Dynamic problems of rate-and-state friction in viscoelasticity

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Experimental background

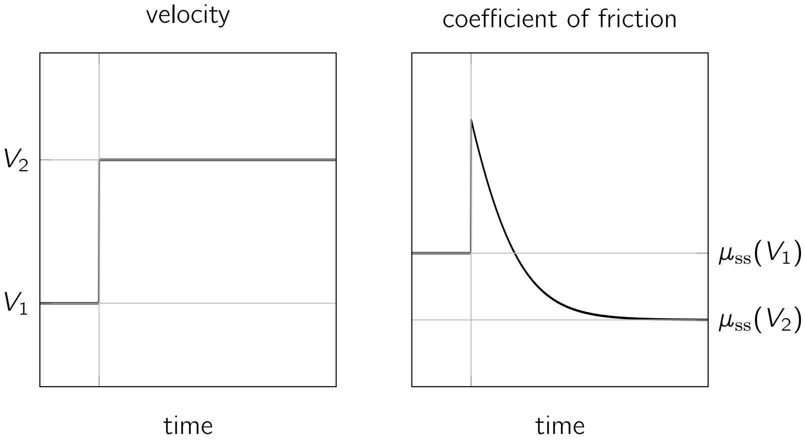


Figure: System response to jump in velocity (after steady-state sliding)

Rate-and-state friction

Widely used law

$$\mu(V,\theta) = \mu_* + a \log \frac{V}{V_*} + b \log \frac{\theta V_*}{L}, \qquad \dot{\theta}(\theta, V) = \begin{cases} 1 - \frac{\theta V}{L} & \text{ageing law} \\ -\frac{\theta V}{L} \log \frac{\theta V}{L} & \text{slip law} \end{cases}$$

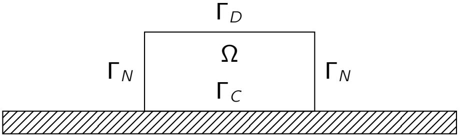
Transformation: $\alpha = \log(\theta V_* / L)$

$$\mu(V,\alpha) = \mu_* + a \log \frac{V}{V_*} + b\alpha, \qquad \dot{\alpha}(\alpha, V) = \begin{cases} \frac{V_* e^{-\alpha} - V}{L} \\ -\frac{V}{L}\left(\log \frac{V}{V_*} + \alpha\right) \end{cases}$$

General setting

- μ is monotone in V for fixed α
- μ is Lipschitz with respect to α (but not θ)
- (unlike θ), α follows a gradient flow for fixed V.
- (ideally): $\dot{\alpha}$ is Lipschitz with respect to V.

A typical continuum mechanical problem



With prescribed $\mathbf{u}(0)$, $\dot{\mathbf{u}}(0)$, and $\alpha(0)$.

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) \quad \text{in } \Omega \quad \text{(linear viscoelasticity)}$$

$$\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{b} = \rho \ddot{\mathbf{u}} \quad \text{in } \Omega \quad \text{(momentum balance)}$$

$$\dot{\mathbf{u}}_n = 0 \quad \text{on } \Gamma_C \quad \text{(bilateral contact)}^1$$

$$\boldsymbol{\sigma}_t = -\lambda \dot{\mathbf{u}}, \quad \lambda = \frac{|\boldsymbol{\sigma}_t|}{|\dot{\mathbf{u}}|} = \frac{|s_n| \mu(|\dot{\mathbf{u}}|, \alpha)}{|\dot{\mathbf{u}}|} \quad \text{on } \Gamma_C \quad \text{with } \lambda = 0 \text{ for } \dot{\mathbf{u}} = 0$$

$$\dots \quad \text{on } \Gamma_{N,D}$$

$$\dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}|, \alpha) \quad \text{on } \Gamma_C \quad \text{(family of ODEs)}$$

with $s_n \approx \sigma_n$, constant in time[1].

[1] Inherited from the rate-and-state friction model

Weak formulation

We get

$$\int_\Omega \rho\ddot{\mathbf{u}}(\mathbf{v}-\dot{\mathbf{u}}) + \int_\Omega \mathcal{B}\boldsymbol{\varepsilon}(\dot{\mathbf{u}})\colon \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}) + \int_\Omega \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u})\colon \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}) + \int_{\Gamma_C} \phi(\mathbf{v},\alpha) \geq \int_{\Gamma_C} \phi(\dot{\mathbf{u}},\alpha) + \ell(\mathbf{v}-\dot{\mathbf{u}})$$

for every $\mathbf{v} \in \mathcal{H}$ with

$$\mathcal{H} = \{\mathbf{v} \in H^1(\Omega)^d \colon \mathbf{v} = 0 \text{ on } \Gamma_D,\ \mathbf{v}_n = 0 \text{ on } \Gamma_C\}$$

or briefly

$$0 \in M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\,\cdot\,,\alpha)(\dot{\mathbf{u}}) - \ell \subset \mathcal{H}^*$$

and

$$\dot\alpha = \dot\alpha(|\dot{\mathbf{u}}|,\alpha) \quad \text{a.e. on } \Gamma_C$$

Time discretisation

Turn

$$0 \in M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\,\cdot\,,\alpha)(\dot{\mathbf{u}}) - \ell, \qquad \dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}|,\alpha)$$

into

$$0 \in M\ddot{\mathbf{u}}_n + C\dot{\mathbf{u}}_n + A\mathbf{u}_n + \partial\Phi(\,\cdot\,,\alpha_n)(\dot{\mathbf{u}}_n) - \ell_n, \qquad \dot{\alpha} = \dot{\alpha}(|\dot{\mathbf{u}}_n|,\alpha)$$

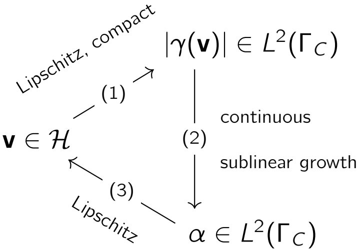
and then (using a time discretisation scheme/solving the ODEs)

$$0 \in (M_n + C + A_n)\dot{\mathbf{u}}_n + \partial\Phi(\,\cdot\,,\alpha_n)(\dot{\mathbf{u}}_n) - \tilde{\ell}_n \qquad \alpha_n = \Psi_{|\dot{\mathbf{u}}_n|}(\alpha_{n-1})$$

$\rightsquigarrow$ A coupling of

1. a convex minimisation problem
2. a family of ordinary differential equations (one-dimensional gradient flows)

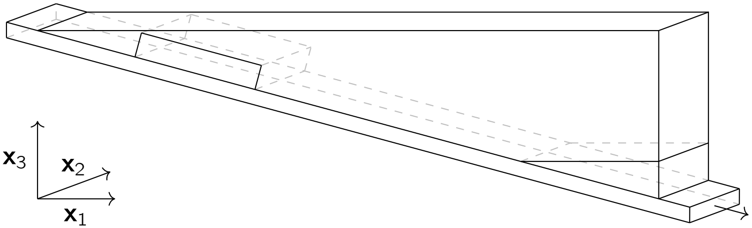
The big picture



$$T\colon \mathcal{H} \to \mathcal{H} \begin{cases} \text{(1) trace map + norm} \\ \text{(2) solve ODEs} \\ \text{(3) convex minimisation} \end{cases}$$

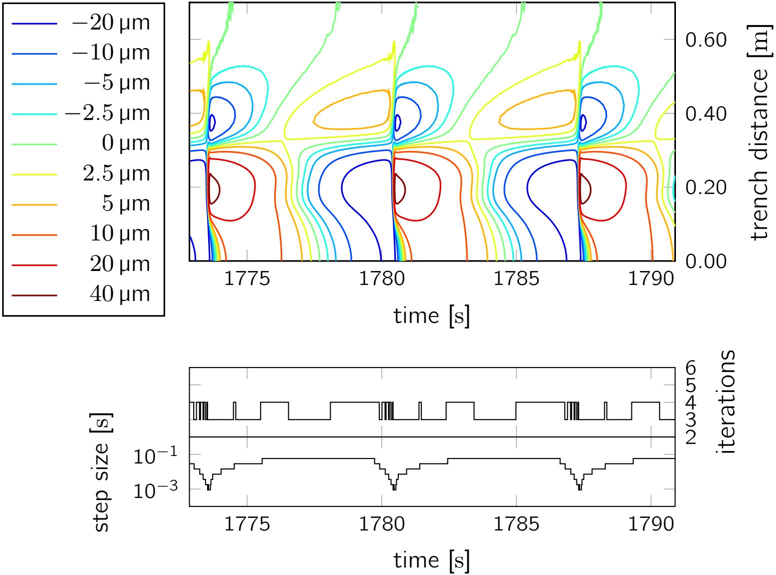
- **Q**: Does T have a fixed point?
  A: Yes, by Banach's/Schauder's fixed point theorem theorem
- **Q**: Is it unique?
  A: Yes/Maybe (depending on the law)
- **Q**: Does $T^n\mathbf{v}$ always converge to a fixed point?
  A: Yes/Maybe (depending on the law)

Application: a simplified subduction zone

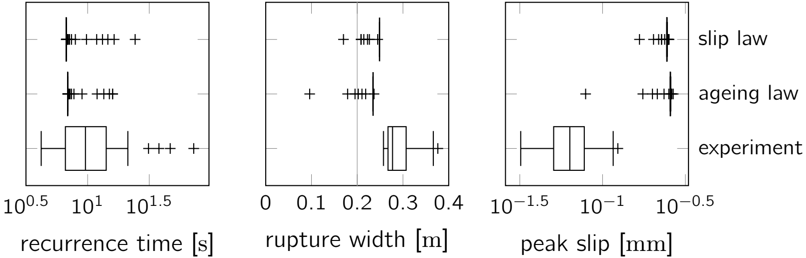


The lower plate moves at a prescribed velocity while the right end of the wedge is held fixed.

Numerical stability: Number of fixed point iterations



Comparison with laboratory data



Recurrence time and rupture width are well reproduced.
Peak slip is off by a factor of approximately 6. The error thus lies within an order of magnitude.

Further reading

E. Pipping, O. Sander and R. Kornhuber. "Variational formulation of rate- and state-dependent friction problems". In: *Zeitschrift für Angewandte Mathematik und Mechanik. Journal of Applied Mathematics and Mechanics* (2013). ISSN: 1521-4001. DOI: 10.1002/zamm.201300062.