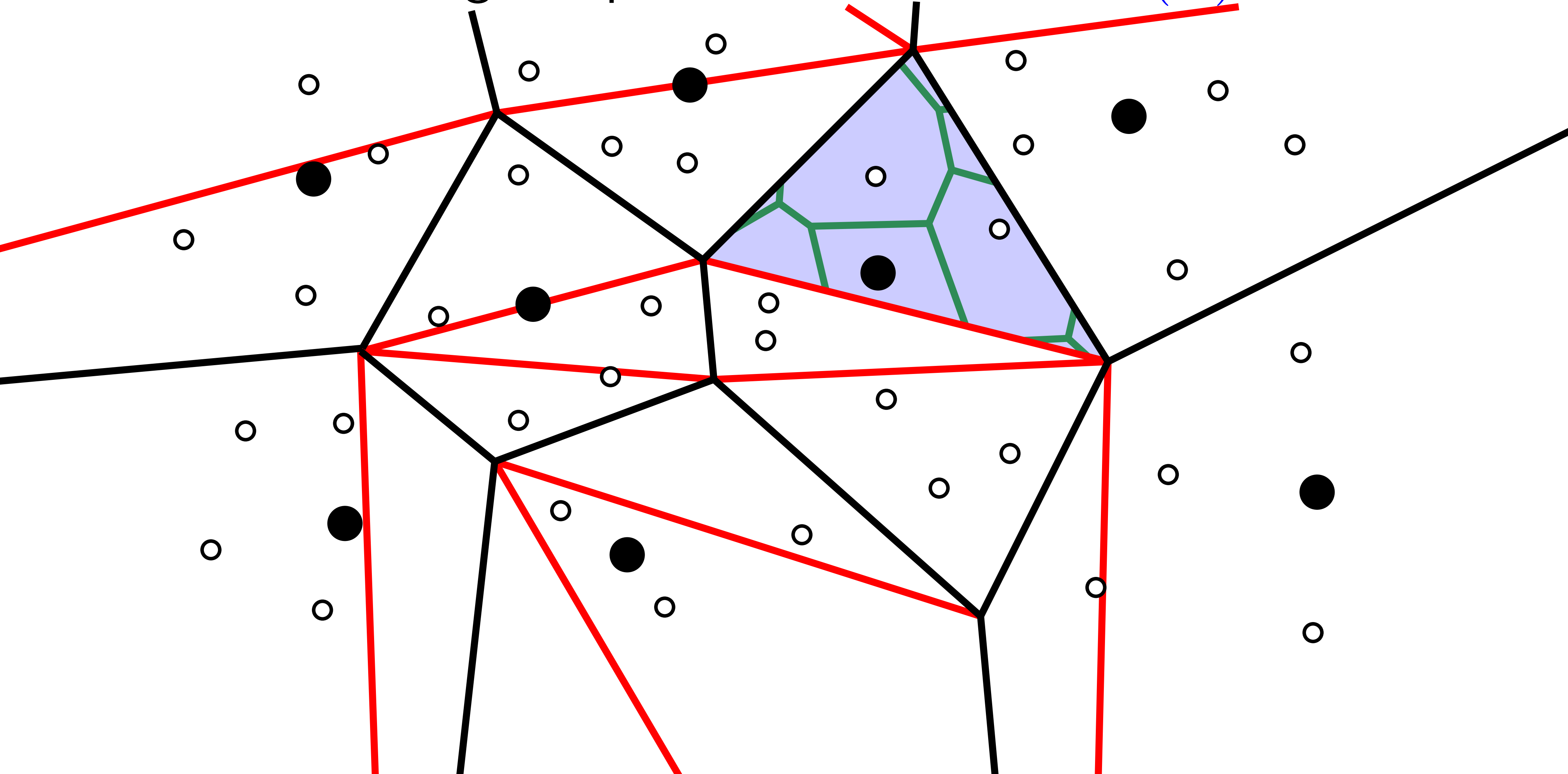

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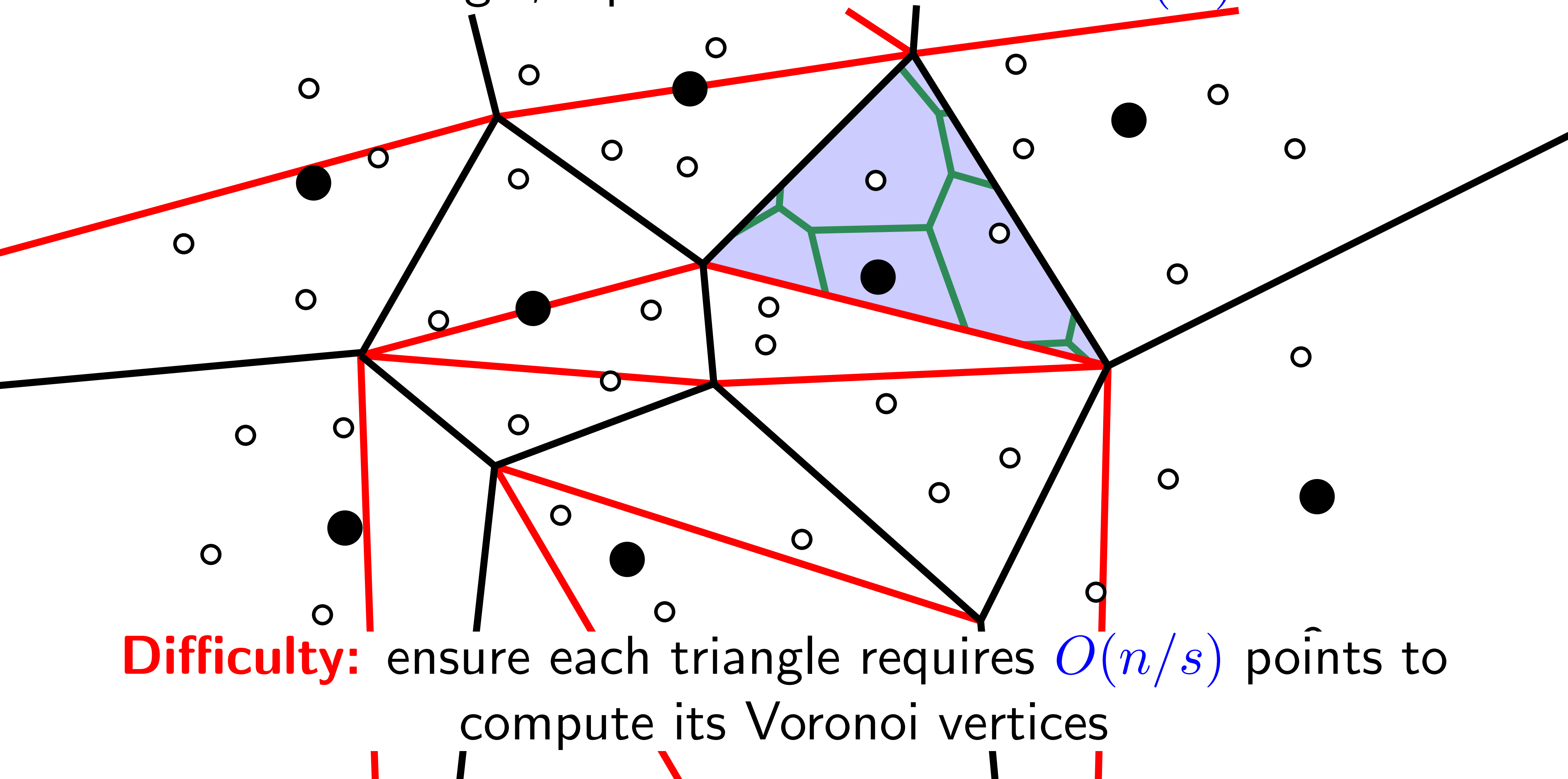

Difficulty: ensure each triangle requires $O(n/s)$ points to compute its Voronoi vertices

Computing the vertices

Given $R_2 \subset P$ such that:

- $|R_2| = O(s)$
- each vertex $v \in VD(R_2)$ has conflict set B_v with $|B_v| = O(n/s)$.

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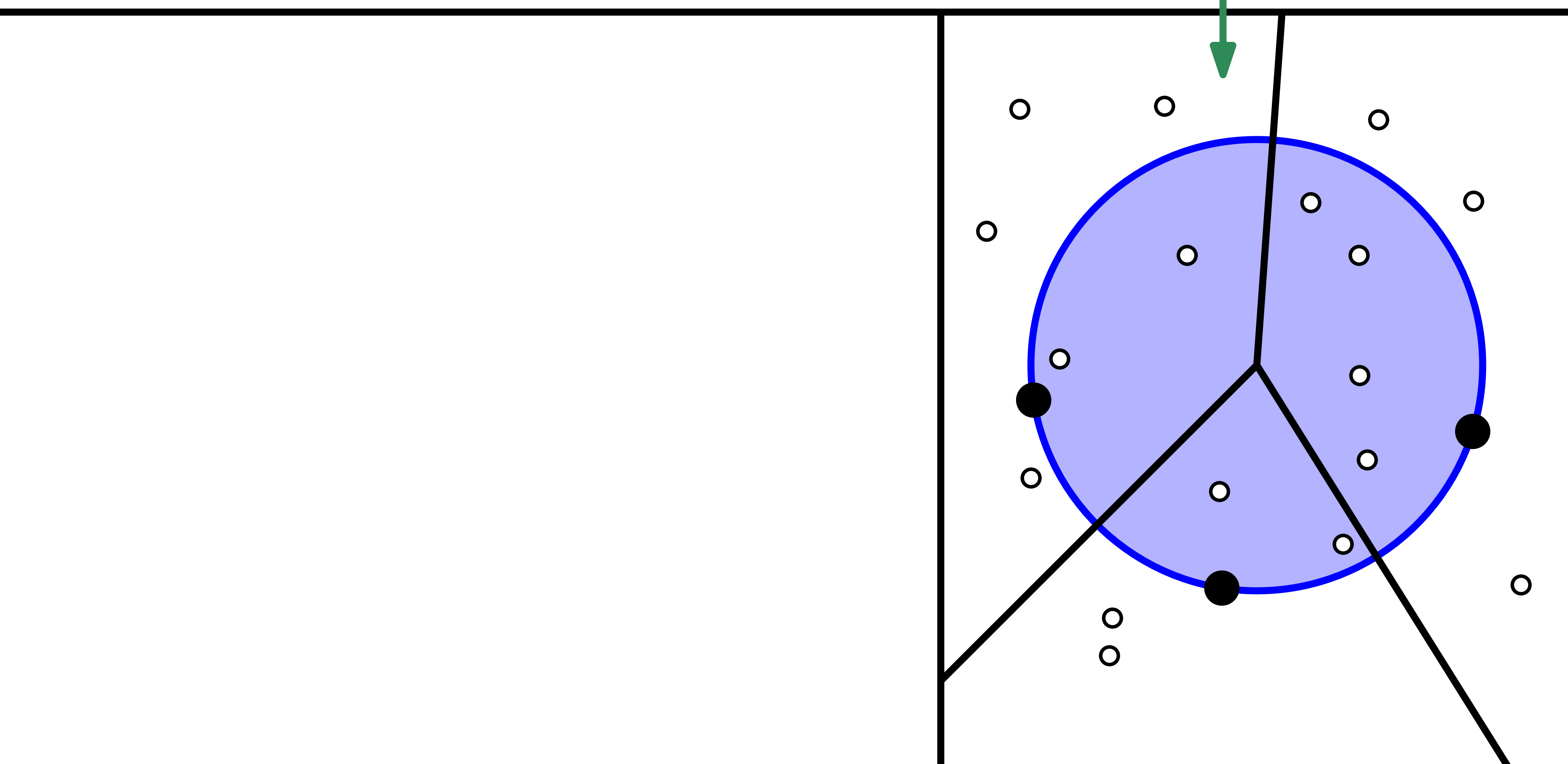

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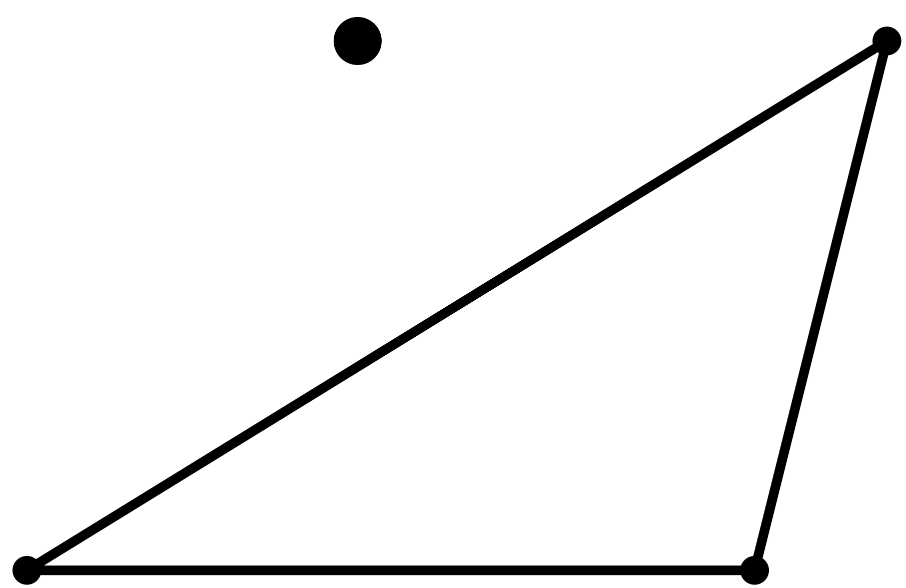
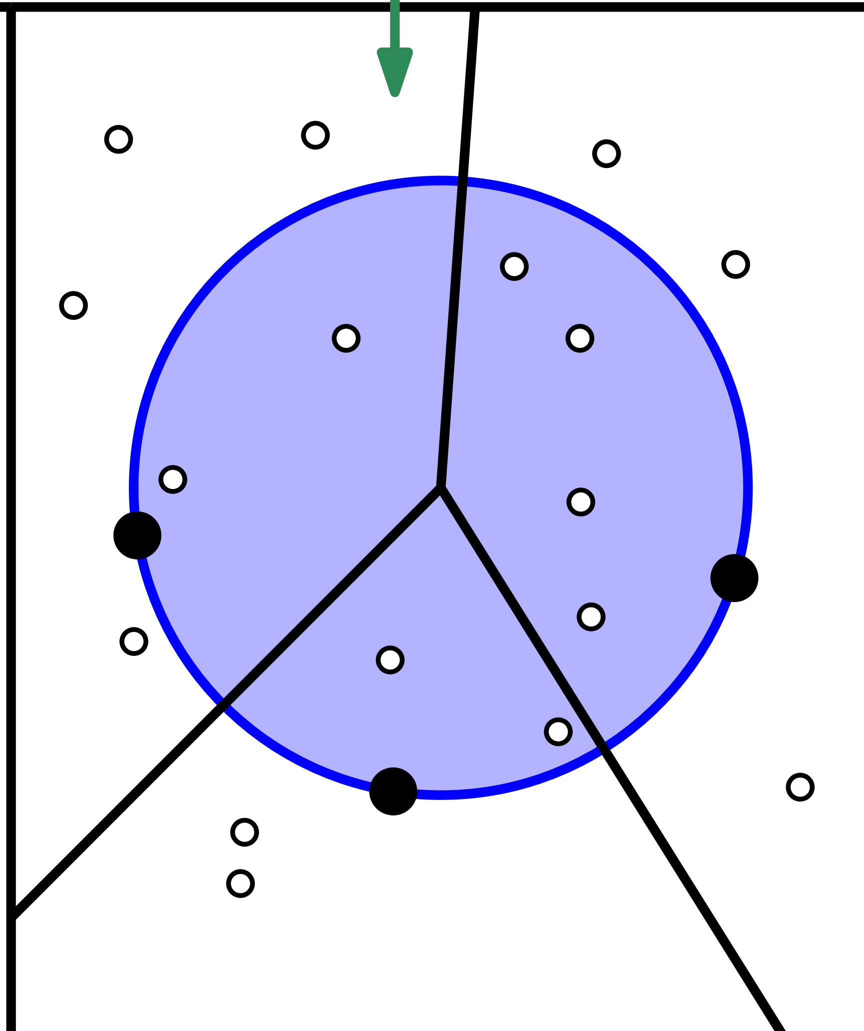
To report Voronoi vertices in $\Delta = \{v_1, v_2, v_3\} \subseteq R_2$ only consider points in $B_{v_1}, B_{v_2}, B_{v_3}$





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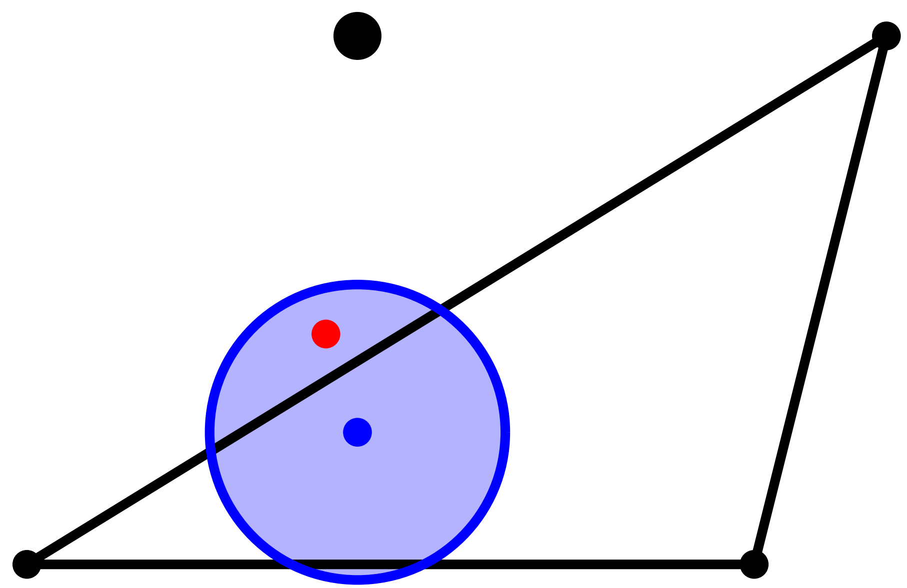

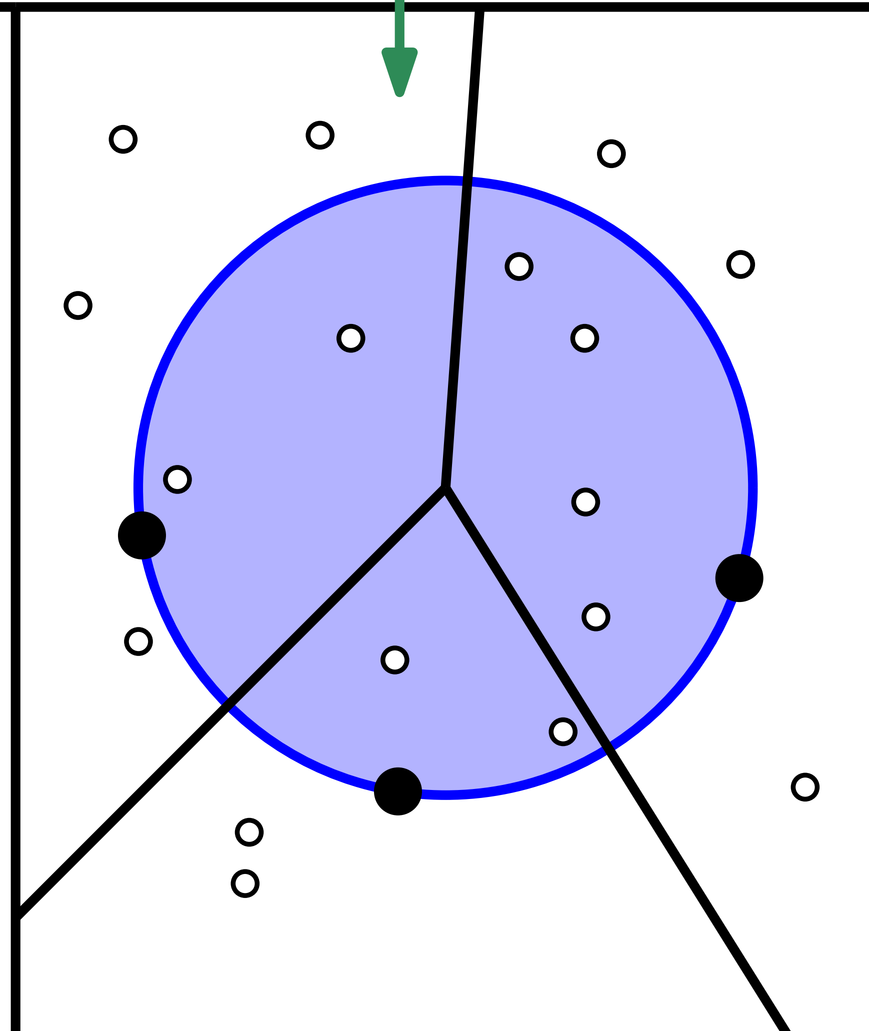

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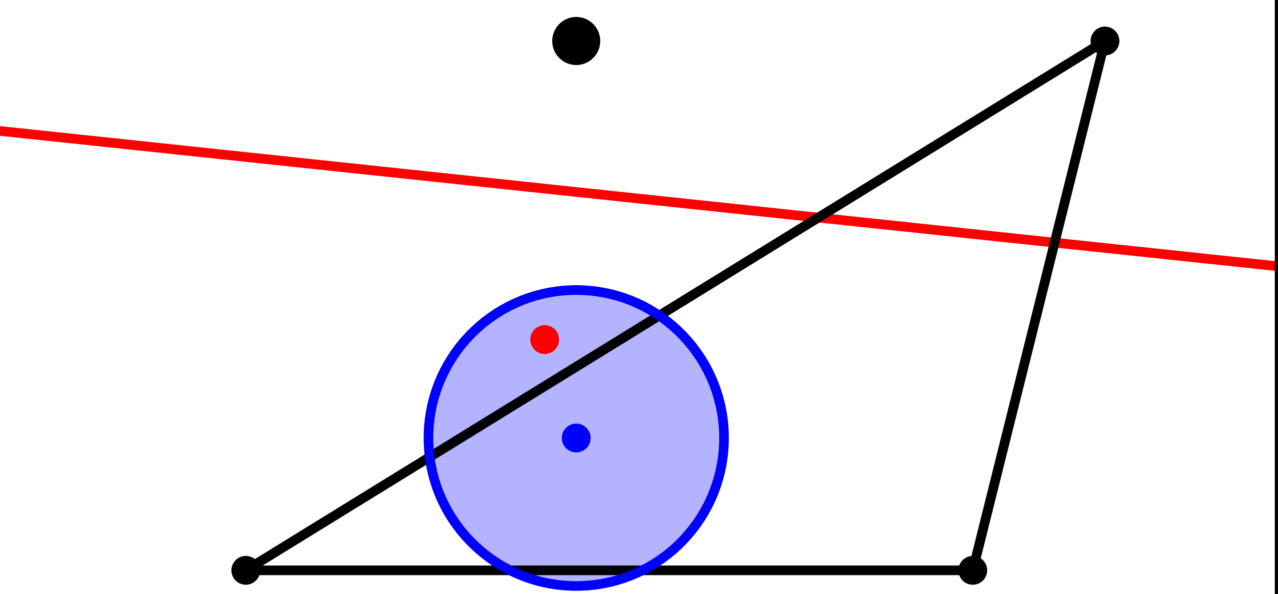

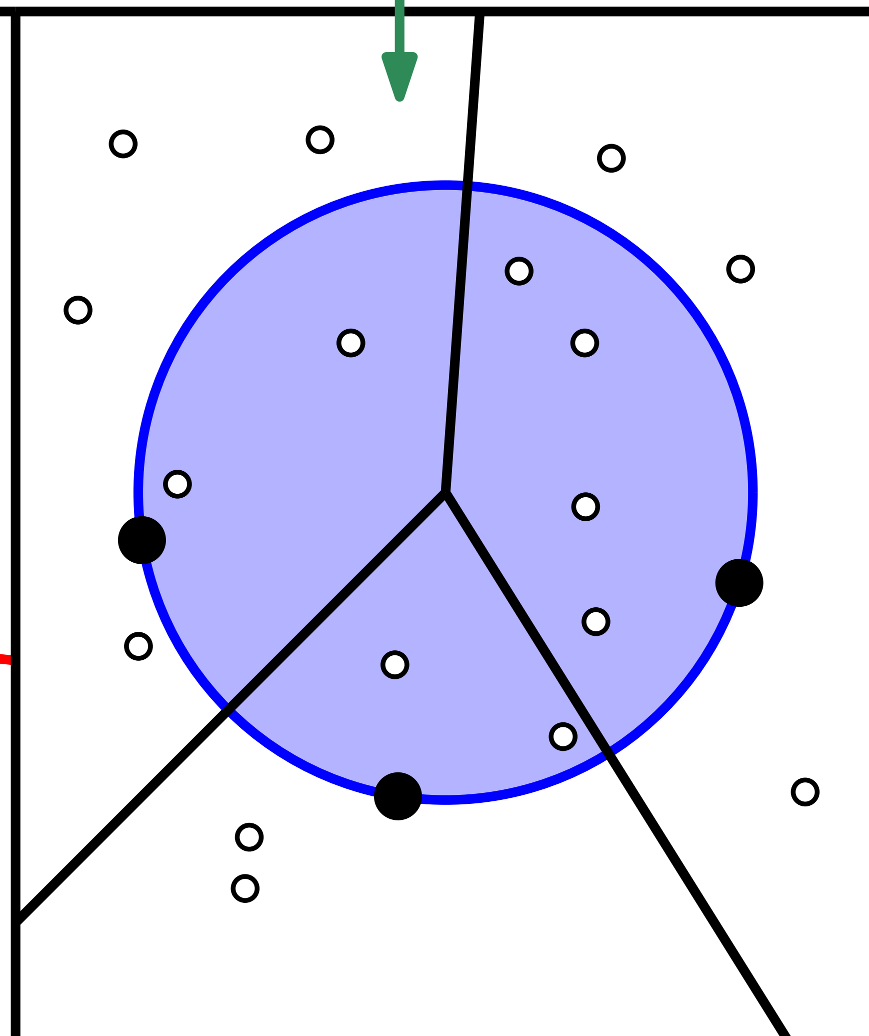

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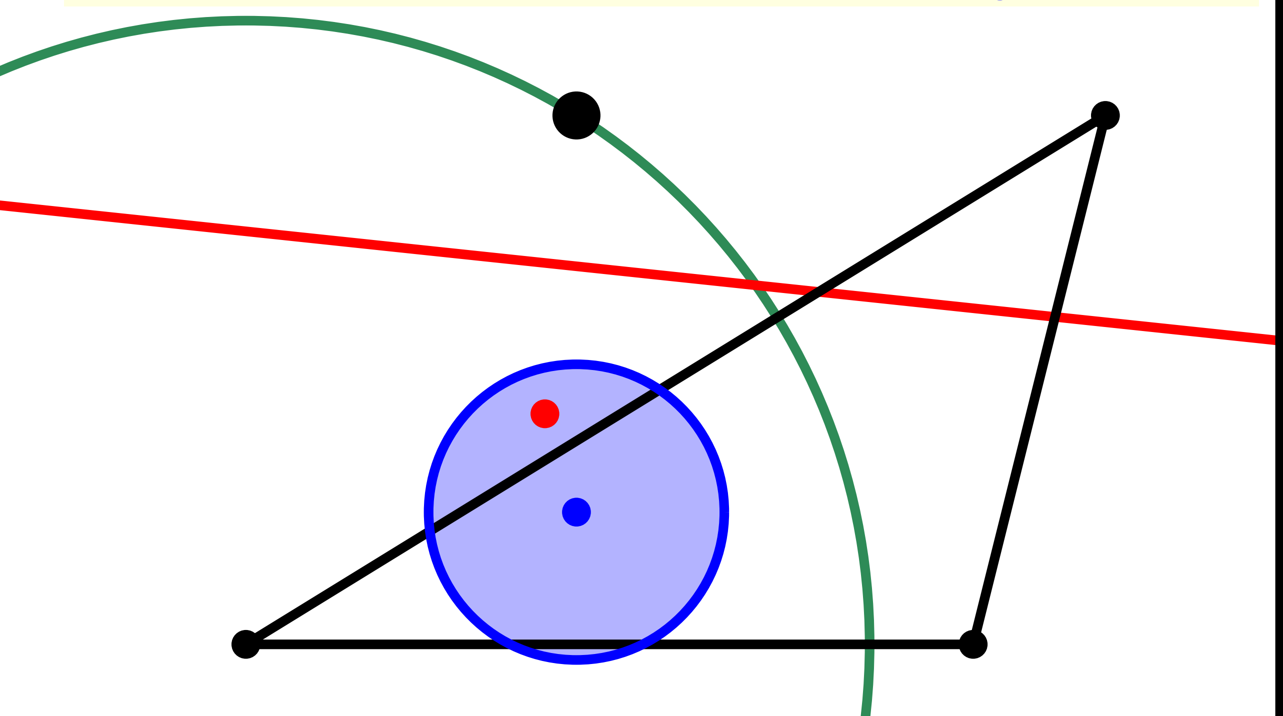
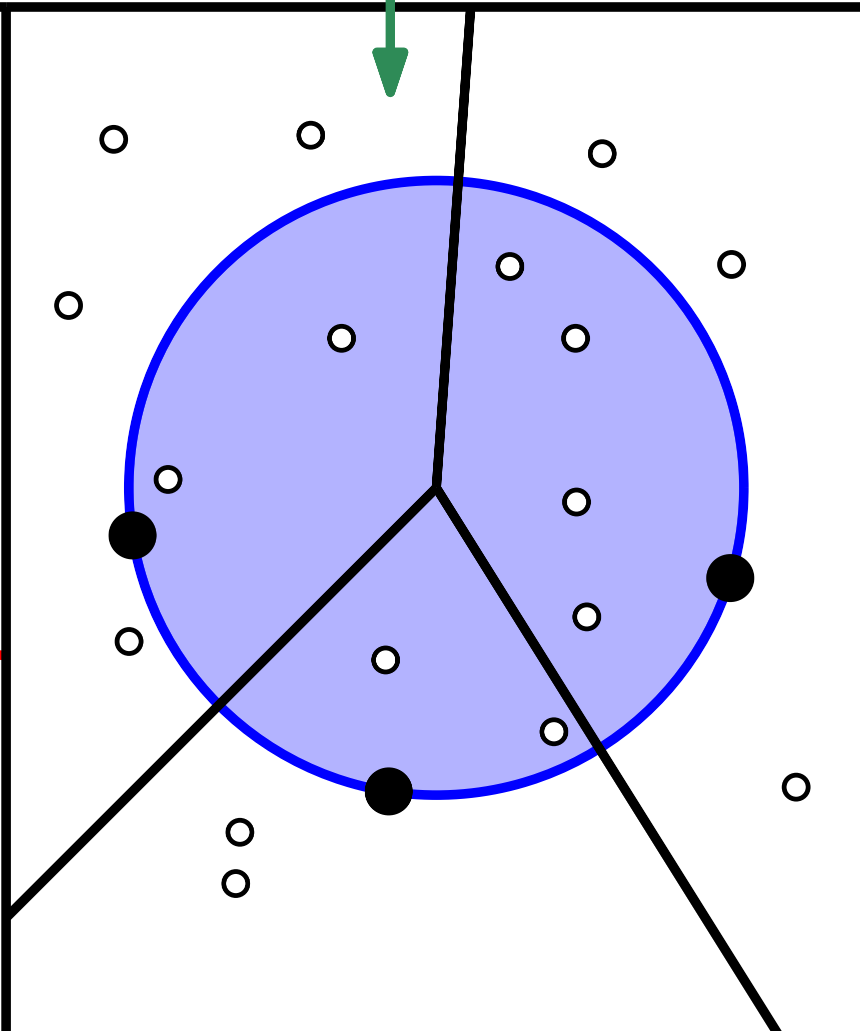
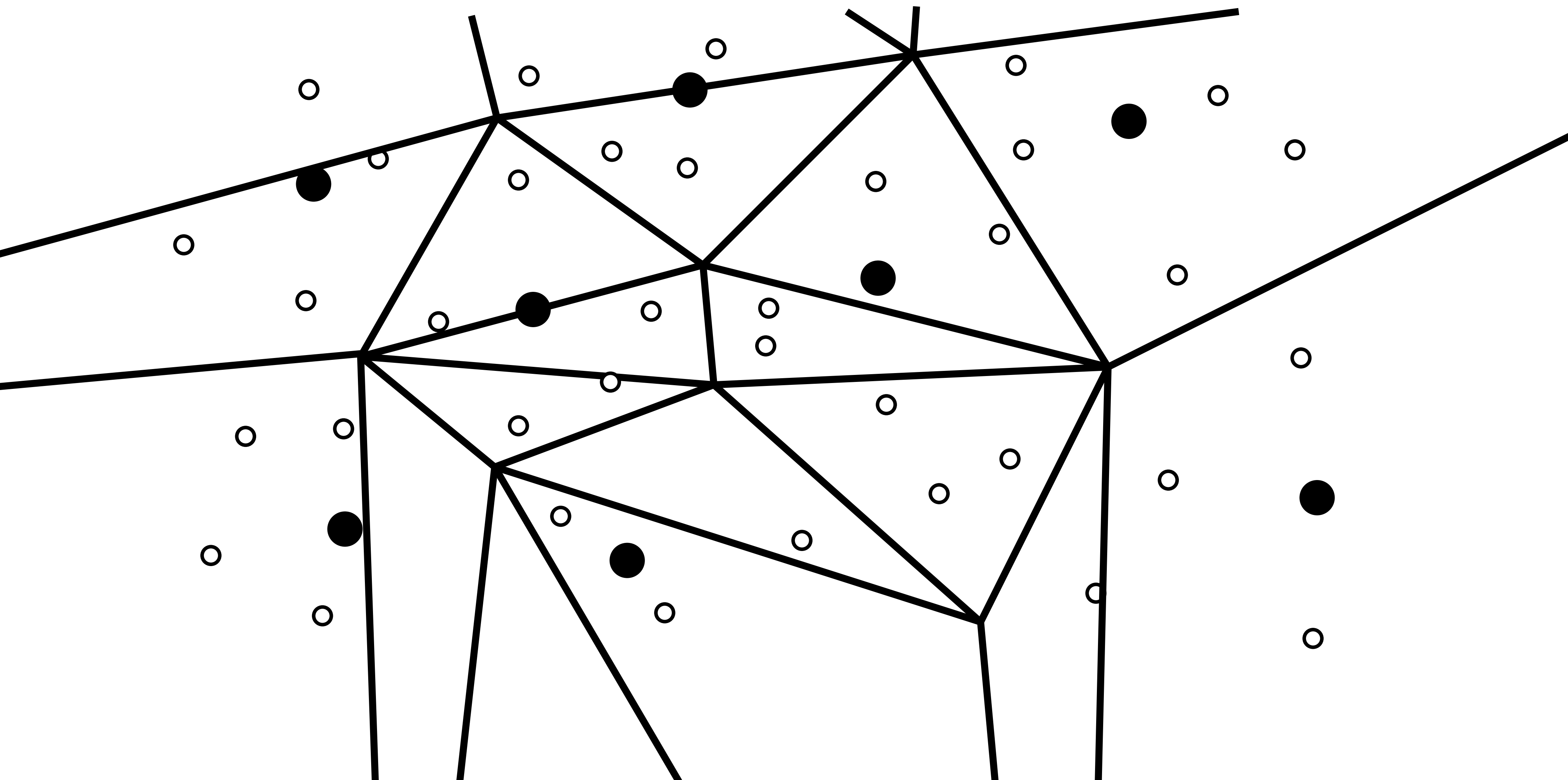




Computing the Voronoi vertices

Problem: $O(s)$ voronoi diagrams of size $O(n/s)$ to compute



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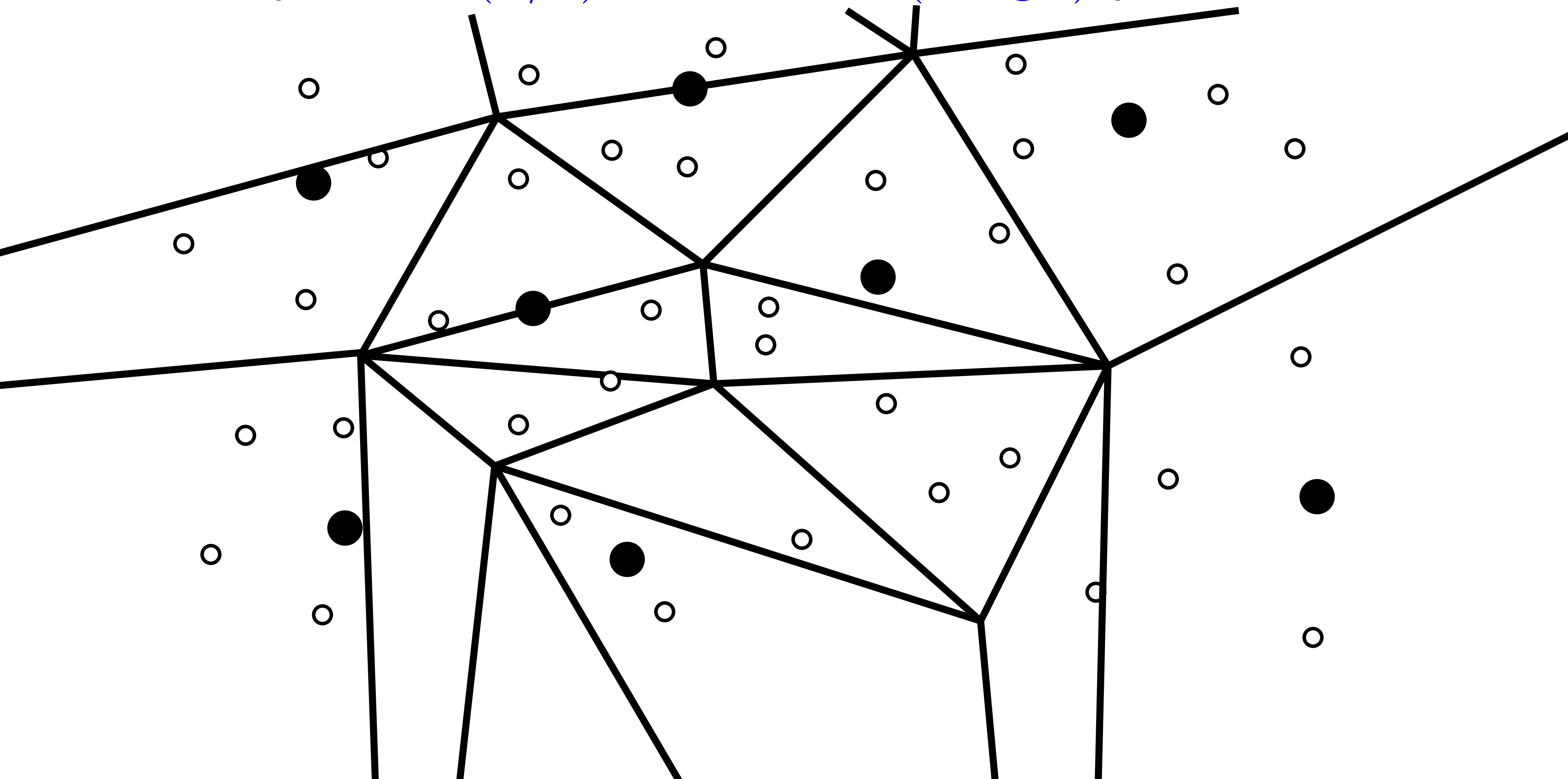
Problem: $O(s)$ voronoi diagrams of size $O(n/s)$ to compute

Solution: Use $O(1)$-memory algorithm on each triangle:

- Allocate each triangle $O(1)$ memory
- Scan points $O(n/s)$ times and $O(n \log s)$ per scan



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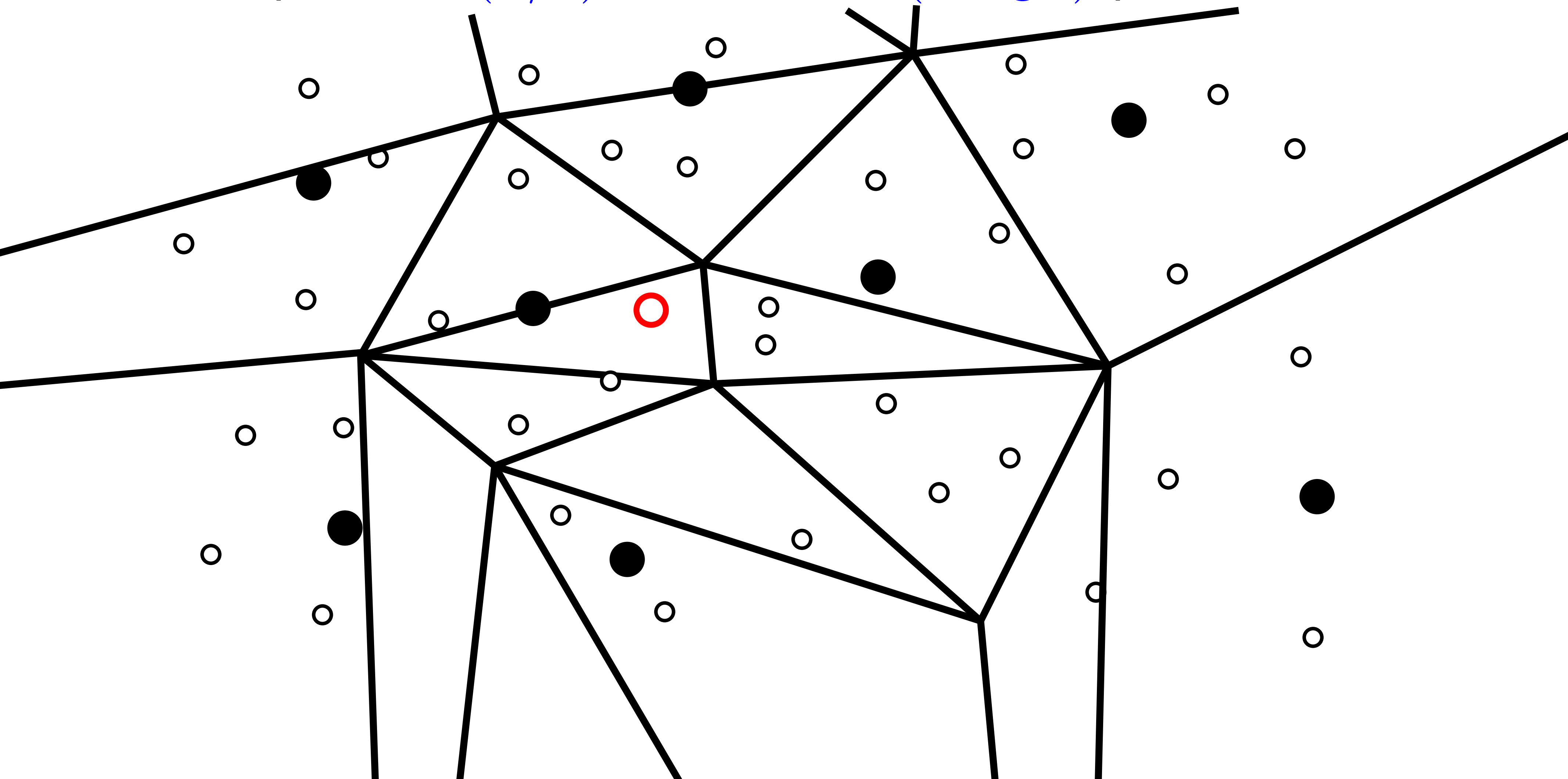

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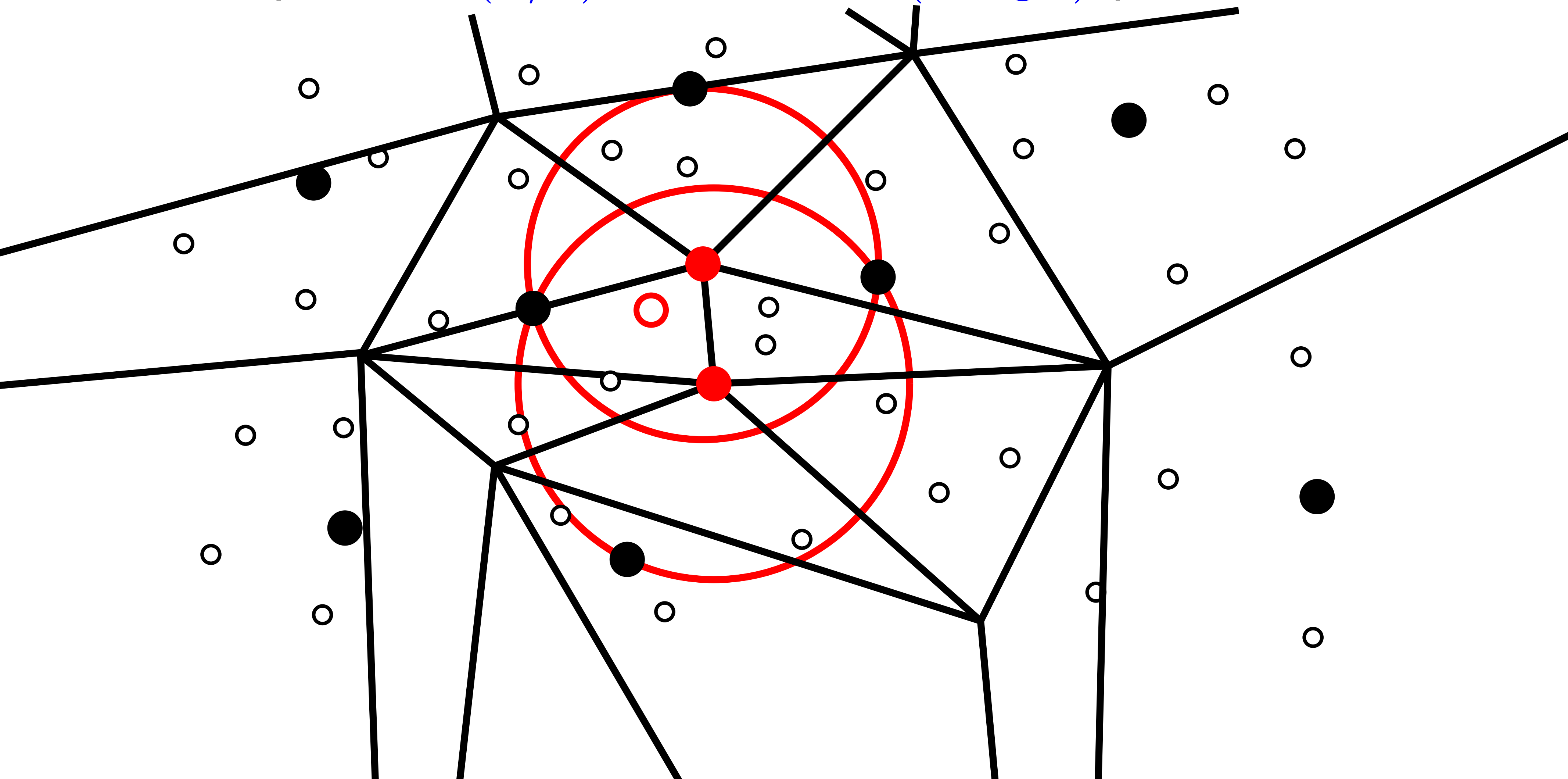

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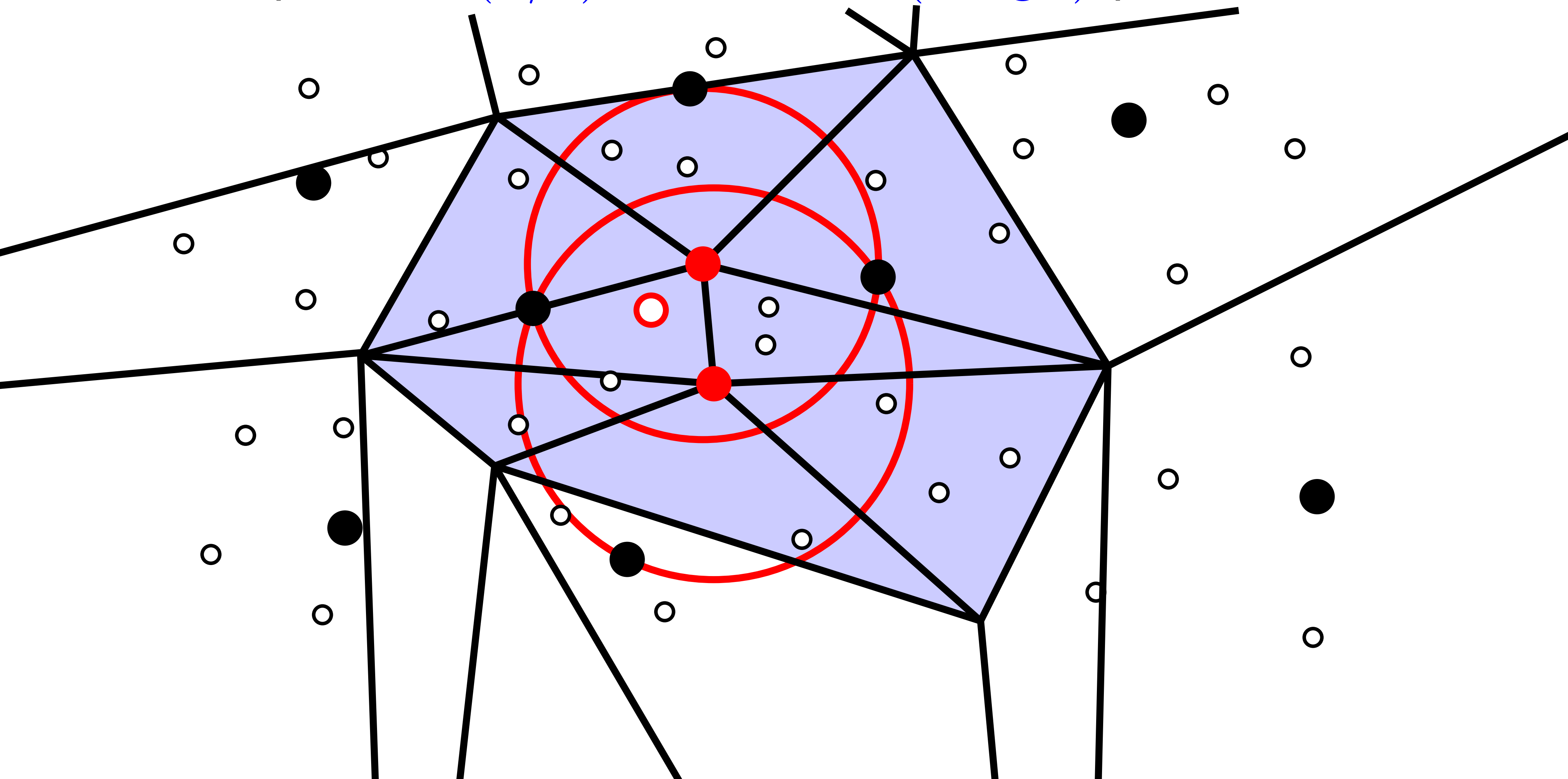
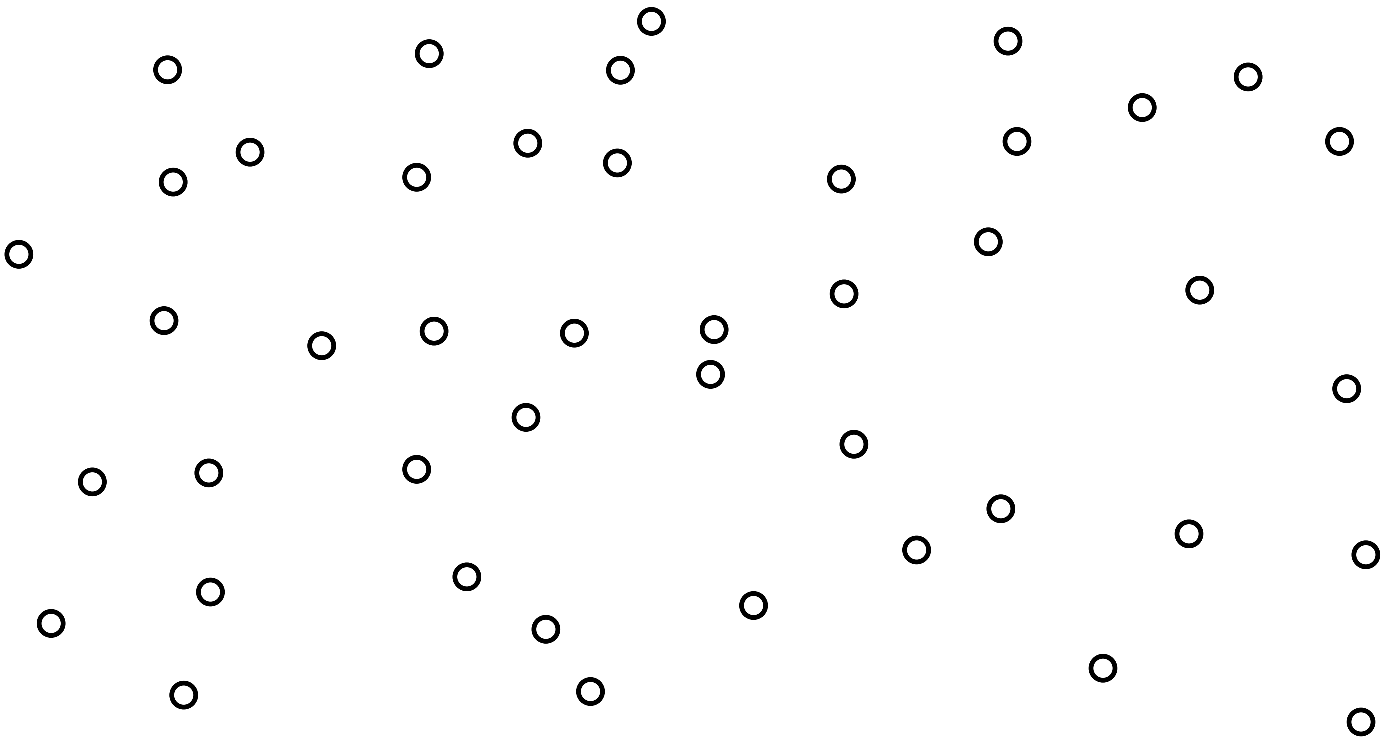


Computing R_2

- Take random set $R \subset P$ of size $\Theta(s)$
- Compute and triangulate $VD(R)$
- Compute sizes of conflict sets B_v for $v \in VD(R)$



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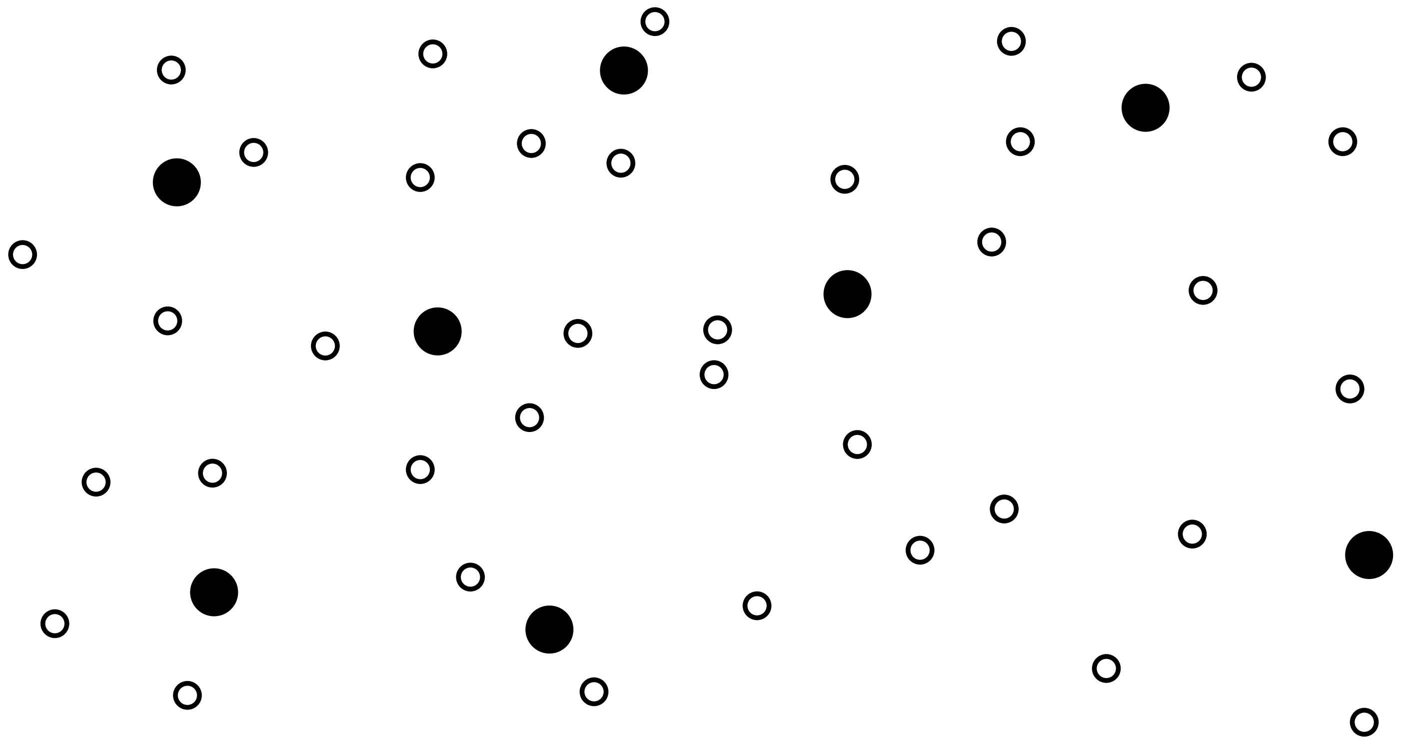

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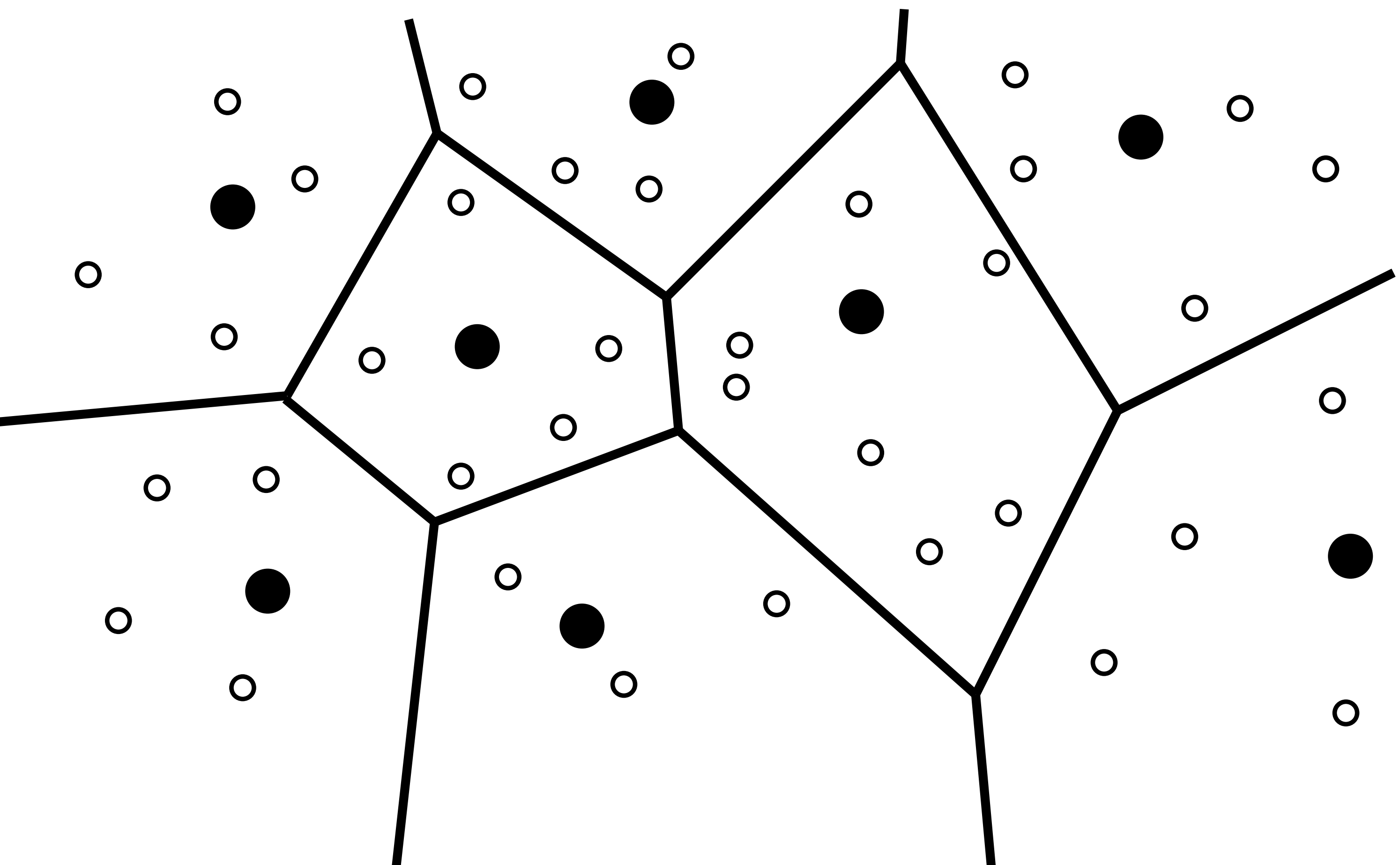

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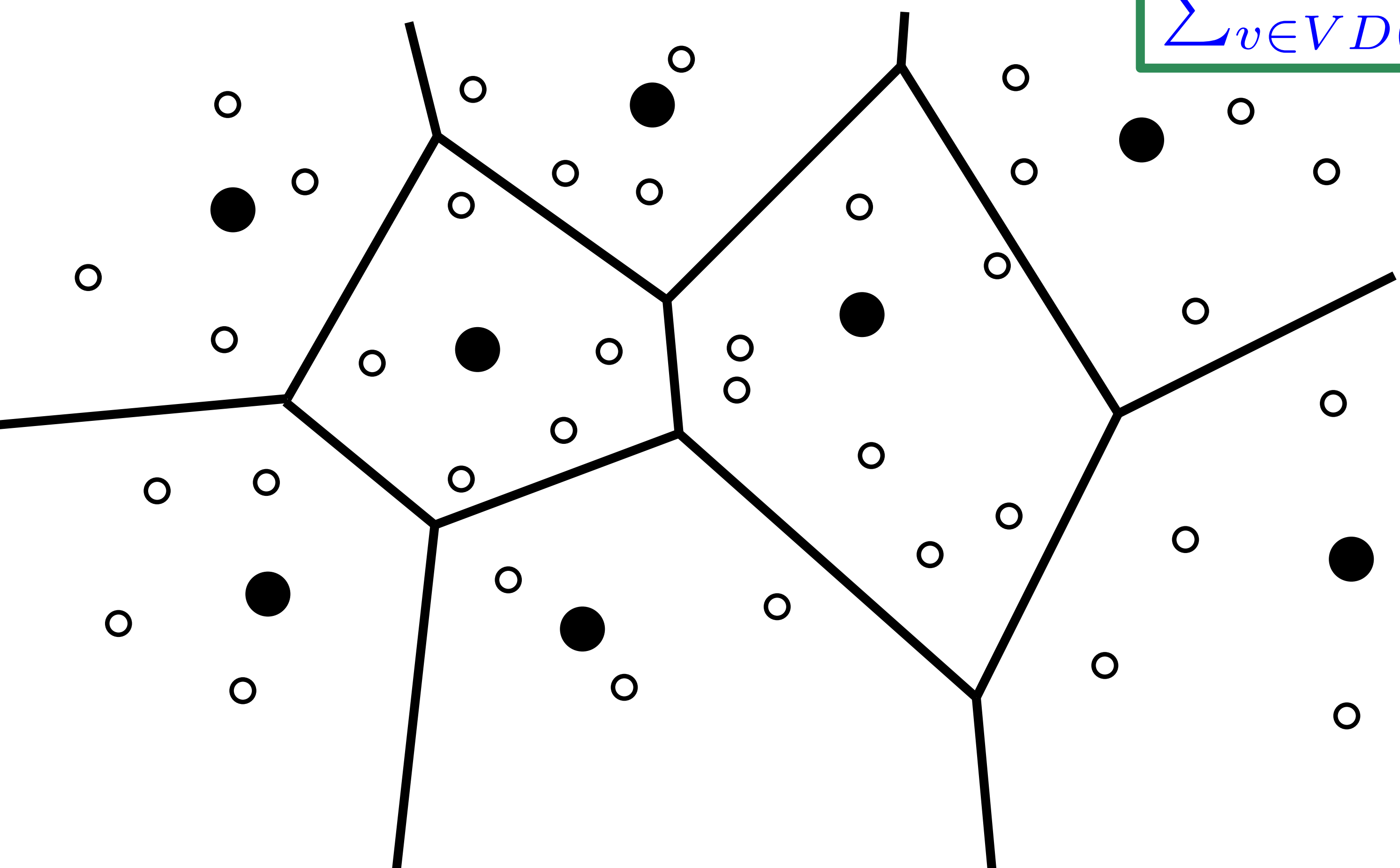
- If size of conflict sets is too large, restart

$$\boxed{\begin{gathered} t_v = |B_v| \cdot s/n \\ \sum_{v \in VD(R)} t_v \log t_v = O(s) \end{gathered}}$$



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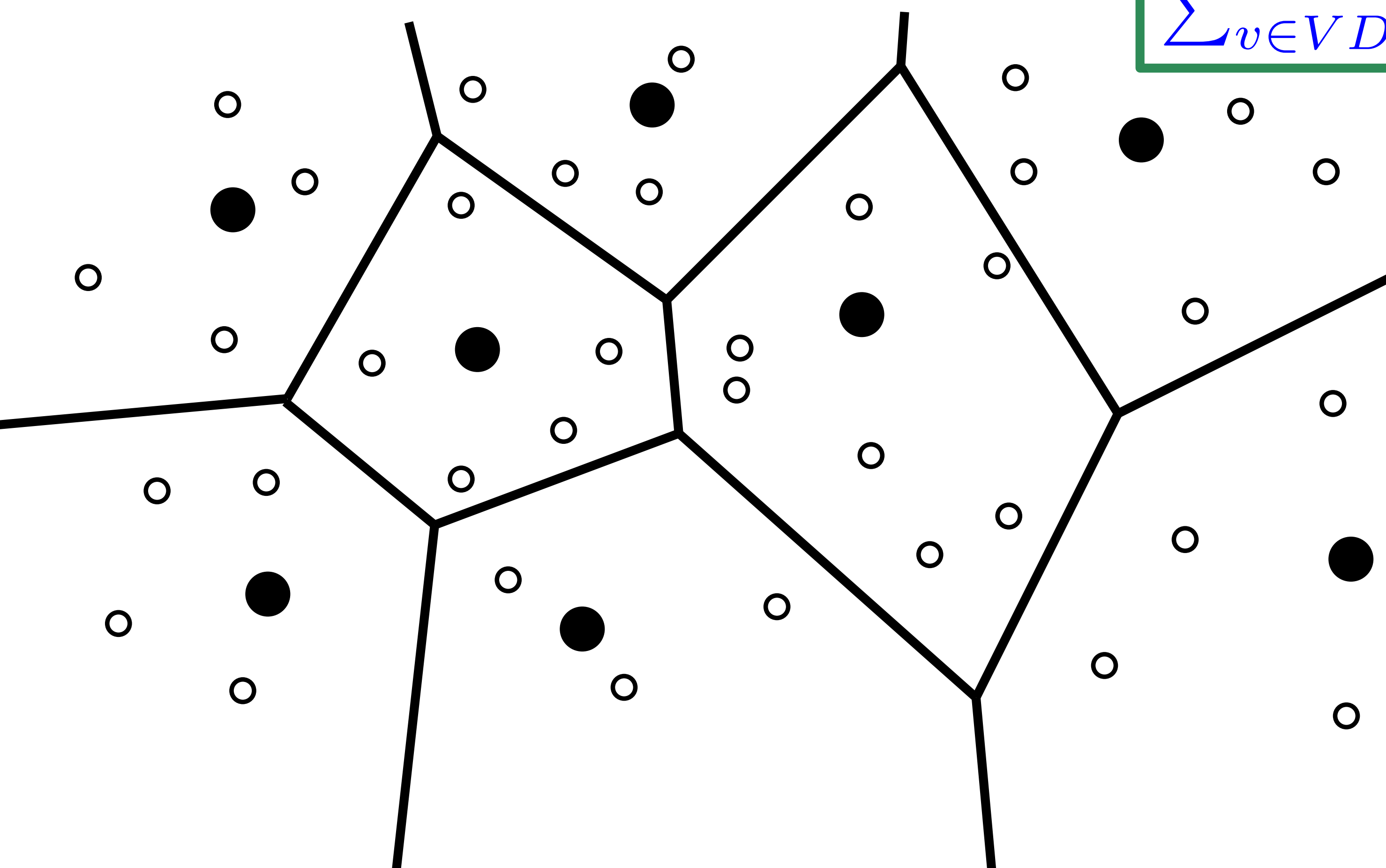
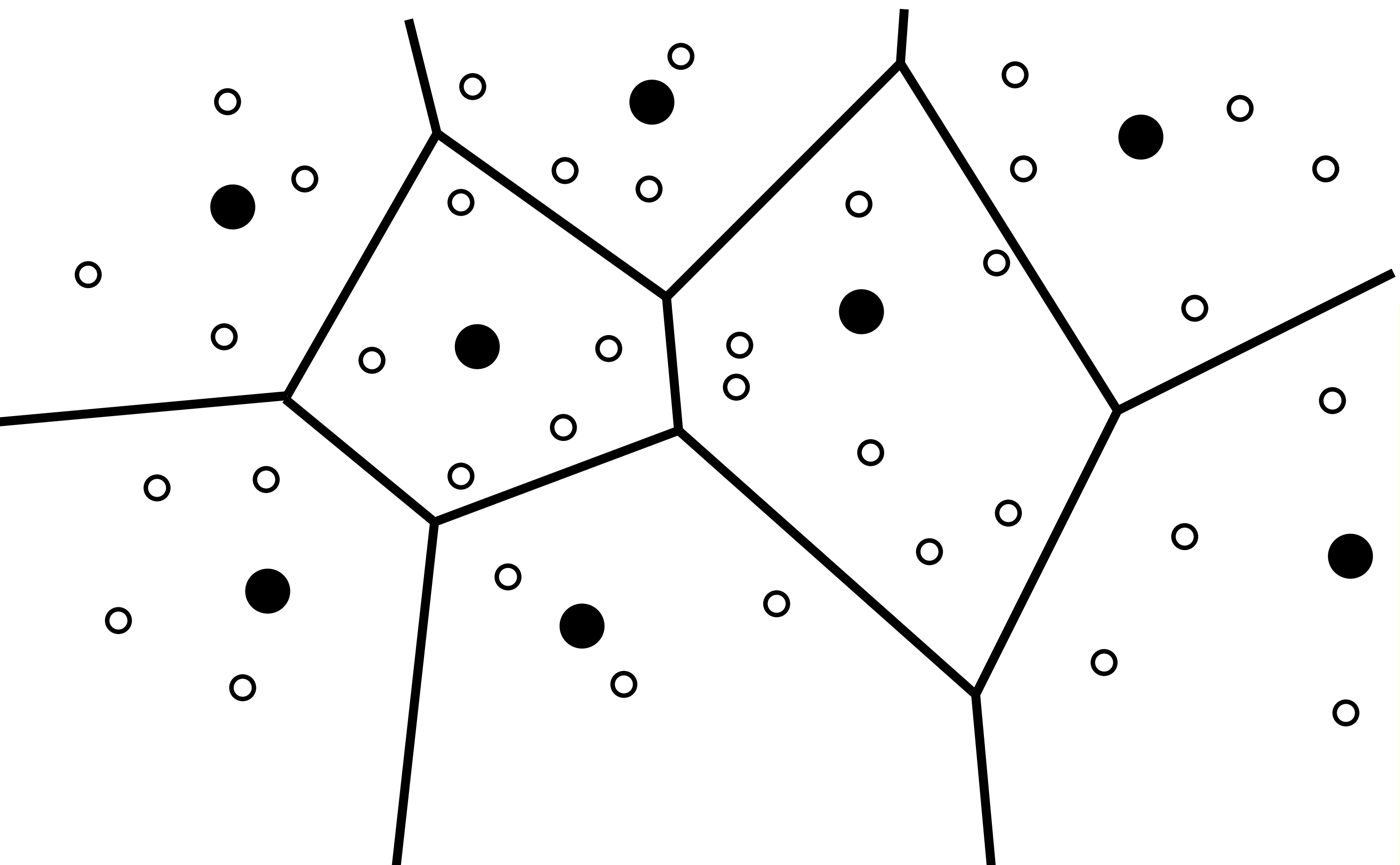


time to sample:
$O(n + s \log s)$
count conflict size:
$O(n \log s)$
total:
$O(n \log s)$

Computing R_2

Problem: For some $v \in VD(R)$ we may have $B_v \gg n/s$

Solution:
- Sample $\Theta(t_v \log t_v)$ extra points from B_v for any $v \in VD(R)$ with $t_v \geq 2$.
- Recompute conflict sizes
- Continue sampling in large conflict sets



1 sampling round: $O(n \log s + s \log s)$
expected #rounds: $O(\log^* s)$
total: $O(n \log s \log^* s)$

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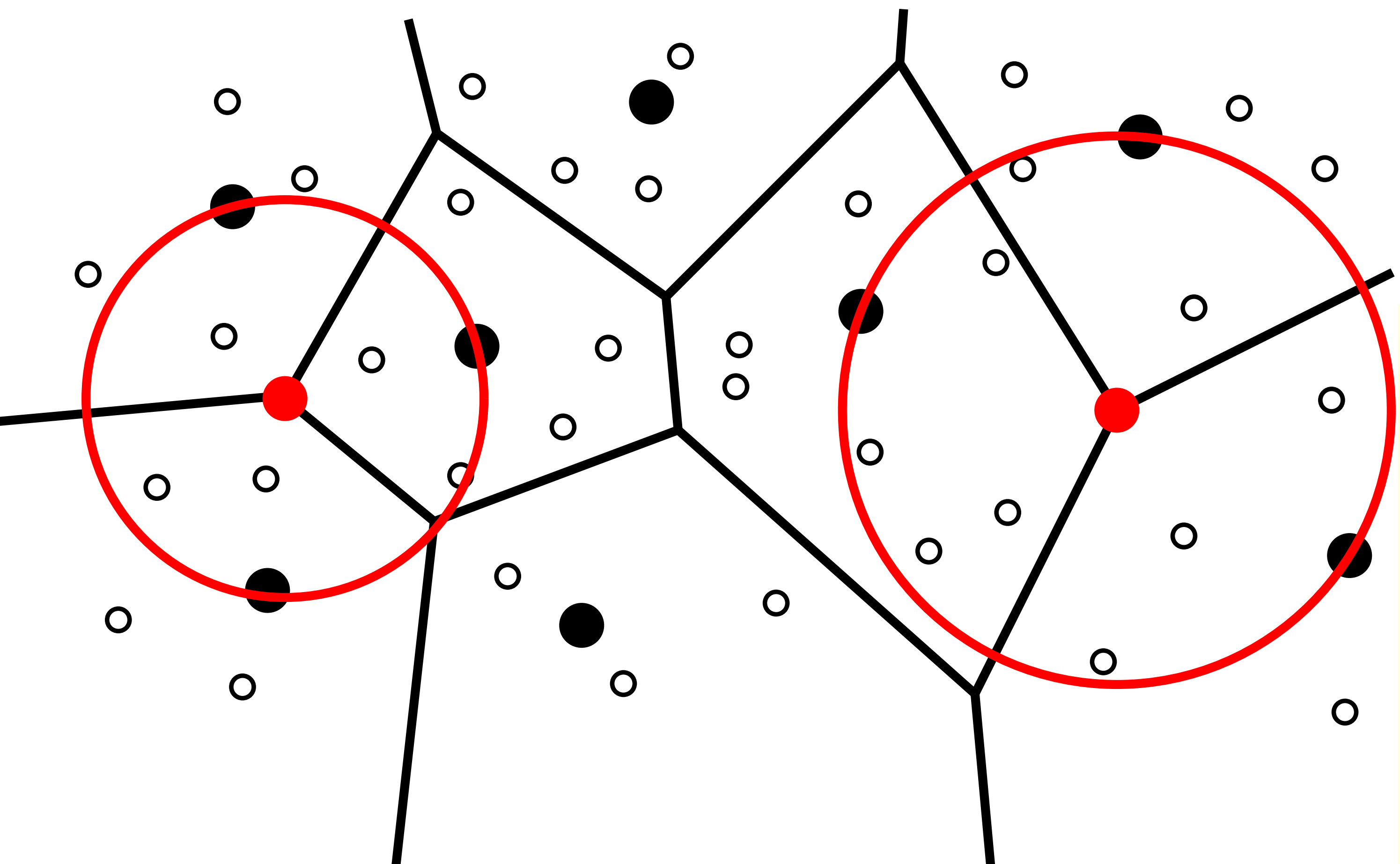

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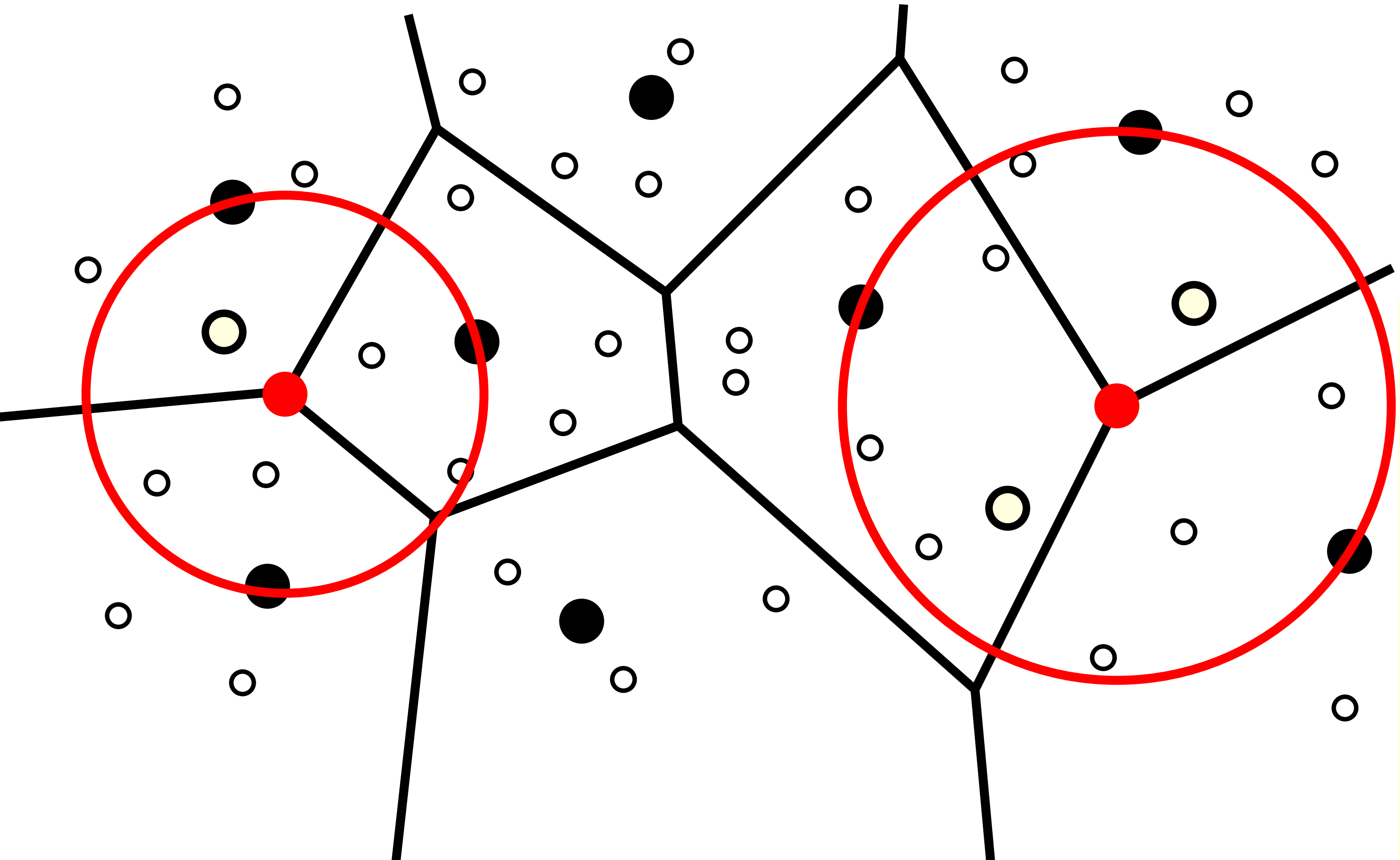

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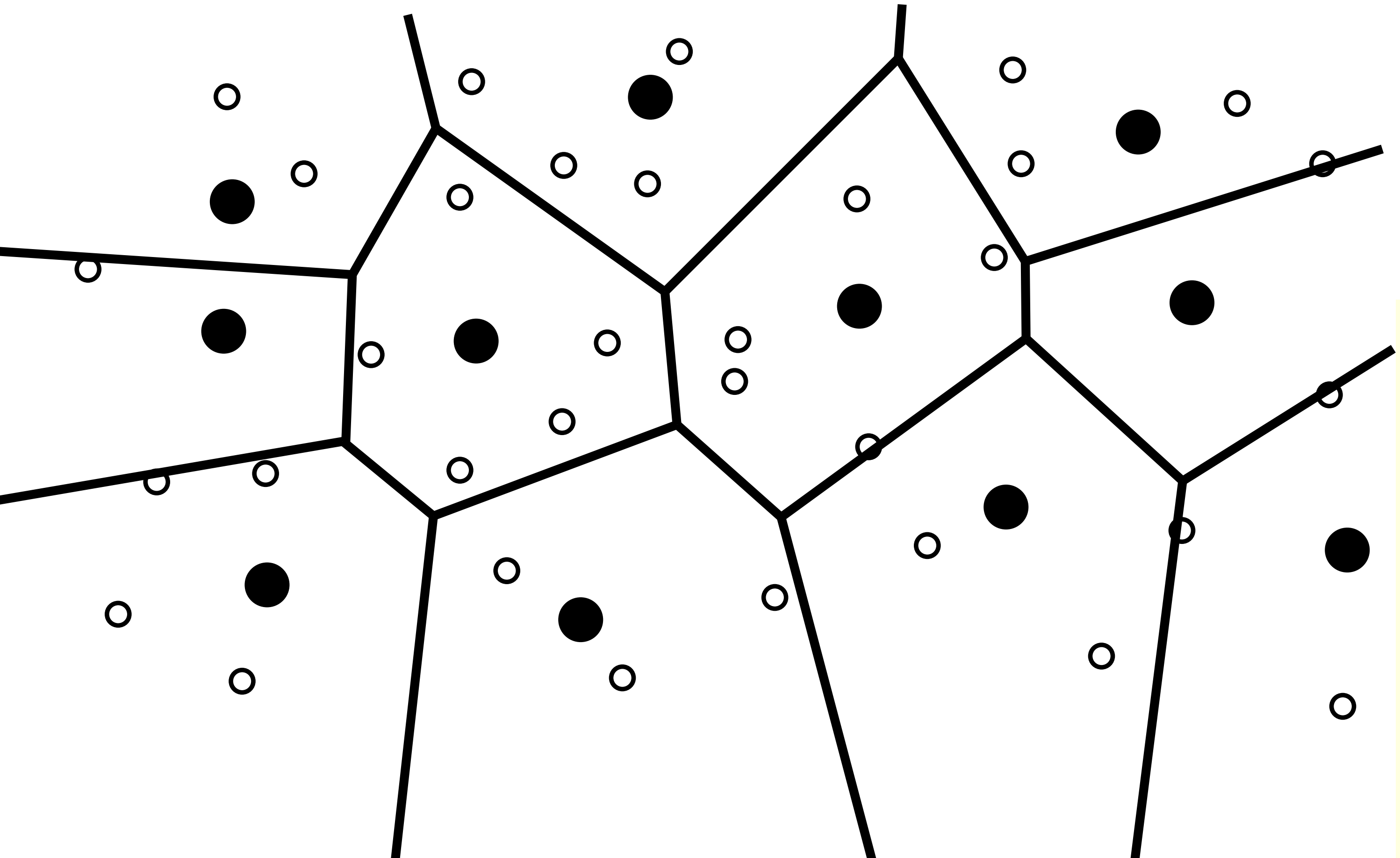

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Putting it together

Computing R_2: expected $O(n \log s \log^* s)$

Computing for each triangle: expected $O((n^2/s) \log s)$

$$\Downarrow$$

$\boxed{\text{Reporting Voronoi diagrams of a set of } n \text{ points in the plane can be done in } O((n^2/s)\log s + n \log s \log^* s) \text{ expected time.}}$

(Almost optimal for both linear and constant memory)

Open Problem: Can we do the same in worst-case time?