Well-Separation and Hyperplane Transversals in High Dimensions*

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Abstract

A family of k point sets in d dimensions is *well-separated* if the convex hulls of any two disjoint subfamilies can be separated by a hyperplane. This notion is instrumental in showing that certain generalized ham-sandwich cuts exist. But how hard is it to check whether a given family of high-dimensional point sets has this property? Starting from this question, we study several algorithmic aspects of the existence of high-dimensional transversals and separations.

1 Introduction

Given a family of k sets $S_1, \dots, S_k$ in $\mathbb{R}^d$, we say that the family is *well-separated* if for any proper index set $I \subset [k]$, with $I \neq \emptyset$ and $I \neq [k]$, the convex hulls of S_I and $S_{[k]\setminus I}$ can be separated by a hyperplane, where we define $S_J = \cup_{j \in J} S_j$, for any proper index set $J \subset [k]$. Well-separation is equivalent to the fact that for any proper index set I, the convex hulls of S_I and $S_{[k]\setminus I}$ do not intersect. A hyperplane h is a *transversal* if $S_i \cap h \neq \emptyset$ for all $i \in [k]$. More generally, an *m-flat* (i.e., an affine subspace of dimension m) is an m-transversal if it intersects all the sets of the family. It turns out that well-separation is intimately related to transversals: a family of sets $S_1, \dots, S_k$ is well-separated if and only if there is no $(k-2)$-transversal of the convex hulls of $S_1, \dots, S_k$. Observe that for any family of $k \leq d$ sets, there always exists a $(k-1)$-transversal. Indeed choose a point from each of the k sets, and consider a $(k-1)$-flat that contains these k points. Furthermore due to Radon's theorem a family of $d+2$ sets in dimension d cannot be well-separated. Radon's theorem states that any set of $d+2$ points in dimension d can be partitioned into two sets with intersecting convex hulls. Questions related to transversals have been studied extensively, mostly from a combinatorial, but also from a computational perspective. For more background, we refer the interested readers to the relevant surveys [2, 10, 11].

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This is an extended abstract of a presentation given at EuroCG'22. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

Well-separation is a strong assumption on set-families, and it should not be a surprise that for many problems, it leads to stronger results and faster algorithms compared to the general case. One such example concerns *Ham-Sandwich cuts.* Given d point sets $P_1, \ldots, P_d$ in $\mathbb{R}^d$, a Ham-Sandwich cut is a hyperplane that simultaneously bisects each point set. While Ham-Sandwich cuts exist for any family of d point sets [16], computing a Ham-Sandwich cut is PPA-complete when the dimension is not fixed [9], meaning that it is unlikely to allow an algorithm that runs in polynomial time in the dimension d. On the other hand, if $P_1, \ldots, P_d$ are well-separated, not only do there exist bisecting hyperplanes, but the Ham-Sandwich theorem can be generalized to hyperplanes cutting off arbitrary given fractions from each point set [5,15]. Moreover, the problem of finding such hyperplanes lies in the complexity class UEOPL [8], a subclass of PPA which is believed to allow for significantly faster algorithms.

From an algorithmic perspective, the main focus of work has been on line transversals in dimensions 2 and 3, see, e.g., [1,4,14]. To the authors' knowledge, in higher dimensions only hyperplane transversals have been studied, where the best known algorithm for deciding whether a set of n polyhedra with m edges has a hyperplane transversal, runs in time $O(nm^{d-1})$ [3]. In particular, there is an exponential dependence in the dimension d. This curse of dimensionality appears in many geometric problems. For several problems, it has been shown that there is probably no hope to get rid of the exponential dependence in the dimension. As an example, we mention a result for Ham-sandwich cuts, due to Knauer, Tiwary and Werner [12]: Given d point sets $P_1, \ldots, P_d$ in $\mathbb{R}^d$ and a point $p \in \mathbb{R}^d$, where d is part of the input, it is $W[1]$-hard (and thus NP-hard) to decide whether there is a Ham-sandwich cut passing through p.

Our Results. A family of k sets in $\mathbb{R}^d$ is well-separated, if and only if their convex hulls have no $(k-2)$-transversal. This fact seems to be well-known, but we could only find some references without proofs, and some proofs of only one direction, for similar definitions of well-separation [6,7]. Therefore, we present a short proof for sake of completeness in the full version. This immediately implies that testing well-separation is in coNP.

In [8], the authors ask what is the complexity of determining whether a family of point sets is well-separated, when d is not fixed. We present several hardness results for finding $(k-2)$-transversals in a family of k sets in $\mathbb{R}^d$. We consider two cases: a) the sets are finite point sets, and b) the sets are convex.

▶ **Theorem 1.1.** *Given a family of $k > d$ point sets in $\mathbb{R}^d$, each consisting of at most two points, it is strongly NP-hard to check whether there is a $(d-1)$-transversal, even in the special case $k = d + 1$.*

Note that this problem is trivial if $k \leq d$, as the answer is always yes. Our result shows that the problem becomes NP-hard for the first value of k for which the problem is non-trivial. We use Theorem 1.1 to show the following:

▶ **Theorem 1.2.** *Given a set of $k > d$ line segments in $\mathbb{R}^d$, it is strongly NP-hard to check whether there is a $(d-1)$-transversal, even in the special case $k = d + 1$.*

Theorem 1.2 implies that testing well-separation is coNP-complete even in the case of $d + 1$ segments in $\mathbb{R}^d$, answering the question from [8].

As a positive result, we can show the existence of the following approximation algorithm. This can be seen as the special case where each point set consists of a single point.

▶ **Theorem 1.3.** *Given a set P of k points in $\mathbb{R}^d$, it is possible to compute in polynomial time in d and k a hyperplane that contains $\Omega(\frac{OPT \log k}{k \log \log k})$ points of P, where OPT denotes the maximum number of points in P that a hyperplane can contain.*

In Section 3, we study the problem through the lens of parametrized complexity. We show a significant difference depending on whether we consider convex sets or finite point sets.

▶ **Theorem 1.4.** *Checking whether a family of $k \leq d+1$ convex hulls of point sets in $\mathbb{R}^d$ has a $(k-2)$-transversal (or equivalently, whether the point sets are well-separated) is FPT with respect to d.*

▶ **Theorem 1.5.** *Given a set of $k > d$ point sets in $\mathbb{R}^d$, it is W[1]-hard with respect to d to check whether there is a $(d-1)$-transversal, even in the special case $k = d+1$.*

Observe that for finite point sets (and more generally for any sets that are not convex), having no $(k-2)$-transversal does not a priori imply well-separation.

2 Hyperplane Transversals in High Dimensions

Let $S_1, \ldots, S_k \subset \mathbb{R}^d$ be k sets in d dimensions, where d is not fixed. Note that we do not assume the sets to be convex. In particular, the sets can even be finite. We consider the decision problem HypTrans: Given sets $S_1, \ldots, S_k$, decide if there is a $(d-1)$-transversal for them. We consider the finite case and the case of line segments. We also consider the optimisation formulation of HypTrans, that we name MaxHyp: Given the sets $S_1, \ldots, S_k$, find a hyperplane that intersects as many of these sets as possible.

We begin with the case that all S_i are finite point sets. We first assume that every S_i contains a single point, for $i = 1, \ldots, k$. Note that in this situation, HypTrans can be solved greedily. We denote by P the point set that is the union of all S_i. Let us denote by OPT the maximum number of points in P that a hyperplane may contain.

▶ **Theorem 1.3.** *Given a set P of k points in $\mathbb{R}^d$, it is possible to compute in polynomial time in d and k a hyperplane that contains $\Omega(\frac{OPT \log k}{k \log\log k})$ points of P, where OPT denotes the maximum number of points in P that a hyperplane can contain.*

Proof. If $k \leq d$, we just output a hyperplane that contains all points of P. Otherwise, let $f(k) = \log k / \log\log k$. If $f(k) < d$, we pick d points from P, and we output a hyperplane through these points. If $f(k) \geq d$, we partition P into disjoint groups of size $f(k)$. In each group, we compute all hyperplanes that go through some d points from the group. Among all hyperplanes for all groups, we output the hyperplane that contains the most points in P. For each group, we have $O(f(k)^d) = O(f(k)^{f(k)}) = O(k)$ hyperplanes to consider. Thus, the algorithm runs in polynomial time in d and k.

We now analyze the approximation guarantee. If $f(k) < d$, then we output a hyperplane with at least $d > f(k) \geq f(k)\mathrm{OPT}/k$ points, since $\mathrm{OPT} \leq k$. If $f(k) \geq d$, we let h be an optimal hyperplane. If h contains at least d points in a single group, then we output an optimal solution. Otherwise, h contains less than d points in each group, so $\mathrm{OPT} \leq d(k/f(k))$. This means that $d \geq f(k)\mathrm{OPT}/k$, and the claim follows from the fact that our solution contains at least d points. ◀

We now restrict ourselves to the situation that every S_i contains at most two points, for $i = 1, \ldots, k$. We will prove that already in this case HypTrans is strongly NP-hard, by reducing from BinPacking. Our reduction will pass through two intermediate problems EqualBinPacking and FlatTrans. We start by defining all the involved problems.

In BinPacking, we are given as input a set of *items* $I = \{I_1, \ldots, I_n\}$ with weights $w(I_i) := w(i) \in \mathbb{Z}_+$, and a set $B = \{B_1, \ldots, B_k\}$ of *bins*, all with the same *capacity* $b \in \mathbb{Z}_+$.

The goal is to decide whether there is a partition of the items into the bins such that in each bin the total weight of the items does not exceed the capacity. In EQUALBINPACKING, we are given the same input, but now the goal is to decide whether there exists a partition of the items into the bins such that in each bin the total weight of the items equals exactly the capacity. Note that BINPACKING can easily be reduced to EQUALBINPACKING by adding the appropriate number of elements of weight 1, so EQUALBINPACKING is strongly NP-hard as well.

Finally, in FLATTRANS, we are given m sets $S_0, \ldots, S_{m-1}$ in $\mathbb{R}^d$, where m and d are both part of the input, and the goal is to decide whether there is an $(m-2)$-transversal. In other words, the question is whether there exists an $(m-2)$-dimensional affine subspace h such that for each $i \in \{0, \ldots, m-1\}$ we have that $S_i \cap h \neq \emptyset$. Note that HYPTRANS with $k = d+1$ is the same as FLATTRANS with $m = d+1$.

▶ **Theorem 2.1.** *FLATTRANS is strongly NP-hard even when $S_0 = \{\mathbf{0}\}$ and any other S_i consists of at most two points.*

Sketch of proof. We reduce from EQUALBINPACKING. Given an input I, B, w, b, where $|I| = n$ and $|B| = k$, to EQUALBINPACKING, we construct an instance of FLATTRANS as follows: First, we set the dimension $d = k + n + kn$ and the number of sets $m = kn + 2$. For any $(i, j) \in [n] \times [k]$ define the vectors

$$v_{i,j}(x) := \begin{cases} w(i), & \text{if } x = j, \\ 1, & \text{if } x = k + i, \\ 1, & \text{if } x = k + n + (i-1)k + j, \\ 0, & \text{else,} \end{cases} \quad \text{and } u_{i,j}(x) := \begin{cases} 0, & \text{if } x = j, \\ 0, & \text{if } x = k + i, \\ 1, & \text{if } x = k + n + (i-1)k + j, \\ 0, & \text{else.} \end{cases}$$

Note that by $x \in \{1, \ldots k + n + kn\}$ we describe the entries of the vector. For example the first entry of $v_{i,j}$ is described by $v_{i,j}(1)$. Further, define the vector $c(x)$ whose entries are $-b$ for $1 \leq x \leq k$ and -1 everywhere else. Now set $S_0 = \{\mathbf{0}\}$, $S_l = \{v_{i,j}, u_{i,j}\}$ for each $l = (i-1)k + j$ (note that this choice of l just gives that the order of the l's corresponds to the lexicographic order of the (i, j)'s) and $S_{kn+1} = \{c\}$. Note that all of this can be done in polynomial time.

In the full version, we show that there is a kn-transversal of the sets $S_0, \ldots, S_{kn+1}$, if and only if there is a valid partition for the EQUALBINPACKING instance. ◀

Now, there is only one reduction remaining:

▶ **Theorem 2.2.** *Let $S_0 = \{\mathbf{0}\}$ and let $S_i \subset \mathbb{R}^d$ be finite for $i = 1, \ldots, m-1$. Then we can construct in polynomial time sets $S'_0, S'_1, \ldots, S'_{d+2} \subset \mathbb{R}^{d+2}$ which can be transversed by a hyperplane if and only if $S_0, S_1, \ldots, S_{m-1} \subset \mathbb{R}^d$ have an $(m-2)$-transversal.*

Sketch of proof. We only show the construction of the sets here. For the complete proof, we refer to the full version. First, for each point p in some set S_i we define the point $p' = (p, 0, 0)$ and place it in the set S'_i. For $m \leq i \leq d+2$, define S'_i as the set consisting only of the point $s'_i = (0, \ldots, 0, 1, i)$. Additionally, let $S'_0 := \{\mathbf{0}\}$. ◀

Theorem 1.1 now follows from combining Theorems 2.1 and 2.2.

Further, we can now show that deciding whether there is a hyperplane transversal for d line segments and the origin in $\mathbb{R}^d$, where d is not fixed, is NP-hard. We will reduce this to the restricted version of HYPTRANS where the sets S_i contain at most two points. This is

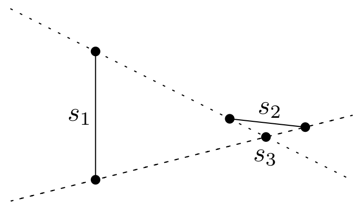

Figure 1 Every hyperplane transversal through s_1, s_2, s_3 must choose an endpoint of s_1 (and of s_2).

done with the help of a gadget that enforces that every hyperplane transversal must use one of the two endpoints of a given line segment. The gadget is shown in Figure 1.

Given a collection of sets of size at most two, for each set we take the line segment formed by its points as s_1, the origin as point s_3, and we construct the corresponding new segment s_2 using the gadget presented in Figure 1. This gives a family S of $2k$ line segments that all lie in a k-dimensional space. In order to prove Theorem 1.2, we need to lift our construction to $\mathbb{R}^{2k}$. This lifting is described in the full version.

3 From the viewpoint of parametrized complexity

Recall that our original motivation comes from determining whether d point sets in $\mathbb{R}^d$ are well-separated. Let us consider those d sets, and let us denote by n the total number of extreme vertices on their respective convex hulls. We say that n is the *convex hull complexity* of the set family. We assume that we are given the extreme points of the convex hull of every set and hence have a finite number of points for every set.

▶ **Theorem 1.4.** *Checking whether a family of $k \leq d+1$ convex hulls of point sets in $\mathbb{R}^d$ has a $(k-2)$-transversal (or equivalently, whether the point sets are well-separated) is FPT with respect to d.*

Sketch of proof. For the $O(2^d)$ choices of index sets $I \subset [k]$, we check with an LP whether the convex hulls of S_I and $S_{[k]\setminus I}$ intersect. ◀

On the other hand, using a framework similar to the one introduced by Marx [13], we show in the full version that

▶ **Theorem 3.1.** *FlatTrans is $W[1]$-hard with respect to the dimension.*

Combining this with Theorem 2.2, we deduce Theorem 1.5.

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