Algorithms for Tolerated Tverberg Partitions

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Abstract. Let P be a d-dimensional n-point set. A partition $\mathcal{T}$ of P is called a *Tverberg* partition if the convex hulls of all sets in $\mathcal{T}$ intersect in at least one point. We say $\mathcal{T}$ is *t-tolerated* if it remains a Tverberg partition after deleting any t points from P. Soberón and Strausz proved that there is always a t-tolerated Tverberg partition with $\lceil n/(d+1)(t+1) \rceil$ sets. However, so far no nontrivial algorithms for computing or approximating such partitions have been presented.

For $d \leq 2$, we show that the Soberón-Strausz bound can be improved, and we show how the corresponding partitions can be found in polynomial time. For $d \geq 3$, we give the first polynomial-time approximation algorithm by presenting a reduction to the (untolerated) Tverberg problem. Finally, we show that it is coNP-complete to determine whether a given Tverberg partition is t-tolerated.

1 Introduction

Let $P \subset \mathbb{R}^d$ be a point set of size n. A point $c \in \mathbb{R}^d$ has *(Tukey) depth* m with respect to P if every closed half-space containing c also contains at least m points from P. A point of depth $\lceil n/(d+1) \rceil$ is called a *centerpoint* for P. The well-known Centerpoint Theorem [10] states that any point set has a centerpoint. Centerpoints are of great interest as they constitute a natural generalization of the median to higher-dimensions and since they are invariant under scaling or translations and robust against outliers.

Chan [1] described a randomized algorithm that finds a d-dimensional centerpoint in expected time $\mathcal{O}(n^{d-1})$. Actually, Chan solves the seemingly harder problem of finding a point with maximum depth, and he conjectures that his result is optimal. Since this is infeasible in higher dimensions, approximation algorithms are of interest. Already in 1993, Clarkson et al. [2] developed a Monte-Carlo algorithm that finds a point with depth $\Omega(n/(d+1)^2)$ in time $\mathcal{O}(d^2(d \log n + \log(1/\delta))^{\log(d+2)})$, where δ is the error-probability. Teng [13] proved that testing whether a given point is a centerpoint is coNP-complete, so we do not know how to verify efficiently the output of the algorithm by Clarkson et al. For a subset of centerpoints, *Tverberg partitions* [14] provide polynomial-time checkable proofs for the depth: a *Tverberg m-partition* for a point set $P \subset \mathbb{R}^d$ is

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a partition $P = T_1 \dot\cup T_2 \dot\cup \dots \dot\cup T_m$ of P into m sets such that $\bigcap_{i=1}^m \mathsf{conv}(T_i) \neq \emptyset$. Each half-space that intersects $\bigcap_{i=1}^m \mathsf{conv}(T_i)$ must contain at least one point from each T_i, so each point in $\bigcap_{i=1}^m \mathsf{conv}(T_i)$ has depth at least m. Tverberg's theorem states that $m = \lceil n/(d+1) \rceil$ is always possible. Thus, there is always a centerpoint with a corresponding Tverberg partition. Miller and Sheehy [7] developed a deterministic algorithm that computes a point of depth $\lceil n/2(d+1)^2 \rceil$ in time $n^{\mathcal{O}(\log d)}$ together with a corresponding Tverberg partition. This was recently improved by Mulzer and Werner [9]. Through recursion on the dimension, they can find a point of depth $\lceil n/4(d+1)^3 \rceil$ and a corresponding Tverberg partition in time $d^{\mathcal{O}(\log d)} n$.

Let $\mathcal{T}$ be a Tverberg m-partition for P. If any nonempty subset $R \subset P$ is removed from P, we do not longer know if $\bigcap_{i=1}^m \mathsf{conv}(T_i \setminus R) \neq \emptyset$. In the worst-case, the maximum number of sets in $\mathcal{T}$ whose convex hulls still have a nonempty intersection is $m - |R|$. This is not always desired. It is therefore of interest to study Tverberg partitions that guarantee $\bigcap_{i=1}^m \mathsf{conv}(T_i \setminus R)$ to be nonempty if the size of R is not "too big". We call a Tverberg partition *t-tolerated* if it remains a Tverberg partition of P even after removing t arbitrary points from P. In 1972, Larman [5] proved that every set of size $2d + 3$ admits a 1-tolerated Tverberg 2-partition. This was motivated by a problem that was proposed to him by McMullen: find the largest number of points that can be brought in convex position by a permissible projective transformation. Colín [4] generalized Larman's result, showing that sets of size $(t + 1)(d + 1) + 1$ always have a t-tolerated Tverberg 2-partition. Later, Montejano and Oliveros conjectured that every set of size $(t + 1)(m - 1)(d + 1) + 1$ admits a t-tolerated Tverberg m-partition [8, Conjecture 4.2]. This was proven by Soberón and Strausz [12] who adapted Sarkaria's proof of Tverberg's theorem [11] to the tolerated setting. Soberón and Strausz also conjectured this bound to be tight [12, Conjecture 1]. Up to now, no exact or approximation algorithms for tolerated Tverberg partitions appear in the literature.

In this paper, we give new bounds for one- and two-dimensional tolerated Tverberg partitions, disproving the Soberón-Strausz-conjecture. We also give efficient algorithms for finding the corresponding partitions. Our bound is tight for $d = 1$. For higher dimensions, we describe an approximation preserving reduction to the untolerated Tverberg problem. Thus, we can apply existing and possible future algorithms for the untolerated Tverberg problem in the tolerated setting. Finally, we show that testing whether a given Tverberg partition has tolerance t is coNP-complete if the dimension is not fixed.

2 Low Dimensions

We start with an algorithm for the one-dimensional case that yields a tight bound. This can be bootstrapped to higher dimensions with a lifting approach similar to [9]. In two dimensions, we also get an improved bound if the size of the desired partition and the tolerance is large enough.

2.1 One Dimension

Let $P \subset \mathbb{R}$ with $|P| = n$, and let $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ be a t-tolerated Tverberg m-partition of P. By definition, there is no subset $R \subset P, |R| = t$ whose removal separates the convex hulls of the sets in $\mathcal{T}$. Bounding the size of the sets in $\mathcal{T}$ gives us more insight into the structure.

Lemma 2.1. *Let $P \subset \mathbb{R}$ with $|P| = n$ and $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ a t-tolerated Tverberg m-partition of P. Then*

(i) for $i = 1, \ldots, m$, we have $|T_i| \geq t + 1$; and
(ii) for $i, j = 1, \ldots, m$, $i \neq j$, we have $|T_i \cup T_j| \geq 2t + 3$.

Proof. (i) Suppose $|T_i| \leq t$. After removing T_i from P, the intersection of the convex hulls of the sets in $\mathcal{T}$ becomes empty, and $\mathcal{T}$ would not be t-tolerated.
(ii) Suppose there are $T_i, T_j \in \mathcal{T}$ with $|T_i \cup T_j| \leq 2t + 2$. By (i), we have $|T_i| = |T_j| = t+1$. Let $p_{\min} = \min(T_i \cup T_j)$ and assume w.l.o.g. that $p_{\min} \in T_i$ (see Figure 1). Then $|T_i \setminus \{p_{\min}\}| = t$, and removing the set $T_i \setminus \{p_{\min}\}$ separates the convex hulls of T_i and T_j. This again contradicts $\mathcal{T}$ being t-tolerated.

□

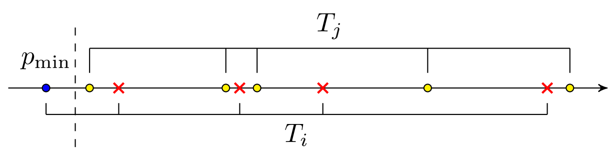

Fig. 1. The convex hulls of two sets of size $t+1$ can be separated by removing t points.

Lemma 2.1 immediately implies a lower bound on the size of any point set that admits a t-tolerated Tverberg m-partition.

Corollary 2.2. *Let $P \subset \mathbb{R}$ with $|P| < m(t + 2) - 1$. Then P has no t-tolerated Tverberg m-partition.*

Now what happens for $|P| = m(t+2)-1$? Note that for $t > 0$ and $m > 2$, we have $m(t+2)-1 < 2(t+1)(m-1)+1$, the bound by Soberón and Strausz. Thus, proving that a t-tolerated Tverberg m-partition exists for any one-dimensional point set of size $m(t + 2) - 1$ would disprove the Soberón-Strausz conjecture.

Let $P \subset \mathbb{R}$ be of size $m(t+2)-1$. By Lemma 2.1, in any t-tolerated Tverberg partition of P, one set has to be of size $t + 1$ and all other sets have to be of size $t + 2$. Let $\mathcal{T} = \{T_1, \ldots, T_m\}$ be a Tverberg m-partition of P such that T_1 contains every mth point of P and each other set T_i $(i \geq 2)$ has one point in each interval defined by the points of T_1; see Fig. 2 for $m = 3$ and $t = 2$. Note that $|T_1| = t + 1$ and $|T_i| = t + 2$ $(i \geq 2)$. We will show that $\mathcal{T}$ is t-tolerated. Intuitively, $\mathcal{T}$ maximizes the interleaving of the sets, making the convex hulls more robust to changes.

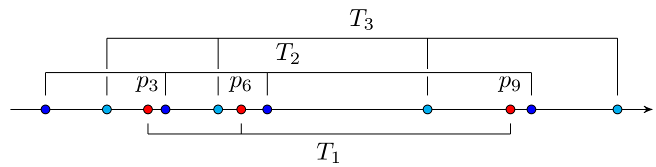

Fig. 2. A 2-tolerated Tverberg 3-partition for 11 $(= 3(2 + 2) - 1)$ points.

Lemma 2.3. *Let $P \subset \mathbb{R}$ with $|P| = m(t + 2) - 1$, and let $\mathcal{T} = \{T_1, \ldots, T_m\}$ be an m-partition of P. Suppose that $|T_1| = t+1$, and write $T_1 = (p_1, p_2, \ldots, p_{t+1})$, sorted from left to right. Suppose that each interval $\mathcal{I} \in \{(-\infty, p_1), (p_1, p_2), \ldots, (p_{t+1}, \infty)\}$ contains one point from each T_i, for $i = 2, \ldots, m$. Then $\mathcal{T}$ is a t-tolerated Tverberg m-partition for P.*

Proof. Suppose there exist $T_i, T_j \in \mathcal{T}$, $(i \neq j)$ and a subset $R \subset P$ of size t such that removing R from P separates the convex hulls of T_i and T_j. Let h be a point that separates $\mathsf{conv}(T_i \setminus R)$ and $\mathsf{conv}(T_j \setminus R)$. Let $T_i^- = T_i \cap (-\infty, h]$ and $T_i^+ = T_i \cap (h, \infty)$, and define T_j^-, T_j^+ similarly. Figure 3 shows the situation. Set $l = |T_1^-| = |T_1 \cap (-\infty, h]|$. By construction of $\mathcal{T}$, both T_i^- and T_j^- contain exactly l or $l + 1$ points.

Since removing R separates the convex hulls of T_i and T_j at h, R must contain either $T_i^- \cup T_j^+$ or $T_j^- \cup T_i^+$. However, we have

$$|T_i^- \cup T_j^+| = |T_i^-| + |T_j| - |T_j^-| \geq \begin{cases} l + |T_j| - (l + 1) = |T_j| - 1 = t + 1 & \text{if } j \neq 1 \\ l + |T_1| - l = |T_1| = t + 1 & \text{if } j = 1 \end{cases}$$

and similarly $|T_j^- \cup T_i^+| \geq t + 1$, a contradiction.

Thus, even after removing t points, the convex hulls of the sets in $\mathcal{T}$ intersect pairwise. Helly's theorem [6, Theorem 1.3.2] now guarantees that the convex hulls of all sets in $\mathcal{T}$ have a common intersection point. Hence, $\mathcal{T}$ is t-tolerated. $\square$

Lemma 2.3 immediately gives a way to compute a t-tolerated Tverberg m-partition in $\mathcal{O}(mt \log mt)$ time for $|P| = m(t + 2) - 1$ by sorting P. However, it is not necessary to know the order of all of P. Algorithm 1 exploits this fact to improve the running time. It repeatedly partitions the point set until it has selected all points whose ranks are multiples of m. These points form the set T_1. Initially, the set Q contains only the input P (line 4). In lines 6–11, we select from each set in Q an element whose rank is a multiple of m (line 8) and we

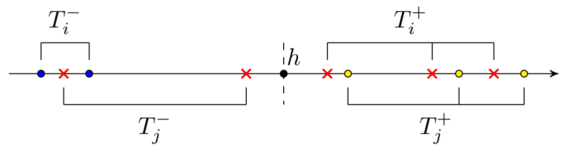

Fig. 3. The convex hulls of two elements in $\mathcal{T}$ are separated after the removal of R. Crosses mark the removed points (i.e., points in R).

split the set at this element. Here, `select`(P, k) is a procedure that returns the element with rank k of P. After termination of both loops in lines 5–11, all remaining sets in Q correspond to points in P between two consecutive points in T_1. In lines 12–14, the points in the sets in Q are distributed equally among the elements T_i ($i \geq 2$) of the returned partition.

Algorithm 1: 1d-Tolerated-Tverberg

input : $P \subset \mathbb{R}$, size of partition m
1 $r \leftarrow m$;
2 **while** $r \leq |P|/2$ **do**
3     $r \leftarrow 2 \cdot r$;
4 $Q \leftarrow \{P\}$; $T_1, T_2, \ldots, T_m \leftarrow \emptyset, \emptyset, \ldots, \emptyset$;
5 **while** $r \geq m$ **do**
6   **foreach** $P' \in Q$ *with* $|P'| \geq r$ **do**
7   remove P' from Q;
8         $p_r \leftarrow$ `select`(P', r);
9         $Q \leftarrow Q \cup \{\{p' \in P' \mid p' < p_r\}, \{p' \in P' \mid p' > p_r\}\}$;
10         $T_1 \leftarrow T_1 \cup \{p_r\}$;
11         $r \leftarrow r/2$;
12 **foreach** $P' \in Q$ **do**
13   **foreach** $j \in \{2, 3, \ldots, m\}$ **do**
14   remove any point from P' and add it to T_j;
15 **return** $\{T_1, T_2, \ldots, T_m\}$;

Theorem 2.4. *Let $P \subset \mathbb{R}$ be a set of size $m(t + 2) - 1$. On input (P, m), Algorithm 1 returns a t-tolerated Tverberg partition for P in time $\mathcal{O}(mt \log t)$.*

Proof. After each iteration of the outer while-loop (lines 5–11), each element $P' \in Q$ has size strictly less than r: initially, Q contains only P and r is strictly greater than $|P|/2$. Hence, both new sets added to Q in line 9 are of size strictly less than r. Since r is halved in each iteration, the invariant is maintained.

We will now check that Lemma 2.3 applies. We only split the sets in Q at elements whose rank is a multiple of m, so the ranks do not change modulo m. By the invariant, after the termination of the outer while-loop in lines 5–11, each set in Q has size strictly less than m. Since the ranks modulo m have not changed, these sets do not contain any element of P whose rank is a multiple of m. Thus, T_1 contains all these elements and the remaining sets in Q after the termination of the outer while-loop in lines 5–11 contain exactly the points of P between two consecutive points of T_1. Lines 12–14, distribute the remaining points among $T_2, \ldots, T_m$. Lemma 2.3 now shows the correctness of the algorithm.

Let us consider the running time. Finding the initial r requires $\mathcal{O}(\log(|P|/m)) = \mathcal{O}(t)$ time. The split-element in line 8 can be found in time $\mathcal{O}(|P'|)$ [3]. Thus, since the sets are disjoint, one iteration of the outer while-loop requires $\mathcal{O}(|P|)$ time, for a total of $\mathcal{O}(\log(|P|/m)|P|) = \mathcal{O}(\log(t)mt)$. By the same argument, both for-loops in lines 12–14 require linear time in the size of P. This results in a total time complexity of $\mathcal{O}(mt\log t)$ as claimed. □

2.2 Higher Dimensions

We use a lifting argument [9] to extend Algorithm 1 to higher-dimensional input. Given a point set $P \subseteq \mathbb{R}^d$ of size n, let h be a hyperplane that splits P evenly (if n is odd, h contains exactly one point of P). We then partition P into $\lfloor n/2 \rfloor$ pairs (p_i^-, p_i^+), where $p_i^- \in h^-$ and $p_i^+ \in h^+$. We obtain a $(d-1)$-dimensional point set with $\lfloor n/2 \rfloor$ elements by mapping each pair to the intersection of the connecting line segment and h.

Let $q_i = p_i^+ p_i^- \cap h$ be the mapped point for (p_i^-, p_i^+) and $\mathcal{T}' = \{T_1', \ldots, T_m'\}$ a t-tolerated Tverberg m-partition of $Q = \{q_1, \ldots, q_{\lfloor n/2 \rfloor}\}$. We obtain a Tverberg m-partition $\mathcal{T}$ with tolerance t for P by replacing each q_i in $\mathcal{T}'$ by its corresponding pair (p_i^-, p_i^+). Thus, we can repeatedly project the set P until Algorithm 1 is applicable. Then, we lift the one-dimensional solution back to higher dimensions.

Algorithm 2 follows this approach. For $d = 1$, Algorithm 1 is applied (lines 1–2). Otherwise, we take an appropriate hyperplane orthogonal to the x_d-axis and compute the lower-dimensional point set (lines 3–7). Finally, the result for $d-1$ dimensions is lifted back to d dimensions (lines 10–11).

Proposition 2.5. *Given a set $P \subset \mathbb{R}^d$ of size $2^{d-1}(m(t+2)-1)$, Algorithm 2 computes a t-tolerated Tverberg m-partition for P in time $\mathcal{O}(2^{d-1}dmt + mt\log t)$.*

Proof. Since the size of P halves in each recursion step, 2^{d-1} points suffice to ensure that Algorithm 1 can be applied in the base case. Each projection and lifting step can be performed in linear time, using a median computation. Since the size of the point set decreases geometrically, the total time for projection and lifting is thus $\mathcal{O}(2^{d-1}dmt)$. Since Algorithm 1 has running time $O(mt\log t)$, the result follows. □

For $d \geq 3$, the bound from Proposition 2.5 is worse than the Soberón-Strausz bound. However, in two dimensions, we have

$$2^{2-1}(m(t+2)-1) < (2+1)(m-1)(t+1)+1 \Leftrightarrow m/(m-3) < t$$

This holds for instance if $t \geq 5$ or $m \geq 7$ and $t \geq 2$. Thus, Algorithm 2 gives a strict improvement over the Soberón-Strausz bound for large enough m and t.

Algorithm 2: DimReduct-Tolerated-Tverberg

input : point set $P \subset \mathbb{R}^d$, tolerance parameter t, size of partition m
output: t-tolerated Tverberg partition for P of size m
if $d = 1$ **then**
  return `1d-Tolerated-Tverberg(`P,m`)`
$h \leftarrow$ hyperplane that halves P according to the x_d-coordinate;
foreach $i \in \{1, 2, \ldots, |P \cap h^-|\}$ **do**
  $p_i^- \leftarrow$ remove any point from P that belongs to $P \cap h^-$;
  $p_i^+ \leftarrow$ remove any point from P that belongs to $P \cap h^+$;
  $q_i \leftarrow$ first $d-1$ coordinates of $p_i^- p_i^+ \cap h$;
$Q \leftarrow \{q_1, q_2, \ldots, q_{|P \cap h^-|}\}$;
$\{T_1', T_2', \ldots, T_m'\} \leftarrow$ `DimReduct-Tolerated-Tverberg(`Q,t,m`)`;
foreach $j \in \{1, 2, \ldots, m\}$ **do**
  $T_j \leftarrow \{p_i^-, p_i^+ \mid q_i \in T_j'\}$;
return $\{T_1, T_2, \ldots, T_m\}$;

3 Reduction to the Untolerated Tverberg Problem

We now show how to use any algorithm that computes (untolerated) approximate Tverberg partitions in order to find tolerated Tverberg partitions. For this, we must increase the tolerance of a Tverberg partition. In the following, we show that one can merge elements of several Tverberg partitions for disjoint subsets of P to obtain a Tverberg partition with higher tolerance for the whole set P. The following lemma is also implicit in the Ph.D. thesis of Colín [4].

Lemma 3.1. *Let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be Tverberg m-partitions for disjoint point sets P_1, $\ldots$,$P_k \subset \mathbb{R}^d$. Let $T_{i,j}$ be the jth element of $\mathcal{T}_i$ and $t_i \geq 0$ the tolerance of $\mathcal{T}_i$. Then $\mathcal{T} = \{T_j = \bigcup_{i=1}^k T_{i,j} \mid j \in \{1, 2, \ldots, m\}\}$ is a Tverberg m-partition of $P = \bigcup_{j=1}^k P_i$ with tolerance $t = \sum_{i=1}^k t_i + k - 1$.*

Proof. Take $R \subseteq P$ with $|R| = t$. As $t = \sum_{i=1}^k t_i + k - 1 < \sum_{i=1}^k (t_i + 1)$, there is an i with $|P_i \cap R| \leq t_i$. Since $\mathcal{T}_i$ is t_i-tolerated, we have $\bigcap_{i=j}^m \mathsf{conv}(T_{i,j} \setminus R) \neq \emptyset$. Because each $T_{i,j}$ is contained in the corresponding T_j of $\mathcal{T}$, the convex hulls of the elements in $\mathcal{T}$ still intersect after the removal of R. □

This directly implies a simple algorithm: we compute untolerated Tverberg partitions for disjoint subsets of P and then merge them using Lemma 3.1.

Corollary 3.2. *Let $P \subseteq \mathbb{R}^d$ and let $\mathcal{A}$ be an algorithm that computes an untolerated Tverberg m-partition for any point set of size $n_{\mathcal{A}}(m)$ in time $T_{\mathcal{A}}(m)$. Then, a $(\lfloor |P|/n_{\mathcal{A}}(m) \rfloor - 1)$-tolerated Tverberg m-partition for P can be computed in time $\mathcal{O}\left(T_{\mathcal{A}}(n_{\mathcal{A}}(m)) \cdot |P|/n_{\mathcal{A}}(m)\right)$.*

Proof. We split P into $\lfloor |P|/n_{\mathcal{A}} \rfloor$ disjoint sets and use $\mathcal{A}$ to obtain for each subset an untolerated Tverberg partition. Applying Lemma 3.1, we obtain a $(\lfloor |P|/n_{\mathcal{A}} \rfloor -$

1)-tolerated Tverberg m-partition. Since the merging step in Lemma 3.1 takes linear time in $|P|$, the total running time is $\mathcal{O}(T_{\mathcal{A}}(n_{\mathcal{A}}(m)) \cdot |P|/n_{\mathcal{A}}(m))$ as claimed. $\square$

Table 3 shows specific values for Corollary 3.2 combined with Miller & Sheehy's and Mulzer & Werner's algorithm.

Algorithm	**Tolerance**	**Running time**
Corollary 3.2 with Miller-Sheehy	$\lfloor \|P\|/2m(d+1)^2 \rfloor - 1$	$m^{\mathcal{O}(\log d)} d^{\mathcal{O}(\log d)} \|P\|$
Corollary 3.2 with Mulzer-Werner	$\lfloor \|P\|/4m(d+1)^3 \rfloor - 1$	$d^{\mathcal{O}(\log d)} \|P\|$

Table 1. Corollary 3.2 combined with existing approximation algorithms for the untolerated Tverberg problem.

Remark 3.3. Lemma 3.1 gives a quick proof of a slightly weaker version of the Soberón-Strausz bound: partition P into $t+1$ disjoint sets of size at least $\lfloor |P|/(t+1) \rfloor$. By Tverberg's theorem, for each subset there exists an untolerated Tverberg partition of size $\lceil \lfloor |P|/(t+1) \rfloor/(d+1) \rceil$. Using Lemma 3.1, we obtain a t-tolerated Tverberg partition of size $\lceil \lfloor |P|/(t+1) \rfloor/(d+1) \rceil \geq \lceil |P|/(t+1)(d+1) \rceil - 1$ of P, which is at most one less than the Soberón-Strausz bound. This weaker bound was also stated by Colín [4, Lemma 3.3.13].

4 Hardness of Tolerance Testing

Teng [13, Theorem 8.4] proved that testing whether a given point is a centerpoint of a given set (TestingCenter) is coNP-complete if the dimension is part of the input. We show the same for the problem of deciding whether a given Tverberg m-partition has tolerance t (TestingToleratedTverberg) by a reduction to TestingCenter. Here, m can be constant.

Lemma 4.1. *Let $P \subset \mathbb{R}^d$ and let $c \in \mathbb{R}^d$. Then c has depth $t+1$ w.r.t. P if and only if for all subsets $R \subset P, |R| \leq t : c \in \mathsf{conv}(P \setminus R)$.*

Proof. "⇒" Suppose there is some $R \subset P, |R| \leq t$ with $c \notin \mathsf{conv}(P \setminus R)$. Then, there is a half-space h^+ that contains c but no points from $\mathsf{conv}(P \setminus R)$. Thus, $c \in h^+$ and $|P \cap h^+| \leq |R| \leq t$, and hence c has depth $\leq t$ w.r.t. P.

"⇐" Assume c has depth $t' \leq t$ w.r.t. P. Let h^+ be a half-space that contains c and t' points from P. Set $R = h^+ \cap P$. Then, $|R| \leq t$ and $c \notin \mathsf{conv}(P \setminus R)$. $\square$

Theorem 4.2. TestingToleratedTverberg *is coNP-complete if the dimension d and the claimed tolerance t are part of the input.*

Proof. Since testing whether a given partition is Tverberg is a simple application of linear programming, the problem lies in coNP.

Let $(P \subset \mathbb{R}^d, c \in \mathbb{R}^d)$ be an input to TestingCenter. We embed the vector space $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ by identifying it with the hyperplane $h : x_{d+1} = 0$. Let ℓ be the line that is orthogonal to h and passes through c. Furthermore, let T^- and T^+ be sets of $t+1$ arbitrary points in $\ell \cap h^-$ and $\ell \cap h^+$, respectively. Set $T = T^- \cup T^+$. We claim that $\{P, T\}$ is a Tverberg 2-partition for $P \cup T$ with tolerance $t = \lceil |P|/(d+1) \rceil - 1$ if and only if c is a centerpoint of P. See Figure 4.

"$\Rightarrow$" Assume $\{P, T\}$ is a t-tolerated Tverberg 2-partition. By construction of T, we have $\mathsf{conv}(P) \cap \mathsf{conv}(T) = \{c\}$. Thus, c lies in the intersection of both convex hulls even if any subset of size at most t is removed. Lemma 4.1 implies that c has depth $t + 1 = \lceil |P|/(d+1) \rceil$ w.r.t. P, so c is a centerpoint for P.

"$\Leftarrow$" Assume c is a centerpoint for P. By definition, c has depth at least $\lceil |P|/(d+1) \rceil = t+1$ w.r.t. P. Lemma 4.1 then implies that c is contained in the convex hull of P even if any t points from P are removed. Since T contains $t+1$ points on both sides of a line through c, c is also contained in $\mathsf{conv}(T)$ if any t points from T are removed. Thus, $\{P, T\}$ is a t-tolerated Tverberg 2-partition for $P \cup T$. $\square$

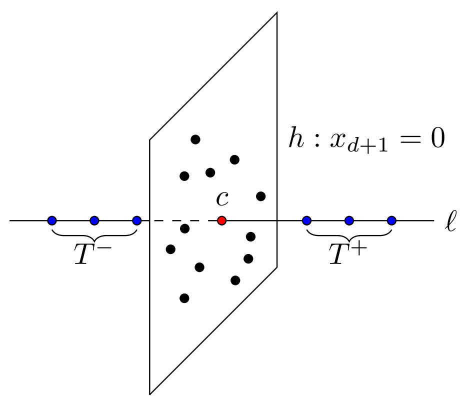

Fig. 4. Reduction of TestingCenter to TestingToleratedTverberg

5 Conclusion

We have shown that each set $P \subset \mathbb{R}$ of size $m(t+2) - 1$ can be partitioned into a t-tolerated Tverberg partition of size m in time $\mathcal{O}(mt \log t)$. This is tight, and it improves the Soberón-Strausz bound in one dimension. Combining this with a lifting method, we could also get improved bounds in two dimensions and an efficient algorithm for tolerated Tverberg partitions in any fixed dimension. However, the running time is exponential in the dimension.

This motivated us to look for a way of reusing the existing technology for the untolerated Tverberg problem. We have presented a reduction to the untolerated Tverberg problem that enables us to reuse the approximation algorithms by Miller & Sheehy and Mulzer & Werner.

Finally, we proved that testing whether a given Tverberg partition is of some tolerance t is coNP-complete. Unfortunately, this does not imply anything about the complexity of finding tolerated Tverberg partitions. It is not even clear whether computing tolerated Tverberg partitions is harder than computing untolerated Tverberg partitions. However, we have shown that given a set $P \subset \mathbb{R}^d$ whose size meets the Soberón-Strausz bound, we can obtain in polynomial time a tolerated Tverberg partition from the untolerated Tverberg partition guaranteed by Tverberg's Theorem of size just one less than stated by the Soberón-Strausz bound.

It remains open whether the bound by Soberón and Strausz is tight for $d > 2$. We believe that our results in one and two dimensions indicate that the bound can be improved also in general dimension. Another open problem is finding a *pruning strategy* for tolerated Tverberg partitions. By this, we mean an algorithm that efficiently reduces the sizes of the sets in a t-tolerated Tverberg partition without deteriorating the tolerance. Such an algorithm could be used to improve the quality of our algorithms. In Miller & Sheehy's and Mulzer & Werner's algorithms, Carathéodory's theorem was used for this task. Unfortunately, this result does not preserve the tolerance of the pruned partitions. Also the generalized tolerated Carathéodory theorem [8] does not seem to help. It remains an interesting problem to develop criteria for superfluous points in tolerated Tverberg partitions.

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