VARIATIONAL FORMULATION OF RATE- AND STATE-DEPENDENT FRICTION PROBLEMS

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Abstract. We propose a variational formulation of rate- and state-dependent models for the dynamic sliding of a linearly elastic block on a rigid surface in terms of two coupled variational inequalities. Classical Dieterich–Ruina models are covered as special cases. We show existence and uniqueness of solutions for the two spatial subproblems arising from time discretisation. Existence of solutions to the coupled spatial problems is established for Dieterich's state equation through a fixed point argument. We conclude with some numerical experiments that suggest mesh independent convergence of the underlying fixed point iteration, and illustrate quasiperiodic occurrence of stick/slip events.

1. Introduction

The Dieterich–Ruina model of rate- and state-dependent friction (RSF) [29] has become a standard for frictional behaviour of solids, in particular in the earth sciences [4, 21, 28]. It is motivated by so-called velocity stepping tests, in which a block is slid along a foundation and subjected to abrupt changes in sliding velocity (see Figure 1.1). The evolution of the coefficient of friction in such tests reveals two effects: A direct increase/decrease that counteracts the increasing/decreasing sliding velocity, and a relaxation effect, similar to the behaviour of viscoelastic solids.

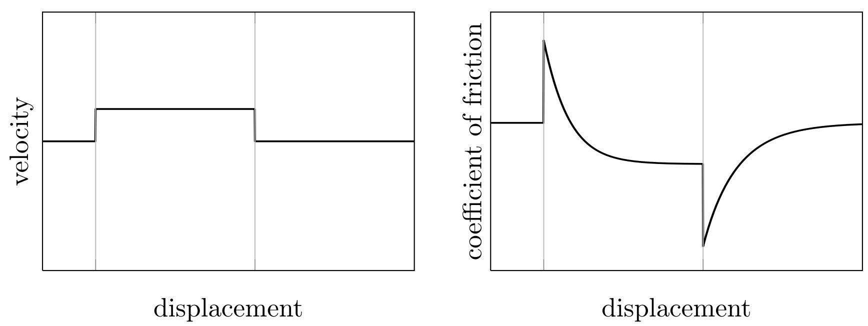

Figure 1.1. An idealised velocity stepping test

In RSF models, the sliding velocity is often called *slip rate*. The direct increase/decrease of the coefficient of friction μ is accounted for through a slip rate-dependence of μ, and the relaxation effect is captured by an additional state variable. While direct slip rate-dependence gives rise to intrinsic instability of stick/slip events, state-dependence of μ has a smoothing effect on the evolution.

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Simpler models, where increasing velocity leads to decreasing friction coefficients directly, rather than with a time delay, have been found to lead to ill-posed problems [11].

RSF models are inherently coupled, since the evolution of the state variable depends on the slip rate through a suitable equation of state, usually a pointwise ordinary differential equation, and the slip rate in turn depends on the state through a continuum mechanical problem involving the rate- and state-dependent friction coefficient.

In spite of widespread practical applications of RSF, the mathematical properties of this model class have hardly been studied. The most thoroughly investigated setup appears to be the spring–block slider with a single degree of freedom [18, 22, 25, 27]. Also, the stability of sliding between two elastic half-spaces has been analysed in [24].

In this paper, we consider dynamic sliding of a linearly elastic body on a rigid surface. Our model involves Newton's second law together with subdifferential inclusions for friction and state evolution. The variational formulation of this general approach amounts to two variational inequalities that describe the evolution of the slip rate and state, respectively. Special cases include Tresca friction [9, 19] and natural extensions of the classical Dieterich–Ruina model [29]. These extensions include non-smooth evolution of state as well as vanishing velocities, which have been treated by means of regularisation in previous simulations [5, 7]. The time-dependent variational inequalities for slip rate and state are discretised in time with the classical Newmark scheme and the backward Euler method, respectively. As a result, two coupled convex minimisation problems have to be solved in each time step. Similar spatial problems would be obtained from other implicit time discretisations. We show existence and uniqueness of solutions for each of these subproblems, so that a corresponding fixed point iteration is well defined. Existence of a fixed point is established in the special case of the Dieterich–Ruina model with Dieterich's state equation. We emphasise that corresponding variants of our theoretical results readily extend to a quasistatic variant of the model.

In our numerical experiments, we use piecewise linear and piecewise constant finite elements for the approximation of velocity and state, respectively. Both for Dieterich's and Ruina's state equation, our numerical computations suggest mesh-independent convergence rates of a discrete version of the fixed point iteration mentioned above. The resulting approximate displacements and velocities eventually enter a regime of quasi-periodic slip events as expected. We observe grid convergence for a fixed spatial mesh and sufficiently high temporal resolution.

2. Rate- and State-Dependent Friction

2.1. **Variational Rate- and State-Dependent Friction.** We consider dynamic sliding of a linearly elastic body on a rigid surface. The body shall be represented by a bounded domain Ω in $\mathbb{R}^d$ with Lipschitz boundary. Here, d stands for the spatial dimension. We assume the boundary of Ω to consist of three subsets Γ_D, Γ_N, and Γ_F with disjoint relative interiors (Figure 2.1). The letter n is used for the unit outer normal vector of Ω wherever it is defined.

Suppose that a body force f acts on all of Ω and a surface force f_N acts on the Neumann boundary section Γ_N. We write $u(x,t)$ for the displacement field, which we assume to be prescribed on the Dirichlet boundary section Γ_D. On the remaining section Γ_F we require the tangential displacement to obey a friction law to be described below. We also assume bilateral contact on Γ_F, i.e. no displacement in the normal direction. This implies that we need not distinguish between the displacement u and its tangential projection u_t on Γ_F.

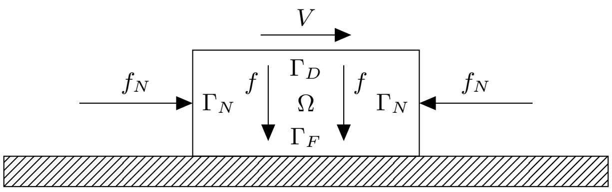

FIGURE 2.1. A model slider

We impose a rate- and state-dependent friction law of the form

$$-\sigma_t \in \partial_{\dot u}\phi(\dot u, \alpha),$$

where $\partial_{\dot u}\phi(\dot u, \alpha)$ denotes the subdifferential of a convex function $\phi(\,\cdot\,, \alpha)$ [26]. Here, we wrote $\sigma_t := \boldsymbol{\sigma} n - (\boldsymbol{\sigma} n \cdot n)n$ for the tangential component of the stress field on the boundary, defined through the stress tensor $\boldsymbol{\sigma}$. The evolution of the solution-dependent state variable α is given by

$$-\dot\alpha \in \partial_\alpha \psi(\alpha, |\dot u|)$$

with a second convex function $\psi(\,\cdot\,, |\dot u|)$.
In summary, we consider the following abstract problem of RSF.

Problem. *Find $u\colon \Omega \times [0,T] \to \mathbb{R}^d$ and $\alpha\colon \Gamma_F \times [0,T] \to \mathbb{R}$ such that*

(2.1) $\quad \boldsymbol{\sigma}(u) = \boldsymbol{\mathcal{C}}\boldsymbol{\varepsilon}(u) \quad$ *in* Ω $\quad$ *(linear elasticity)*

(2.2) $\quad \operatorname{Div}\boldsymbol{\sigma}(u) + f = \rho\ddot u \quad$ *in* Ω $\quad$ *(balance of momentum)*

with boundary conditions

$u = 0 \quad$ *on* Γ_D

$\boldsymbol{\sigma}(u)n = f_N \quad$ *on* Γ_N

$u \cdot n = 0 \quad$ *on* Γ_F

(2.3) $\quad -\sigma_t \in \partial_{\dot u}\phi(\dot u, \alpha) \quad$ *on* Γ_F $\quad$ *(friction law)*

and such that α satisfies

(2.4) $\quad -\dot\alpha \in \partial_\alpha\psi(\alpha, |\dot u|) \quad$ *on* Γ_F $\quad$ *(state evolution)*

for all $t \in [0,T]$, where $\rho > 0$ is the constant material density, $\boldsymbol{\mathcal{C}}$ is the tensor of elasticity, and $\boldsymbol{\varepsilon}$ is the linearised strain tensor. In addition, we impose initial conditions on the displacement u, velocity $\dot u$, and state α.

Remark. We have assumed homogeneous Dirichlet boundary conditions, i.e., $u = 0$ on Γ_D. This assumption serves mainly to simplify the presentation; in Section 6, we consider a numerical experiment with inhomogeneous Dirichlet boundary conditions.

Assuming that the state α is known, this problem can be written as a variational inequality for $u \in H^1((0,T), H) \cap H^2((0,T), H^*)$ with

$$H := \{w \in H^1(\Omega)^d : w|_{\Gamma_D} = 0,\ w_n|_{\Gamma_F} = 0\}.$$

Consider the balance of momentum equation (2.2). After testing with $v - \dot u(t)$, $v \in H$, and using (2.1) as well as the boundary conditions for u and u_n, we obtain

$$\langle \rho\ddot u, v - \dot u\rangle + a(u, v - \dot u) = \ell(v - \dot u) + \int_{\Gamma_F} \sigma_t(u)\cdot(v - \dot u) \quad \forall v \in H, \tag{2.5}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the dual pairing of H and H^*. Also, we have set

$$a(v,w) := \int_\Omega \boldsymbol{\mathcal{C}}\boldsymbol{\varepsilon}(v) : \boldsymbol{\varepsilon}(w) \quad \text{as well as} \quad \ell(v) := \int_\Omega f\cdot v + \int_{\Gamma_N} f_N \cdot v.$$

We now recall that (2.3) can be equivalently written as

$$\sigma_t(x) \cdot [v - \dot{u}(x)] + \phi(v, \alpha) \geq \phi(\dot{u}(x), \alpha) \quad \forall v \in \mathbb{R}^d \tag{2.6}$$

at any point $x \in \Gamma_F$. Testing (2.6) with traces of functions from H yields the weaker form

$$\Phi(v, \alpha) \geq \Phi(\dot{u}, \alpha) - \int_{\Gamma_F} \sigma_t \cdot (v - \dot{u}) \quad \forall v \in H$$

with

$$\Phi(v, \alpha) := \int_{\Gamma_F} \phi(v, \alpha).$$

Here, we assume ϕ to be chosen such that Φ is well-defined. Combined with (2.5) this leads to a variational formulation of the elastic problem with given state α.

Problem (R)**.** *For given α, find $u \in H^1((0,T), H) \cap H^2((0,T), H^*)$ such that for almost every $t \in [0,T]$, we have*

$$\langle \rho\ddot{u}, v - \dot{u} \rangle + a(u, v - \dot{u}) + \Phi(v, \alpha) \geq \Phi(\dot{u}, \alpha) + \ell(v - \dot{u}) \quad \forall v \in H.$$

Analogously, we can formulate a variational problem for the state variable α under the assumption that $|\dot{u}|$ is known. To that end, we test (2.4) with functions $\beta \in L^2(\Gamma_F)$ and define a functional through

$$\Psi(\beta, V) = \int_{\Gamma_F} \psi(\beta, V),$$

with ψ such that Ψ is well-defined, to obtain the following variational formulation of (2.4).

Problem (S)**.** *For given $|\dot{u}|$, find $\alpha \in H^1((0,T), L^2(\Gamma_F))$ such that for almost every $t \in [0,T]$, we have*

$$(\dot{\alpha}, \beta - \alpha)_{L^2(\Gamma_F)} + \Psi(\beta, |\dot{u}|) \geq \Psi(\alpha, |\dot{u}|) \quad \forall \beta \in L^2(\Gamma).$$

Here, $(\,\cdot\,,\,\cdot\,)_{L^2(\Gamma_F)}$ denotes the scalar product in $L^2(\Gamma_F)$. The variational formulation of the coupled rate- and state-dependent friction problem finally reads

Problem (RSF)**.** *Find $u \in H^1((0,T), H) \cap H^2((0,T), H^*)$ and $\alpha \in H^1((0,T), L^2(\Gamma_F))$ such that for almost every $t \in [0,T]$, we have*

$$\begin{aligned} \langle \rho\ddot{u}, v - \dot{u} \rangle + a(u, v - \dot{u}) + \Phi(v, \alpha) &\geq \Phi(\dot{u}, \alpha) + \ell(v - \dot{u}) && \forall v \in H, \\ (\dot{\alpha}, \beta - \alpha)_{L^2(\Gamma_F)} + \Psi(\beta, |\dot{u}|) &\geq \Psi(\alpha, |\dot{u}|) && \forall \beta \in L^2(\Gamma). \end{aligned}$$

2.2. **Tresca Friction.** Coulomb friction [9, 19] postulates that tangential stress σ_t and velocity $\dot{u}$ are related according to

$$-\sigma_t = \mu|\sigma_n|\frac{\dot{u}}{|\dot{u}|} \quad \text{if } \dot{u} \neq 0 \quad \text{and} \quad |\sigma_t| \leq \mu|\sigma_n| \quad \text{if } \dot{u} = 0, \tag{2.7}$$

with a given friction coefficient $\mu \geq 0$. Tresca friction is obtained by replacing the solution-dependent normal stress $\sigma_n < 0$ by a given parameter $\bar{\sigma}_n$. Since the subdifferential of the Euclidean norm $|\cdot|$ is given by

$$\partial|\cdot|(\dot{u}) = \begin{cases} \{\dot{u}/|\dot{u}|\} & \text{if } \dot{u} \neq 0 \\ \{x \in \mathbb{R}^d \colon |x| \leq 1\} & \text{if } \dot{u} = 0, \end{cases}$$

the friction law (2.7) with $\sigma_n = \bar{\sigma}_n$ can be equivalently written in the form (2.3). The convex function ϕ is then given by

$$\phi(\dot{u}) = \mu|\bar{\sigma}_n||\dot{u}|. \tag{2.8}$$

A state-dependent extension of classical Tresca friction (2.8) can be introduced by replacing $\phi(\dot{u})$ with $\phi(\dot{u}, \alpha) = \mu(\alpha)|\bar{\sigma}_n||\dot{u}|$, involving a state-dependent friction coefficient $\mu(\alpha)$ and a state evolution law of the form (2.4).

2.3. **The Dieterich–Ruina Model.** The Dieterich–Ruina model of RSF in its most common form goes back to [29] and consists of two parts: An equation that relates the coefficient of friction μ to the slip rate $V := |\dot{u}|$ and a state θ, as well as a state evolution equation. The former is most commonly stated as [1, 3, 10, 20, 23, 30]

$$\mu := \frac{|\sigma_t|}{|\sigma_n|} = \mu_0 + a\log\left(\frac{V}{V_0}\right) + b\log\left(\frac{V_0\theta}{L}\right) \tag{2.9}$$

with positive parameters μ_0, a, b, V_0, and $L \in \mathbb{R}$.

For the second equation, multiple proposals have been made. The two most popular laws are given by

$$\dot{\theta} = 1 - \frac{V}{L}\theta \qquad \text{(Dieterich's law)}$$

and

$$\dot{\theta} = -\frac{V}{L}\theta\log\left(\frac{V}{L}\theta\right) \qquad \text{(Ruina's law).}$$

Both can be used to describe some phenomena but not others [1, 20].

As a common feature, both of these state equations provide increasing state θ for small slip rate V and vice versa. A variety of state equations with this characteristic property might be useful. Consider, e.g., the most simple but non-smooth law

$$\dot{\theta} = \begin{cases} +V & \text{if } \theta V < 1 \\ -V & \text{if } \theta V > 1 \end{cases} \quad \text{and} \quad |\dot{\theta}| \leq V \quad \text{if } \theta V = 1. \tag{2.10}$$

We replace the solution-dependent normal stress σ_n in (2.9) with a parameter $\bar{\sigma}_n$ (as is done in Tresca friction) and assume collinearity of velocity and stress, i.e.,

$$-|\sigma_t|\dot{u} = |\dot{u}|\sigma_t. \tag{2.11}$$

Then the Dieterich–Ruina model becomes a special case of the framework set forth in Section 2.1.

To show that, we first relate (2.9) and (2.3). It is obvious that (2.9) is not meaningful for very low velocities V, since for fixed θ and $V \to 0$ the right-hand side tends to $-\infty$, whereas the left-hand side remains non-negative. This problem has been circumvented in the literature by means of regularisation [5, 7]; in what follows, we follow a variational approach.

To give a precise bound from which on velocities become inadmissible to (2.9), we set

$$V_m(\theta) := V_0\exp\left(-\frac{\mu_0 + b\log(\theta V_0/L)}{a}\right),$$

so that (2.9) becomes

$$|\sigma_t|/|\bar{\sigma}_n| = a\log(V/V_0) - a\log(V_m/V_0).$$

This formulation makes it clear that we must have $V \geq V_m$ for (2.9) to make sense. A straightforward extension of (2.9) to velocities that fall short of V_m is given by

$$\mu = \begin{cases} \mu_0 + a\log(V/V_0) + b\log(V_0\theta/L) & \text{if } V \geq V_m(\theta) \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

In conjunction with the collinearity assumption (2.11), this expression can be reformulated as a subdifferential inclusion of type (2.3). Indeed, for $V \geq V_m(\theta)$, we have

$$|\sigma_t| = a|\bar{\sigma}_n|\log(V/V_m(\theta)) = \frac{\partial\varphi}{\partial V}(V,\theta),$$

where φ is given by

$$\varphi(V,\theta) := a|\bar{\sigma}_n|[V\log(V/V_m(\theta)) - V + V_m(\theta)], \quad V \geq V_m(\theta).$$

This function is convex and non-decreasing in V, a property that is shared by the extension

$$\varphi(V,\theta) = \begin{cases} a|\bar\sigma_n|[V\log(V/V_m(\theta)) - V + V_m(\theta)] & \text{if } V \geq V_m(\theta) \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

corresponding to (2.12), too, since $\varphi(\,\cdot\,,\theta)$ and $\partial\varphi/\partial V(\,\cdot\,,\theta)$ vanish at $V_m(\theta)$. If we now define

$$\phi(\,\cdot\,,\theta) \coloneqq \varphi(|\cdot|,\theta), \tag{2.14}$$

we obtain

$$-\sigma_t = \frac{\partial\varphi}{\partial V}(V,\theta)\frac{\dot u}{|\dot u|} = \frac{\partial\phi}{\partial\dot u}(\dot u,\theta)$$

by virtue of (2.11) and the chain rule, and thus a smooth case of the subdifferential inclusion (2.3) as desired.

It remains to be shown that the evolution of θ can be written as a subdifferential inclusion of type (2.4). If we set $\alpha \coloneqq \log\theta$ and rewrite Dieterich's law in terms of α, it becomes

$$-\dot\alpha = \frac{V}{L} - e^{-\alpha} = \frac{d}{d\alpha}\psi_d(\alpha,V) \quad \text{with } \psi_d(\alpha,V) = \frac{V}{L}\alpha + e^{-\alpha}. \tag{2.15}$$

With the same substitution, Ruina's law turns into $\dot\alpha = -V/L(\alpha + \log(V/L))$, or

$$-\dot\alpha = \frac{d}{d\alpha}\psi_r(\alpha,V) \quad \text{with } \psi_r(\alpha,V) = \frac{V}{L}\left(\frac{1}{2}\alpha^2 + \log\left(\frac{V}{L}\right)\alpha\right). \tag{2.16}$$

For the discontinuous law (2.10), we set $\alpha \coloneqq \theta$ to obtain

$$-\dot\alpha \in \partial_\alpha\psi_{dc}(\alpha,V) \quad \text{with} \quad \psi_{dc}(\alpha,V) = |\alpha V - 1|. \tag{2.17}$$

Since the functions $\psi_d(\,\cdot\,,V)$, $\psi_r(\,\cdot\,,V)$, and $\psi_{dc}(\,\cdot\,,V)$ are convex, the corresponding state equations are again special cases of (2.4).

3. Time-Discretisation

As a first step towards the numerical solution of the coupled variational Problem (RSF) stated in Subsection 2.1, we now consider time-discretisations of the Subproblems (R) and (S). For simplicity, we assume the interval $[0,T]$ to be partitioned uniformly into N subintervals $[t_{n-1},t_n]$, each of length $\tau = T/N$.

To Subproblem (R) we apply the classical Newmark scheme, which we can write as

$$\dot u_n = \dot u_{n-1} + \frac{\tau}{2}(\ddot u_{n-1} + \ddot u_n) \tag{3.1}$$

$$u_n = u_{n-1} + \tau\dot u_{n-1} + \frac{\tau^2}{4}(\ddot u_{n-1} + \ddot u_n) \tag{3.2}$$

for the spatial approximations $u_n \coloneqq u(t_n)$ and $0 < n \leq N$. Note that we can also write (3.1) as

$$\ddot u_n = \frac{2}{\tau}(\dot u_n - \dot u_{n-1}) - \ddot u_{n-1}, \tag{3.3}$$

which we can insert into (3.2) to obtain

$$u_n = u_{n-1} + \frac{\tau}{2}(\dot u_n + \dot u_{n-1}). \tag{3.4}$$

It is easy to see that an application of Newmark's method in this form to Subproblem (R) leads to a variational inequality over H where the sole unknown is $\dot u_n$:

$$a_\tau(\dot u_n, v - \dot u_n) + \Phi(v,\alpha) \geq \Phi(\dot u_n,\alpha) + \ell_n(v - \dot u_n) \quad \forall v \in H$$

with

$$a_\tau(\,\cdot\,,\,\cdot\,) \coloneqq \frac{2}{\tau}(\rho\,\cdot\,,\,\cdot\,)_{L^2(\Omega)} + \frac{\tau}{2}a(\,\cdot\,,\,\cdot\,)$$

and

$$\ell_n(\,\cdot\,) \coloneqq \ell(\,\cdot\,) + \left(\rho\left(\frac{2}{\tau}\dot u_{n-1} + \ddot u_{n-1}\right),\,\cdot\,\right)_{L^2(\Omega)} - a\Big(u_{n-1} + \frac{\tau}{2}\dot u_{n-1},\,\cdot\,\Big).$$

This variational problem for $\dot u_n$ can also be written as a minimisation problem; the corresponding energy functional $\mathcal{J}(\,\cdot\,,\alpha)$ is given by

$$\mathcal{J}(v,\alpha) = \tfrac{1}{2}a_\tau(v,v) + \Phi(v,\alpha) - \ell_n(v).$$

The Newmark time-discretisation of Subproblem (R) thus leads to the following spatial problems.

Problem (R_τ)**.** *For given state α, find $\dot u_n \in H$ such that*

$$\mathcal{J}(\dot u_n,\alpha) \le \mathcal{J}(v,\alpha) \quad \forall v \in H.$$

The displacement u_n can then be computed from $\dot u_n$ using (3.4)*.*

Next, we apply the backward Euler scheme to the L^2-gradient flow Subproblem (S). The spatial approximations $\alpha_n \coloneqq \alpha(t_n)$, $0 < n \le N$, then satisfy the variational inequality

$$(\alpha_n, \beta - \alpha_n)_{L^2(\Gamma_F)} + \tau\Psi(\beta,|\dot u|) \ge \tau\Psi(\alpha_n,|\dot u|) + (\alpha_{n-1}, \beta - \alpha_n)_{L^2(\Gamma_F)} \tag{3.5}$$

for all $\beta \in L^2(\Gamma_F)$. Since (3.5) can be equivalently written as a minimisation problem for the convex energy functional $\mathcal{E}(\,\cdot\,,|\dot u|)$ given by

$$\mathcal{E}(\beta,|\dot u|) = \tfrac{1}{2}(\beta,\beta)_{L^2(\Gamma_F)} + \tau\Psi(\beta,|\dot u|) - (\alpha_{n-1},\beta)_{L^2(\Gamma_F)},$$

we obtain the following spatial problem in each time step.

Problem (S_τ)**.** *For given slip rate $V = |\dot u|$, find $\alpha_n \in L^2(\Gamma_F)$ such that*

$$\mathcal{E}(\alpha_n, V) \le \mathcal{E}(\beta, V) \quad \forall \beta \in L^2(\Gamma_F).$$

The spatial problems of the time-discretised coupled Problem (RSF) finally read as follows.

Problem (RSF_τ)**.** *Find $\dot u_n \in H$ and $\alpha_n \in L^2(\Gamma_F)$ such that*

$$\mathcal{J}(\dot u_n,\alpha_n) \le \mathcal{J}(v,\alpha_n) \quad \forall v \in H$$
$$\mathcal{E}(\alpha_n,|\dot u_n|) \le \mathcal{E}(\beta,|\dot u_n|) \quad \forall \beta \in L^2(\Gamma_F).$$

4. Existence and Uniqueness of Solutions of the Spatial Subproblems

Existence and uniqueness of solutions of the spatial Subproblems (R_τ) and (S_τ) will be derived from the following general results on convex minimisation and superposition operators.

Lemma 4.1. *Let $\mathcal{V}$ be a Hilbert space. If we assume that $b(\,\cdot\,,\,\cdot\,)$ is a symmetric, continuous and $\mathcal{V}$-elliptic bilinear form, $j\colon \mathcal{V} \to \mathbb{R}\cup\{+\infty\}$ a proper, convex, and lower semicontinuous functional, and l a bounded linear functional on $\mathcal{V}$, then the problem of minimising*

$$v \mapsto \tfrac{1}{2}b(v,v) + j(v) - l(v)$$

over $\mathcal{V}$ admits a unique solution.

Proof. See [14, Lemma 4.1]. □

It is a straightforward consequence of Fatou's Lemma that integral operators preserve lower semicontinuity.

Lemma 4.2. *Assume that $f\colon \Gamma_F \times \mathbb{R}^d \to \mathbb{R}$ is a non-negative function, such that $f(x,\,\cdot\,)$ is lower semicontinuous for almost every $x \in \Gamma_F$. Then the superposition operator*

$$\int_{\Gamma_F} f(x,\,\cdot\,)\,dx\colon L^2(\Gamma_F) \to \mathbb{R} \cup \{+\infty\}$$

is lower semicontinuous.

Proof. See [12, Theorem 6.49]. □

Now we are ready to show existence and uniqueness of a solution of Subproblem (R_τ) under the additional assumption that

(A.1) $\boldsymbol{\mathcal{C}}$ is elliptic, i.e. there exists a $c > 0$ such that a.e. in Ω, we have

$$\boldsymbol{\mathcal{C}}\boldsymbol{\tau}\colon \boldsymbol{\tau} \geq c|\boldsymbol{\tau}|^2 \quad \text{for every symmetric tensor } \boldsymbol{\tau}.$$

Proposition 4.3. *Assume that (A.1) holds, u_{n-1}, $\dot{u}_{n-1}$, $\ddot{u}_{n-1} \in H$, and $\ell \in H^*$. Then the Subproblem (R_τ) with ϕ corresponding to Tresca friction* (2.8) *or the Dieterich–Ruina model* (2.14) *has a unique solution for any given state $\alpha \in L^2(\Gamma_F)$.*

Proof. Both for Tresca friction (2.8) and the Dieterich–Ruina model (2.14) the superposition operator $\Phi(\,\cdot\,,\alpha)$ is convex, because so is $\phi(\,\cdot\,,\alpha)$. It is proper, because $\phi(0,\alpha) = 0$. Since in both cases ϕ is continuous and non-negative, Lemma 4.2 implies that $\Phi(\,\cdot\,,\alpha)$ is lower semicontinuous. The bilinear form $a_\tau(\,\cdot\,,\,\cdot\,)$ is symmetric, continuous, and, by assumption (A.1) in conjunction with Korn's second inequality [31], elliptic on H. Hence, the claim follows from Lemma 4.1. □

We now consider existence and uniqueness for Subproblem (S_τ) under the assumption that

(A.2) $\log L \in L^\infty(\Gamma_F)$.

Proposition 4.4. *Assume $\alpha_{n-1} \in L^2(\Gamma_F)$. Then the Subproblem ($S_\tau$) with ψ corresponding to the state evolution laws of Dieterich* (2.15), *Ruina* (2.16), *or* (2.17) *has a unique solution for any given slip rate $V = |\dot{u}| \in L^{2+\delta}(\Gamma_F)$ with $\delta \geq 0$, $\delta > 0$, or $\delta \geq 0$, respectively.*

Proof. For fixed $V = |\dot{u}| \geq 0$, the convexity of the functionals $j \coloneqq \tau\Psi(\,\cdot\,,V) = \tau\int_{\Gamma_F}\psi(\,\cdot\,,V)$ follows immediately from the convexity of $\psi(\,\cdot\,,V)$. From $\psi_r(0,V) = 0$, $\psi_d(0,V) = 1$, and $|\psi_{dc}(0,V)| \leq 1$, we conclude that Ψ is proper.
To show that Ψ is lower semicontinuous, we decompose ψ into its linear and nonlinear parts. For ψ_d and ψ_r, this leads to

$$\begin{aligned} \psi_d &= \psi_{d,1} + \psi_{d,2}, &\quad \psi_{d,1}\colon \alpha &\mapsto \frac{V}{L}\alpha, &\quad \psi_{d,2}\colon \alpha &\mapsto e^{-\alpha}, \\ \psi_r &= \psi_{r,1} + \psi_{r,2}, &\quad \psi_{r,1}\colon \alpha &\mapsto \frac{V}{L}\log(V/L)\alpha, &\quad \psi_{r,2}\colon \alpha &\mapsto \frac{1}{2}\frac{V}{L}\alpha^2. \end{aligned}$$

Since $\psi_{d,1}$ and $\psi_{r,1}$ multiply their arguments with functions that lie in L^2 by assumption (namely $V \in L^{2+\delta}(\Gamma_F)$ and (A.2)), their respective contribution to Ψ is continuous in α. A similar observation shows that Ψ is continuous if $\psi = \psi_{dc}$.
From the non-negativity of $\psi_{d,2}$ and $\psi_{r,2}$ we conclude that the corresponding integrals are well-defined and that Lemma 4.2 applies.
Because of $\rho > 0$, the bilinear form $(\,\cdot\,,\,\cdot\,)_{L^2(\Gamma_F)}$ is symmetric, continuous, and elliptic on $L^2(\Gamma_F)$. Since we assumed $\alpha_{n-1} \in L^2(\Gamma_F)$, the claim now follows from Lemma 4.1. □

5. Existence of solutions for a Coupled Spatial Dieterich–Ruina Problem

Since the Dieterich–Ruina model of RSF with Dieterich's or Ruina's state evolution law leads to subproblems that are uniquely solvable, we can define a solution operator for each problem, namely

$$S\colon L^{2+\delta}(\Gamma_F) \to L^2(\Gamma_F) \qquad \text{corresponding to Subproblem } (\mathrm{S}_\tau)$$

and

$$R\colon L^2(\Gamma_F) \to H \qquad \text{corresponding to Subproblem } (\mathrm{R}_\tau),$$

for any $\delta \geq 0$. In this manner, we map scalar velocities on Γ_F to states (using S) and states to velocity fields on Ω (using R). Since such velocity fields can be transformed into scalar velocities on Γ_F through the map

$$|\gamma_F|\colon H \to L^p(\Gamma_F), \qquad v \mapsto (\Gamma_F \ni x \mapsto |v(x)|),$$

we can close the circle here. Before we discuss for which δ this is possible, we consider the consequences. We first compose the two solution operators to obtain

$$RS \coloneqq R \circ S \,:\, L^{2+\delta}(\Gamma_F) \to H.$$

If $|\gamma_F|$ now maps to $L^{2+\delta}(\Gamma_F)$, we have a self-map $RS \circ |\gamma_F|\colon H \to H$. By construction, a fixed point of this map and its corresponding state solve both Subproblem (S_τ) and Subproblem (R_τ) simultaneously and thus the coupled Problem (RSF_τ).
Now back to δ: Since the trace operator $\gamma\colon H \to L^p(\Gamma_F)^d$ is well-defined and compact for $1 \leq p < p^*$, so is $|\gamma_F|$, with $p^* = 4$ in the three-dimensional case and $p^* = \infty$ in two dimensions [6]. Any $\delta < p^* - 2$ is thus admissible.
In what follows, we make the necessary arrangements for an application of Schauder's theorem which guarantees that $RS \circ |\gamma_F|$ has a fixed point. To that end, we are forced to restrict ourselves to Dieterich's state evolution law for reasons that become clear as we proceed. We also make the assumptions

- (A.3) $\bar{\sigma}_n \in L^\infty(\Gamma_F)$,
- (A.4) $V_0 \in L^\infty(\Gamma_F)$ and $\log V_0 \in L^\infty(\Gamma_F)$,
- (A.5) $a \in L^\infty(\Gamma_F)$,
- (A.6) $b \in L^\infty(\Gamma_F)$, and
- (A.7) $\mu_0 \in L^\infty(\Gamma_F)$.

5.1. **Towards boundedness and continuity of RS.** Since for any two states α, $\beta \in L^2(\Gamma_F)$ we have

$$\begin{aligned} &\Phi(R(\beta),\alpha) - \Phi(R(\alpha),\alpha) + \Phi(R(\alpha),\beta) - \Phi(R(\beta),\beta) \\ &\quad\geq a_\tau(R(\alpha)-R(\beta), R(\alpha)-R(\beta)) \\ &\quad\geq C\|R(\alpha)-R(\beta)\|^2_{1,2,\Omega} \end{aligned} \tag{5.1}$$

for a constant $C > 0$ by H-ellipticity of $a_\tau(\,\cdot\,,\,\cdot\,)$, upper bounds for the term

$$\begin{aligned} &\Phi(R(\beta),\alpha) - \Phi(R(\alpha),\alpha) + \Phi(R(\alpha),\beta) - \Phi(R(\beta),\beta) \\ &\qquad = \int_{\Gamma_F} \phi(R(\beta),\alpha) - \phi(R(\alpha),\alpha) + \phi(R(\alpha),\beta) - \phi(R(\beta),\beta) \end{aligned}$$

also yield upper bounds for the term $\|R(\alpha) - R(\beta)\|^2$. By means of such bounds we establish that RS is bounded (i.e. it maps bounded sets into bounded sets) and continuous.

Proposition 5.1. *We have*

$$\phi(U,\alpha) - \phi(V,\alpha) + \phi(V,\beta) - \phi(U,\beta) \le a|\bar\sigma_n|\bigl[\max(U,V)|\log(V_m(\alpha)/V_m(\beta))| + |V_m(\alpha) - V_m(\beta)|\bigr] \tag{5.2}$$

with α, β, U, $V \in \mathbb{R}$ and $0 \le U$, V.

Proof. Since the function φ from (2.13) is defined piecewise depending on whether the first argument is smaller than V_m or not, and we consider four terms at once, we can distinguish 16 cases, each of which yields an explicit expression for (5.2). Given that (5.2) remains unchanged if we swap U and V simultaneously with α and β, some cases are analogous to others. For brevity, we write

$$(*) = [\phi(U,\alpha) - \phi(V,\alpha) + \phi(V,\beta) - \phi(U,\beta)]/a|\bar\sigma_n|$$

and find:

(1) $U \le V_m(\alpha)$, $V \le V_m(\alpha)$, $V \le V_m(\beta)$, $U \le V_m(\beta)$: Implies $(*) = 0$.
(2) $U \le V_m(\alpha)$, $V \le V_m(\alpha)$, $V \le V_m(\beta)$, $U \ge V_m(\beta)$: Implies $(*) \le 0$.
(3) $U \le V_m(\alpha)$, $V \le V_m(\alpha)$, $V \ge V_m(\beta)$, $U \le V_m(\beta)$: Implies

$$(*) = V\log(V/V_m(\beta)) - V + V_m(\beta) \le V\log(V_m(\alpha)/V_m(\beta)).$$

(4) $U \le V_m(\alpha)$, $V \le V_m(\alpha)$, $V \ge V_m(\beta)$, $U \ge V_m(\beta)$: Implies

$$V_m(\alpha) \ge V,\, U \ge V_m(\beta)$$

and thus

$$\begin{aligned}(*) &= V\log(V/V_m(\beta)) - U\log(U/V_m(\beta)) + U - V\\ &\le V\log(V_m(\alpha)/V_m(\beta)) + |V_m(\alpha) - V_m(\beta)|.\end{aligned}$$

(5) $U \le V_m(\alpha)$, $V \ge V_m(\alpha)$, $V \le V_m(\beta)$, $U \le V_m(\beta)$: Analogous to (2).
(6) $U \le V_m(\alpha)$, $V \ge V_m(\alpha)$, $V \le V_m(\beta)$, $U \ge V_m(\beta)$: Implies

$$U = V_m(\alpha) = V = V_m(\beta) = U$$

and thus $(*) = 0$.
(7) $U \le V_m(\alpha)$, $V \ge V_m(\alpha)$, $V \ge V_m(\beta)$, $U \le V_m(\beta)$: Implies

$$(*) = V\log(V_m(\alpha)/V_m(\beta)) + V_m(\beta) - V_m(\alpha).$$

(8) $U \le V_m(\alpha)$, $V \ge V_m(\alpha)$, $V \ge V_m(\beta)$, $U \ge V_m(\beta)$: Implies

$$\begin{aligned}(*) &= V\log(V_m(\alpha)/V_m(\beta)) - U\log(U/V_m(\beta)) + U - V_m(\alpha)\\ &\le V\log(V_m(\alpha)/V_m(\beta)).\end{aligned}$$

(9) $U \ge V_m(\alpha)$, $V \le V_m(\alpha)$, $V \le V_m(\beta)$, $U \le V_m(\beta)$: Analogous to (3).
(10) $U \ge V_m(\alpha)$, $V \le V_m(\alpha)$, $V \le V_m(\beta)$, $U \ge V_m(\beta)$: Analogous to (7).
(11) $U \ge V_m(\alpha)$, $V \le V_m(\alpha)$, $V \ge V_m(\beta)$, $U \le V_m(\beta)$: Implies

$$U = V_m(\alpha) = V = V_m(\beta) = U$$

and thus $(*) = 0$.
(12) $U \ge V_m(\alpha)$, $V \le V_m(\alpha)$, $V \ge V_m(\beta)$, $U \ge V_m(\beta)$: Implies

$$\begin{aligned}(*) &= U\log(V_m(\beta)/V_m(\alpha)) - V\log(V_m(\beta)/V) + V_m(\alpha) - V\\ &\le U\log(V_m(\beta)/V_m(\alpha)) - V\log(V_m(\beta)/V_m(\alpha)) + V_m(\alpha) - V\\ &\le V_m(\alpha) - V \le V_m(\alpha) - V_m(\beta).\end{aligned}$$

(13) $U \ge V_m(\alpha)$, $V \ge V_m(\alpha)$, $V \le V_m(\beta)$, $U \le V_m(\beta)$: Analogous to (4).
(14) $U \ge V_m(\alpha)$, $V \ge V_m(\alpha)$, $V \le V_m(\beta)$, $U \ge V_m(\beta)$: Analogous to (8).
(15) $U \ge V_m(\alpha)$, $V \ge V_m(\alpha)$, $V \ge V_m(\beta)$, $U \le V_m(\beta)$: Analogous to (12).

(16) $U \geq V_m(\alpha)$, $V \geq V_m(\alpha)$, $V \geq V_m(\beta)$, $U \geq V_m(\beta)$: Implies

$$(*) = (U - V)\log(V_m(\beta)/V_m(\alpha)).$$

The claim now follows by taking the maximum of the above bounds. □

In the following, we write $\|\cdot\|_{k,p,M}$ for the canonical $W^{k,p}(M)$ norm. The pointwise bound obtained in Proposition 5.1 yields the integral bound[1]

$$\begin{aligned}
&\Phi(R(\beta),\alpha) - \Phi(R(\alpha),\alpha) + \Phi(R(\alpha),\beta) - \Phi(R(\beta),\beta) \\
&\quad\leq \|a\bar{\sigma}_n \max(|R(\alpha)|,|R(\beta)|)\log(V_m(\alpha)/V_m(\beta))\|_{0,1,\Gamma_F} \\
&\qquad + \|a\bar{\sigma}_n[V_m(\alpha) - V_m(\beta)]\|_{0,1,\Gamma_F} \\
&\quad\leq \|\max(|R(\alpha)|,|R(\beta)|)\|_{0,2,\Gamma_F}\|a\bar{\sigma}_n \log(V_m(\alpha)/V_m(\beta))\|_{0,2,\Gamma_F} \\
&\qquad + \|a\bar{\sigma}_n\|_{0,\infty,\Gamma_F}\|V_m(\alpha) - V_m(\beta)\|_{0,1,\Gamma_F} \\
&\quad\leq C_1\|\max(|R(\alpha)|,|R(\beta)|)\|_{0,2,\Gamma_F}\|\beta - \alpha\|_{0,2,\Gamma_F} \\
&\qquad + C_2\|e^{-(b/a)\alpha} - e^{-(b/a)\beta}\|_{0,1,\Gamma_F}
\end{aligned}$$

with the constants $C_1 := \|b\bar{\sigma}_n\|_{0,\infty,\Gamma_F}$ and

$$C_2 := \|a\bar{\sigma}_n\|_{0,\infty,\Gamma_F}\|V_0\exp(-\mu_0/a - (b/a)\log(V_0/L))\|_{0,\infty,\Gamma_F},$$

both of which are finite by assumptions (A.2)–(A.7). In conjunction with (5.1), we thus have

$$\begin{aligned}
C\|R(\alpha) - R(\beta)\|^2_{1,2,\Omega} &\leq C_1\|\max(|R(\alpha)|,|R(\beta)|)\|_{0,2,\Gamma_F}\|\beta - \alpha\|_{0,2,\Gamma_F} \\
&\quad + C_2\|e^{-(b/a)\alpha} - e^{-(b/a)\beta}\|_{0,1,\Gamma_F}.
\end{aligned} \tag{5.3}$$

On the one hand, $\alpha = S(V)$ and $\beta = 0$ now turn (5.3) into

$$\begin{aligned}
&C\big(\|RS(V)\|_{1,2,\Omega} - \|R(0)\|_{1,2,\Omega}\big)^2 \\
&\quad\leq C\|RS(V) - R(0)\|^2_{1,2,\Omega} \\
&\quad\leq C_1\|\max(|RS(V)|,|R(0)|)\|_{0,2,\Gamma_F}\|S(V)\|_{0,2,\Gamma_F} \\
&\qquad + C_2\|e^{-(b/a)S(V)} - 1\|_{0,1,\Gamma_F} \\
&\quad\leq C_1\big(\|RS(V)\|_{0,2,\Gamma_F} + \|R(0)\|_{0,2,\Gamma_F}\big)\|S(V)\|_{0,2,\Gamma_F} \\
&\qquad + C_2\|e^{-(b/a)S(V)}\|_{0,1,\Gamma_F} + C_2\|1\|_{0,1,\Gamma_F},
\end{aligned} \tag{5.4}$$

which we can use to bound the growth of RS. On the other hand, once we know that RS is bounded and show that $V_n \to V$ in $L^{2+\delta}(\Gamma_F)$ implies $S(V_n) \to S(V)$ in $L^2(\Gamma_F)$ and $e^{-(b/a)S(V_n)} \to e^{-(b/a)S(V)}$ in $L^1(\Gamma_F)$, it follows from (5.3) that RS is continuous. To that end, we need to investigate S more thoroughly.

5.2. An explicit formulation for the state problem.

In this section it is shown that S can also be viewed as a superposition operator.

We first observe that an application of the backward Euler scheme to Dieterich's law (2.15) turns it into

$$\alpha_n - \tau e^{-\alpha_n} = \alpha_{n-1} - \frac{\Delta U_n}{L}, \tag{5.5}$$

with $\Delta U_n := \tau V_n$. This prompts us to investigate the abstract problem

$$z - \tau e^{-z} = r, \tag{5.6}$$

for which it is convenient to introduce the Lambert W function [8].

Definition. For $z \in [0,\infty)$, we uniquely define $W(z)$ by $W(z)e^{W(z)} = z$.

[1] Here and in what follows, we do not distinguish between $v \in H$ and $\gamma_F(v) \in L^p(\Gamma_F)^d$ if the intended meaning is clear from the context.

We can then write $z = W(\tau e^{-r}) + r$ for (5.6) and

$$\alpha_n = W(\tau e^{\Delta U_n/L-\alpha_{n-1}}) - (\Delta U_n/L - \alpha_{n-1})$$

for (5.5). If we furthermore define $s(V) \coloneqq W(\tau e^V) - V$ and $s^\alpha(V) \coloneqq s(V-\alpha)$, then (5.5) takes the form

$$\alpha_n = s^{\alpha_{n-1}}(\Delta U_n/L).$$

Proposition 5.2. *The function s is Lipschitz continuous and so is s^α. As a consequence, T_{s^α} is a well-defined Lipschitz continuous operator from $L^p(\Gamma_F)$ to $L^p(\Gamma_F)$ whenever $\alpha \in L^p(\Gamma_F)$.*

Proof. It is straightforward to show $W(z)' = W(z)/[z(1+W(z))]$, which implies

$$s'(z) = \frac{W(\tau e^z)}{1+W(\tau e^z)} - 1 = -\frac{1}{1+W(\tau e^z)}$$

and thus $|s'| < 1$ since we have $W(z) > 0$ for $z > 0$. □

Given that Subproblem (S_τ) with $\alpha_{n-1} \in L^2(\Gamma_F)$ has a unique solution over $L^2(\Gamma_F)$ by Proposition 4.4, and $T_{s^{\alpha_{n-1}}}$ maps to $L^2(\Gamma_F)$ by Proposition 5.2, the two operators S and $T_{s^{\alpha_{n-1}}}$ must coincide.

5.3. **Growth of S.** As a consequence of (5.4), the growth of RS is dominated by the growth of S and $e^{-(b/a)S}$. In this section, we show that S has asymptotically logarithmic growth, so that we can shift our attention to $e^{-(b/a)S}$. To this end, we first show that s has logarithmic growth on the positive real axis.

Lemma 5.3. *We have $|s(z)| \le \log(z/\tau)$ for $\tau \le 1$ and $z \ge 1 - \log\tau$.*

Proof. Since $z \ge \tau$ implies $W(\tau e^z) \le z$, we can assume $|s(z)| = -s(z)$. For $\tau = 1$, we have

$$y - W(e^y) \le \log y \iff (y - \log y)e^{y-\log y} \le e^y \iff y - \log y \le y \iff 1 \le y.$$

For the general case, with $\tau \le 1$ and $z + \log\tau \ge 1$ the above implies

$$-s(z) + \log\tau = (z + \log\tau) - W(\tau e^z) \le \log(z + \log\tau) \le \log z$$

from which the claim immediately follows. □

The operator S inherits this property for non-negative arguments from s. To show this, we can use Jensen's inequality. The application is not straightforward, however, since S is parametrised with a state α, and s^2 is not concave on all of $\mathbb{R}$. We first address the second concern.

Lemma 5.4. *Let $\tau > 0$ be arbitrary. Then there is a $z_0(\tau) \ge 0$ such that*

$$\frac{d}{dz}\bigl(s(z)^2\bigr) \ge 0 \quad \text{and} \quad \frac{d^2}{dz^2}\bigl(s(z)^2\bigr) \le 0$$

for $z \ge z_0(\tau)$.

Proof. We find

$$\frac{d}{dz}\bigl(s(z)^2\bigr) = 2\cdot\frac{z - W(\tau e^z)}{1+W(\tau e^z)} \ge 0 \iff z \ge W(\tau e^z) \iff z \ge \tau,$$

so that we can choose any $z_0 \ge \tau$ to make s^2 non-decreasing from z_0 on. We also find

$$\frac{d^2}{dz^2}\bigl(s(z)^2\bigr) = 2\cdot\frac{1+W(\tau e^z)+(W(\tau e^z)-z)W(\tau e^z)}{(1+W(\tau e^z))^3} \le 0$$

$$\iff \frac{1+W(\tau e^z)}{W(\tau e^z)} \le z - W(\tau e^z), \tag{5.7}$$

which must be true from some $z_0 \geq 0$ on since the left-hand side of (5.7) converges to 1 from above and the right-hand side goes to infinity with $z \to \infty$. □

Since now s^2 is concave and non-decreasing on an interval $[z_0, \infty)$, we can bound integrals over $s(z)^2$ with $z \geq z_0$.

Lemma 5.5. *Assume $\tau \leq 1$. Then there is a constant C such that for any $V \in L^p(\Gamma_F)$ and $\alpha \in L^2(\Gamma_F)$ with $Z(V,\alpha) := \|V\|_{0,p,\Gamma_F} + \|\alpha\|_{0,2,\Gamma_F} \geq C$ and $M := \{x \in \Gamma_F : V(x) - \alpha(x) \geq z_0\}$, we have*

$$\left(\int_M s(V-\alpha)^2\right)^{1/2} \leq \lambda(\Gamma_F)^{1/2} \log\left(\frac{C(\Gamma_F)}{\tau} Z(V,\alpha)\right) + 2/e,$$

where λ denotes the $(d-1)$-dimensional Lebesgue measure.

Proof. Since the case $\lambda(M) = 0$ is trivially covered by choosing $C \geq \tau/C(\Gamma_F)$, we assume $\lambda(M) > 0$ and define a function g by

$$g(z) := \begin{cases} s(z)^2 & \text{if } z \geq z_0 \\ s(z_0)^2 + (z - z_0)\frac{d}{dz}(s(z)^2)|_{z=z_0} & \text{otherwise,} \end{cases}$$

so that g is non-decreasing, concave, and coincides with s^2 on $[z_0, \infty)$. By construction, we now have

$$\begin{aligned}
\frac{1}{\lambda(M)} \int_M s(V-\alpha)^2 &= \int_M g(V-\alpha) \frac{d\lambda}{\lambda(M)} \\
&\leq g\left(\int_M V - \alpha \frac{d\lambda}{\lambda(M)}\right) \\
&\leq g\left(\frac{C(\Gamma_F)}{\lambda(M)} Z(V,\alpha)\right) \\
&= s\left(\frac{C(\Gamma_F)}{\lambda(M)} Z(V,\alpha)\right)^2 \qquad \text{if } \frac{C(\Gamma_F)}{\lambda(\Gamma_F)} Z(V,\alpha) \geq z_0 \\
&\leq \log\left(\frac{C(\Gamma_F)}{\tau\lambda(M)} Z(V,\alpha)\right)^2 \qquad \text{if } \frac{C(\Gamma_F)}{\lambda(\Gamma_F)} Z(V,\alpha) \geq 1 - \log\tau
\end{aligned}$$

by Jensen's inequality, Lemma 5.3, and Lemma 5.4. This means

$$\left(\int_M s(V-\alpha)^2\right)^{1/2} \leq \lambda(M)^{1/2} \log\left(\frac{C(\Gamma_F)}{\tau\lambda(M)} Z(V,\alpha)\right)$$

whenever $Z(V,\alpha) \geq \tilde{C} := \max\{z_0, 1 - \log\tau\}\lambda(\Gamma_F)/C(\Gamma_F)$. We conclude

$$\begin{aligned}
\left(\int_M s(V-\alpha)^2\right)^{1/2} &\leq \lambda(M)^{1/2} \log\left(\frac{C(\Gamma_F)}{\tau} Z(V,\alpha)\right) - \lambda(M)^{1/2} \log(\lambda(M)) \\
&\leq \lambda(\Gamma_F)^{1/2} \log\left(\frac{C(\Gamma_F)}{\tau} Z(V,\alpha)\right) + 2/e
\end{aligned}$$

whenever $Z(V,\alpha) \geq \max\{\tilde{C}, \tau/C(\Gamma_F)\}$. □

To show that S has asymptotically logarithmic growth, we now only need to make sure that points x with $V(x) < z_0$ can be neglected.

Lemma 5.6. *We have $s(z)^2 \leq z^2 + 1/\tau$ for all z and $\tau \leq 1$.*

Proof. The substitution $z = y/\tau + \log(y/\tau^2)$ reduces the claim to

$$2\frac{y}{\tau}\log(\tau^2) - \frac{y^2}{\tau^2} \leq 2\frac{y}{\tau}\log y + \frac{1}{\tau} \tag{5.8}$$

with $y > 0$. Since the left-hand side of (5.8) is negative, this is implied by

$$0 \leq 2y \log y + 1,$$

which is obvious. □

Combining the above observations yields the result.

Proposition 5.7. *Assuming $\tau \leq 1$ and $\alpha \in L^p(\Gamma_F)$, we have $\|s^\alpha(V)\|_{0,2,\Gamma_F} \in O(\log\|v\|_{0,p,\Gamma_F})$ for $\|V\|_{0,p,\Gamma_F} \to \infty$.*

Proof. We first observe

$$\|s^\alpha(V)\|_{0,2} \leq \left(\int_{V-\alpha<z_0} s(V-\alpha)^2\right)^{1/2} + \left(\int_{V-\alpha\geq z_0} s(V-\alpha)^2\right)^{1/2}$$

and

$$\begin{aligned}\left(\int_{V-\alpha<z_0} s(V-\alpha)^2\right)^{1/2} &\leq \left(\int_{V-\alpha<z_0} (V-\alpha)^2 + 1/\tau\right)^{1/2} \\ &\leq \left(\int_{V-\alpha<0} \alpha^2 + 1/\tau\right)^{1/2} + \left(\int_{0\leq V-\alpha<z_0} z_0^2 + 1/\tau\right)^{1/2} \\ &\leq \|\alpha\|_{0,2,\Gamma_F} + 2\|1/\tau\|_{0,1,\Gamma_F}^{1/2} + \|z_0\|_{0,2,\Gamma_F}\end{aligned}$$

with arbitrary $V \in L^p(\Gamma_F)$ by virtue of Lemma 5.6. Now choose C in accordance with Lemma 5.5. Either we have $\|V\|_{0,p,\Gamma_F} + \|\alpha\|_{0,2,\Gamma_F} \geq C$, so that

$$\left(\int_{V-\alpha\geq z_0} s(V-\alpha)^2\right)^{1/2} \leq \lambda(\Gamma_F)^{1/2} \log\left(\frac{C(\Gamma_F)}{\tau}\left[\|V\|_{0,p,\Gamma_F} + \|\alpha\|_{0,2,\Gamma_F}\right]\right) + 2/e,$$

which means

$$\|s^\alpha(V)\|_{0,2,\Gamma_F} \in O(\log\|V\|_{0,p,\Gamma_F}) \quad \text{with } \|V\|_{0,p,\Gamma_F} \to \infty$$

as claimed, or $\|V-\alpha\|_{0,2,\Gamma_F}$ is bounded by a constant and so is $\|s(V-\alpha)\|_{0,2,\Gamma_F}$ by Proposition 5.2, so that the claim is trivially true. □

Remark. In the same manner, we obtain $\|s^\alpha(|v|)\|_{0,2,\Gamma_F} \in O(\log\|v\|_{1,2,\Omega})$ with $\|v\|_{1,2,\Omega} \to \infty$.

5.4. **Growth of $e^{-(b/a)S}$.** We now bound the growth of $e^{-S} := T_{\exp(-s^{\alpha_{n-1}})}$; again by investigating the underlying scalar map.

Proposition 5.8. *The function e^{-s} is Lipschitz continuous.*

Proof. We have

$$\left|\frac{d}{dz}e^{-s(z)}\right| = \frac{e^{z-W(\tau e^z)}}{1+W(\tau e^z)} \leq 1/\tau$$

since

$$\tau e^{z-W(\tau e^z)} = W(\tau e^z) \leq 1 + W(\tau e^z).$$ □

As a consequence, the operator e^{-S} is obviously well-defined and Lipschitz continuous from $L^p(\Gamma_F)$ to $L^p(\Gamma_F)$ whenever $\alpha_{n-1} \in L^p(\Gamma_F)$. We also have

$$\|e^{-rS(V)}\|_{0,1,\Gamma_F} = \|e^{-S(V)}\|_{0,r,\Gamma_F}^r \in O(\|V\|_{0,r,\Gamma_F}^r) \tag{5.9}$$

for any $r \leq p$.

Corollary 5.9. *From α_{n-1}, $e^{-\alpha_{n-1}} \in L^p(\Gamma_F)$ and $u \in L^p(\Gamma_F)^d$ it follows that α_n, $e^{-\alpha_n} \in L^p(\Gamma_F)$. In other words, regularity of the state variable is carried over from one time step to the next.*

Proof. This is an immediate consequence of Propositions 5.2 and 5.8. □

It should be noted that the analogue of Corollary 5.9 for Ruina's law does not seem to hold.

5.5. **Growth and continuity of RS.** With Proposition 5.7 and Proposition 5.8 we have bounded the growth of the right-hand side of (5.4). In this section, we collect the implications for the operator RS. We first need a technical lemma.

Lemma 5.10. *From $f(t)^2 \in O(f(t)\log t + t^r)$ with $r > 0$ it follows that*

$$f(t) \in O(t^{r/2}) \quad \text{with } t \to \infty.$$

Proof. Straightforward. □

Proposition 5.11. *Assume $b/a \leq r$ with $r \leq 2$. We then have*

$$\|RS(V)\|_{1,2,\Omega} \in O(\|V\|^{r/2}_{0,r,\Gamma_F})$$

for $\|V\|_{0,r,\Gamma_F} \to \infty$. In particular, RS is then bounded.

Proof. From (5.4) and (5.9) as well as Proposition 5.7 we deduce

$$\begin{aligned}\|RS(V)\|^2_{1,2,\Omega} &\in O\big(\|RS(V)\|_{1,2,\Omega}\|S(V)\|_{0,2,\Gamma_F} + \|e^{-(b/a)S(V)}\|_{0,1,\Gamma_F}\big)\\ &\subseteq O\big(\|RS(V)\|_{1,2,\Omega}\log\|V\|_{0,r,\Gamma_F} + \|V\|^r_{0,r,\Gamma_F}\big).\end{aligned}$$

The claim now follows from Lemma 5.10. □

Remark. In a similar fashion, we conclude

$$\|RS(|u|)\|_{1,2,\Omega} \in O(\|u\|^{r/2}_{1,2,\Omega})$$

for $b/a \leq r < p^*$.

Corollary 5.12. *If we, furthermore, assume $r < 2$, we obtain*

$$\|RS(|u|)\|_{1,2,\Omega} \in o(\|u\|_{1,2,\Omega}),$$

so that $RS \circ |\gamma_F|$ is a self-map on sufficiently large balls in H.

The assumption $b/a \leq r < 2$ is not unreasonable. The literature has this to say:

> "Laboratory experiments generally show $a \approx 2(b - a)$" [27].

which implies $b/a \approx 3/2$. From another source:

> "Laboratory values of a/b are typically larger than 0.5" [1].

In order to apply Schauder's theorem, we now only need to show that RS is continuous.

Proposition 5.13. *The operator $RS\colon L^p(\Gamma_F) \to H$ with $p \geq 2$ is continuous whenever $b/a \leq p$ and $\alpha_{n-1} \in L^p(\Gamma_F)$.*

Proof. Let V_n converge to $V \in L^p(\Gamma_F)$. We know that the sequence $RS(V_n)$ is bounded from Proposition 5.11, that $S(V_n)$ converges in L^2 from Proposition 5.2, and that $e^{-(b/a)S(V)}$ converges in L^1 from Proposition 5.8. In summary, the right-hand side of (5.3) converges and so must the left-hand side. □

We thus have that for $b/a \leq r < 2$, the operator $RS \circ |\gamma_F|$ is a continuous compact self-map on large balls in H. By Schauder's fixed point theorem [13, Corollary 11.2], such balls contain fixed points of $RS \circ |\gamma_F|$. We conclude:

Theorem 5.14. *If we have $\|b/a\|_{0,\infty,\Gamma_F} \leq r < 2$ as well as $\alpha_{n-1} \in L^2(\Gamma_F)$ and $e^{-\alpha_{n-1}} \in L^2(\Gamma_F)$, then the time-discrete coupled problem corresponding to the n^{th} time step with Dieterich's state evolution law has a solution. The new state satisfies $\alpha_n \in L^2(\Gamma_F)$ and $e^{-\alpha_n} \in L^2(\Gamma_F)$.*

Corollary 5.15. *If we have* $\|b/a\|_{0,\infty,\Gamma_F} \leq r < 2$ *as well as* $\alpha_0 \in L^2(\Gamma_F)$ *and* $e^{-\alpha_0} \in L^2(\Gamma_F)$*, then the time-discrete coupled problem with Dieterich's state evolution law has a solution for every time step.*

6. Numerical Experiments

6.1. **A model problem.** We consider a two-dimensional $5\,\mathrm{m} \times 1\,\mathrm{m}$ slider Ω as depicted in Figure 2.1 and assume it to consist of a St. Venant–Kirchhoff material. The body force shall represent gravity, hence we set $f = -\rho g \cdot e_2$. Here and in the following e_1, e_2 denote the unit vectors in $\mathbb{R}^2$.

At the upper part Γ_D of the slider, we impose Dirichlet boundary conditions prescribing the velocity V_D in the direction of e_1, given by

$$V_D(t) = \begin{cases} 2 \times 10^{-4}\,\mathrm{m/s} \cdot \frac{4t}{T} & \text{if } t \leq T/4 \\ 2 \times 10^{-4}\,\mathrm{m/s} & \text{otherwise.} \end{cases}$$

The bottom Γ_F obeys a Dieterich–Ruina friction model, and on Γ_N homogeneous Neumann conditions $f_N = 0$ are imposed.

Initially, the configuration is selected to be stress-free, so that $u_0 = u(\,\cdot\,, 0) \in H$ solves

$$a(u_0, v) = \ell(v) \quad \forall v \in H.$$

In addition, we choose $\dot{u}(\,\cdot\,, 0) = V_D(0) \cdot e_1$ on Ω, which is consistent with the aforementioned Dirichlet boundary condition, as well as $\alpha(\,\cdot\,, 0) = -10$ on Γ_F. The remaining parameters are listed in Table 6.1.

We select the final time $T = 15\,\mathrm{s}$ and the time step $\tau = T/N$ with $N = 10^4$, if not stated otherwise. Spatial discretisation of Subproblem (R_τ), i.e., of the velocity $\dot{u}_n$, is carried out with respect to a triangular grid $\mathcal{T}_j$ arising from j successive uniform refinements of a regular initial grid $\mathcal{T}_0$ with $2{\times}6$ vertices and corresponding piecewise linear finite elements $\mathcal{S}_j \subset H$. To discretise the nonlinearity Φ, lumping is used. For the states α_n, $n = 1, \ldots, N$, we use piecewise constant finite elements on the dual of the trace grid $\mathcal{T}_j \cap \Gamma_F$. The implementation is based on the Dune libraries [2].

6.2. **Convergence properties of the fixed point iteration.** In light of the theoretical considerations in Section 4, we first investigate the convergence of the fixed point iteration

$$\dot{u}^{\nu+1}_{n,j} = R_j\big(\omega^\nu \alpha^{\nu+1}_{n,j} + (1-\omega^\nu)\alpha^\nu_{n,j}\big), \quad \alpha^{\nu+1}_{n,j} = S_j\big(|\gamma_F(\dot{u}^\nu_{n,j})|\big), \qquad \nu = 0, 1, \ldots$$

for the algebraic spatial problems (see Section 6.1) numerically. We select the relaxation parameter $\omega^\nu = 1$, if $\nu = 0, 1$ or

$$\|\alpha^{\nu+1}_{n,j} - \alpha^\nu_{n,j}\|_{L^2(\Gamma_F)} \leq \frac{1}{2}\|\alpha^\nu_{n,j} - \alpha^{\nu-1}_{n,j}\|_{L^2(\Gamma_F)}, \qquad \nu = 2, 3, \ldots,$$

and $\omega^\nu = 0.5$ otherwise.

The evaluation of S_j, i.e., the solution of a discrete version of (S_τ) is computed pointwise by a bisection method up to a pointwise absolute error of 10^{-12}.

Parameter	Value	Parameter	Value
Mass density ρ	$5 \times 10^3\,\mathrm{kg/m^2}$	Ref. velocity V_0	$1 \times 10^{-6}\,\mathrm{m/s}$
Poisson's ratio ν	0.3	Ref. friction coeff. μ_0	0.6
Young's modulus E	$5 \times 10^7\,\mathrm{N/m}$	Rate-effect coeff. a	0.010
Gravity g	$9.81\,\mathrm{N/kg}$	State-effect coeff. b	0.015
Prescr. normal stress $\bar{\sigma}_n$	$49\,050\,\mathrm{N/m}$	Characteristic slip dist. L	$1 \times 10^{-5}\,\mathrm{m}$

Table 6.1. Material parameters

The evaluation of the discrete operator R_j, i.e., the solution of the discrete counterpart of the smooth, convex minimisation problem (R_τ) on the finite-dimensional space $\mathcal{S}_j \subset H$ is performed iteratively by Truncated Nonsmooth Newton Multigrid iterations (TNNMG) [15, 16, 17] with an absolute error tolerance of 10^{-10} with respect to the norm $\|\cdot\|$ given by

$$\|v\| = \left((\boldsymbol{\mathcal{C}}\boldsymbol{\varepsilon}(v), \boldsymbol{\varepsilon}(v))_{L^2(\Omega)} + (\rho v, v)_{L^2(\Omega)}\right)^{1/2}, \quad v \in H.$$

The overall fixed point iteration is stopped once the criterion

$$\|u_{n,j}^{\nu+1} - u_{n,j}^{\nu}\| \leq 10^{-10}$$

is satisfied.

Both for Dieterich's law (left) and Ruina's law (right), Figure 6.1 shows the average (solid) and maximum number (dashed) of required fixed point iterations per time step for the spatial problems arising in 10^4 time steps over the number of refinement levels j of the underlying triangulation $\mathcal{T}_j$, $j = 2, \dots, 8$. In both cases, the number of required fixed point iterations appears to saturate with decreasing mesh size, suggesting mesh-independent convergence. A theoretical justification is the subject of future research.

We emphasize that under relaxation, i.e. $\omega^\nu < 1$, occurred only for Ruina's law, only for $j = 5$, and only in a single time step, which is responsible for the prominent peak in Figure 6.1.

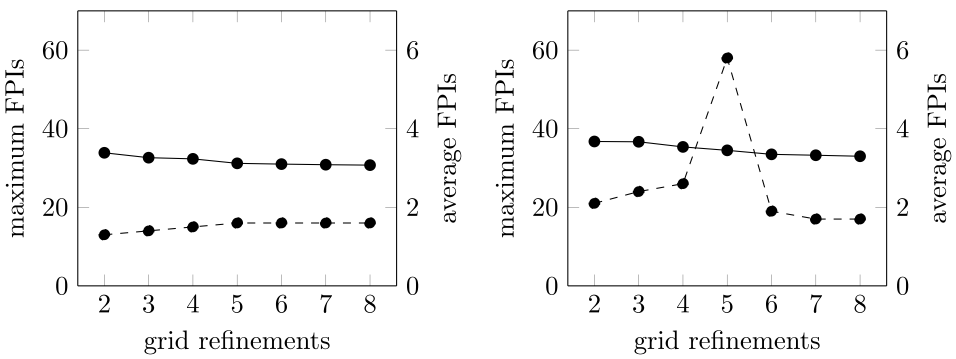

FIGURE 6.1. Average (solid) and maximum (dashed) number of fixed point iterations per time step required for 10^4 time steps over the number of refinement levels j for Dieterich's law (left) and Ruina's law (right).

6.3. **Convergence properties of the discretisation.** During the time interval $[0, T]$, the evolution of the body Ω goes through two phases: It first shears without slipping (roughly for $t_n \in [0, 2T/3]$), and then it enters a regime of quasiperiodic slip events, both for Dieterich's and Ruina's law.

We concentrate on Dieterich's law and the quasiperiodic regime. To get an idea of the total movement of the object we examine the displacement $\gamma_F(u_j)(x_0, \cdot)$ and sliding velocity $|\gamma_F(\dot{u}_j)(x_0, \cdot)|$ at the centre x_0 of Γ_F. Discretisation is carried out using the Newmark/finite element methodology described in Sections 3 and 6.1. The triangulation $\mathcal{T}_j$, $j = 4$ is fixed and we consider a sequence of decreasing time step sizes $\tau_i = T/N_i$, $N_i = 10^{4+i}$, $i = 0, 1, 2$. The resulting approximate displacements and velocities are shown in Figures 6.2 and 6.3, respectively.

Note the quasiperiodic steps (displacement) and peaks (velocity) associated with slip events. In addition to these low-frequency events, high-frequency oscillations

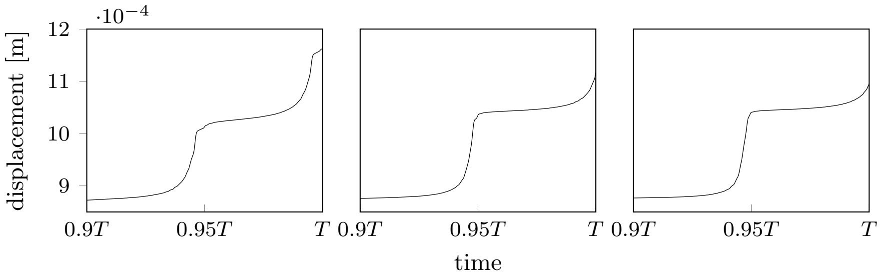

FIGURE 6.2. Dieterich's law: Displacement at the centre x_0 of the frictional boundary Γ_F for the time step sizes $\tau_i = T/10^{4+i}$, $i = 0, 1, 2$ (left to right).

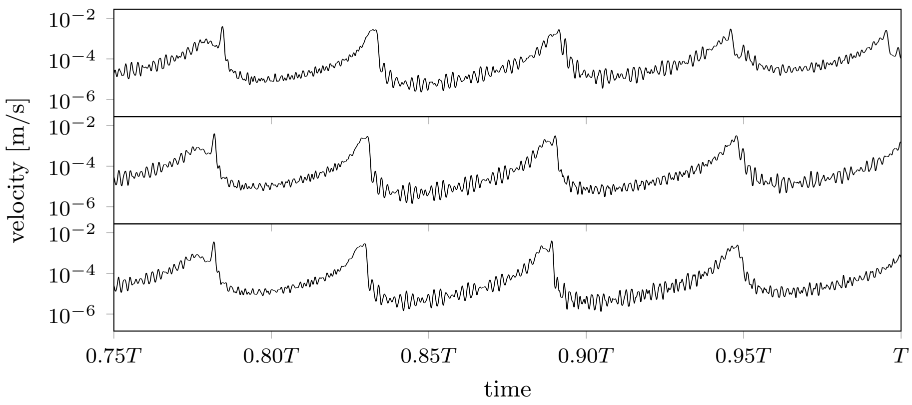

FIGURE 6.3. Dieterich's law: Velocity at the centre x_0 of the frictional boundary Γ_F for the time step sizes $\tau_i = T/10^{4+i}$, $i = 0, 1, 2$ (top to bottom).

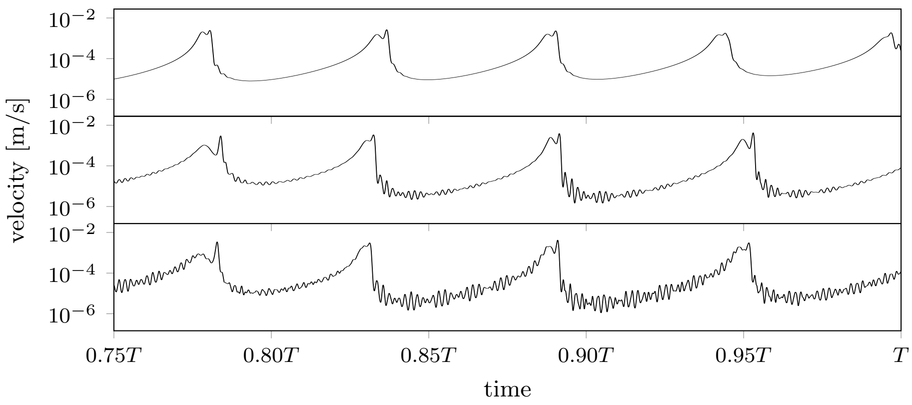

FIGURE 6.4. Same as Figure 6.3, for a computation based on the backward Euler method.

in the velocity are clearly visible. To ascertain that these oscillations are no time discretisation artifacts (even though for $i = 1, 2$ elastic waves are fully resolved), we redid the computation with the highly dissipative backward Euler method in place of the Newmark scheme. The results are shown in Figure 6.4. Again, high-frequency

oscillations can be observed for sufficiently small time steps and the approximations appear to converge to the same solution as for the Newmark scheme.
We found the same qualitative behaviour for Ruina's law.

6.4. **Slip events.** While quasiperiodic low-frequency slip events were illustrated through the velocity at a single point on the frictional boundary Γ_F, we now present the evolution of a slip event in a small time interval along the whole of Γ_F. The time interval is selected to be $[0.6725T, 0.6775T]$ and has a length of $0.005T = 0.075\,\mathrm{s}$. We use a time step size of $\tau = T/10^6$, and $j = 4$ uniform spatial mesh refinements. Figure 6.5 shows the velocity on the boundary Γ_F (horizontal axis), evolving in time (vertical axis): A wave forms at the centre of the boundary, spreads towards the end points, and is arrested before reaching either.

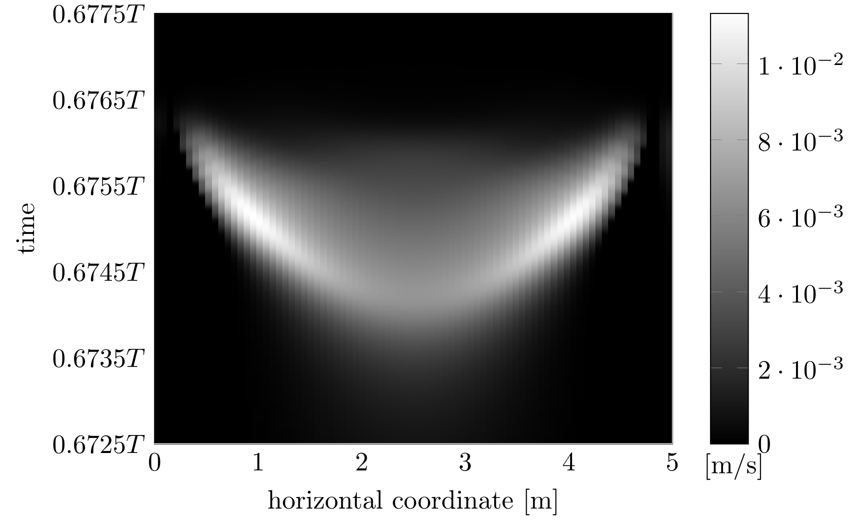

FIGURE 6.5. Evolution of the velocity on the frictional boundary Γ_F for a single slip event.

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