Algorithms for Tolerant Tverberg Partitions

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Abstract. Let P be a d-dimensional n-point set. A partition $\mathcal{T}$ of P is called a *Tverberg* partition if the convex hulls of all sets in $\mathcal{T}$ intersect in at least one point. We say that $\mathcal{T}$ is *t-tolerant* if it remains a Tverberg partition after deleting any t points from P. Soberón and Strausz proved that there is always a t-tolerant Tverberg partition with $\lceil n/(d+1)(t+1)\rceil$ sets. However, no nontrivial algorithms for computing or approximating such partitions have been presented so far.
For $d \leq 2$, we show that the Soberón-Strausz bound can be improved, and we show how the corresponding partitions can be found in polynomial time. For $d \geq 3$, we give the first polynomial-time approximation algorithm by presenting a reduction to the regular Tverberg problem (with no tolerance). Finally, we show that it is coNP-complete to determine whether a given Tverberg partition is t-tolerant.

Keywords: Tverberg partition; tolerant Tverberg partition; high-dimensional approximation; coNP-completeness.

1 Introduction

Let $P \subset \mathbb{R}^d$ be a point set of size n. A point $c \in \mathbb{R}^d$ has *(Tukey) depth m* with respect to P if every closed half-space containing c also contains at least m points from P. A point of depth $\lceil n/(d+1)\rceil$ is called a *centerpoint* for P. The well-known centerpoint theorem [10] states that every point set has a centerpoint. Centerpoints are of great interest since they constitute a natural generalization of the median to higher-dimensions and are invariant under scaling or translations and robust against outliers.

Chan [1] described a randomized algorithm that finds a d-dimensional centerpoint in expected time $\mathcal{O}(n^{d-1})$. In fact, Chan solves the seemingly harder problem of finding a point with maximum depth, and he conjectures that his result is optimal. Since this is infeasible in higher dimensions, approximation algorithms are of interest. Already in 1993, Clarkson et al. [2] developed a Monte-Carlo algorithm that finds a point with depth $\Omega(n/(d+1)^2)$ in time $\mathcal{O}(d^2(d\log n + \log(1/\delta))^{\log(d+2)})$, where δ is the error-probability. Teng [15] proved that it is

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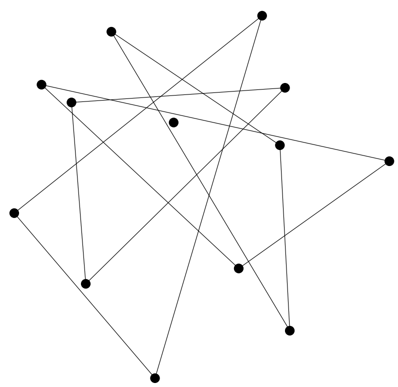
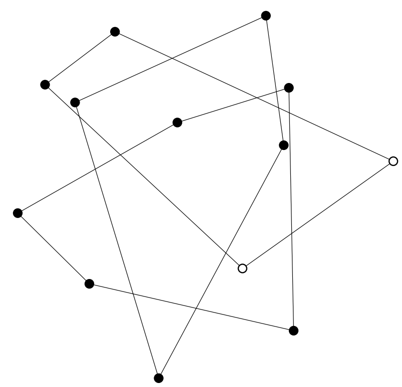
coNP-complte to test whether a given point is a centerpoint. Thus, we do not know how to verify efficiently the output of the algorithm by Clarkson et al. For a subset of centerpoints, *Tverberg partitions* [16] provide polynomial-time checkable proofs for the depth: a *Tverberg m-partition* for a point set $P \subset \mathbb{R}^d$ is a partition $P = T_1 \dot\cup T_2 \dot\cup \cdots \dot\cup T_m$ of P into m sets such that $\bigcap_{i=1}^m \mathsf{conv}(T_i) \neq \emptyset$. Each half-space that intersects $\bigcap_{i=1}^m \mathsf{conv}(T_i)$ must contain at least one point from each T_i, so each point in $\bigcap_{i=1}^m \mathsf{conv}(T_i)$ has depth at least m. Tverberg's theorem states that depth $m = \lceil n/(d+1) \rceil$ can always be achieved. Thus, there is always a centerpoint with a corresponding Tverberg partition. Miller and Sheehy [7] developed a deterministic algorithm that computes a point of depth $\lceil n/2(d+1)^2 \rceil$ in time $n^{\mathcal{O}(\log d)}$ together with a corresponding Tverberg partition. This was recently improved by Mulzer and Werner [9]. Through recursion on the dimension, they can find a point of depth $\lceil n/4(d+1)^3 \rceil$ and a corresponding Tverberg partition in time $d^{\mathcal{O}(\log d)} n$.



(a) 0-tolerant Tverberg 5-partition



(b) 1-tolerant Tverberg 3-partition

Fig. 1. (a) A regular Tverberg partition for 13 points. One set of the partition consists of a single point. The removal of any point would separate the convex hulls. (b) A 1-tolerant Tverberg partition for the same point set. The Tverberg partition is not 2-tolerant since the removal of both white points would separate the convex hulls.

Let $\mathcal{T}$ be a Tverberg m-partition for P. If any nonempty subset $R \subset P$ is removed from P, we no longer know if $\bigcap_{i=1}^m \mathsf{conv}(T_i \setminus R) \neq \emptyset$. In the worst-case, the maximum number of sets in $\mathcal{T}$ whose convex hulls still have a nonempty intersection is $m - |R|$. Thus, after removing only m points, the convex hulls of sets in $\mathcal{T}$ could be pairwise disjoint and do not longer serve as a depth-certificate for points in the intersection of the convex hulls. In order to give stronger guarantees if points are removed, we study Tverberg partitions that remain Tverberg partitions even after removing t arbitrary points from P. We call a Tverberg partition *t-tolerant* if $\bigcap_{i=1}^m \mathsf{conv}(T_i \setminus R)$ is nonempty for any subset $R \subset P$ of size at most t. To distinguish tolerant Tverberg partitions from Tverberg partitions with no tolerance, we call the latter *regular* Tverberg partitions. Furthermore, we say that a set R *separates* the convex hulls of the sets in $\mathcal{T}$ if $\bigcap_{i=1}^{m} \mathsf{conv}(T_i \setminus R) = \emptyset$. See Figure 1 for two examples. In 1972, Larman [5] proved that every set of size $2d+3$ admits a 1-tolerant Tverberg 2-partition. This was motivated by a problem proposed by McMullen: find the largest number n such that any n-point set can be made convex by applying a permissible projective transformation. Here, permissible means that no point in the set is mapped to a point at infinity. Larman also showed that there are sets of size $d + \Theta(\sqrt{d})$ that do not have a 1-tolerant Tverberg 2-partition if $d \geq 2$. This lower bound was later improved by Ramírez Alfonsín [11] to $5/3d + 4/3$ for $d \geq 4$. García-Colín [4] generalized Larman's upper bound, showing that sets of size $(t+1)(d+1)+1$ always have a t-tolerant Tverberg 2-partition, and asked for a general bound to guarantee the existence of t-tolerant Tverberg m-partitions. Later, Montejano and Oliveros [8] conjectured that every set of size $(t+1)(m-1)(d+1)+1$ admits a t-tolerant Tverberg m-partition. This was proved by Soberón and Strausz [14] who adapted Sarkaria's proof of Tverberg's theorem [12] to the tolerant setting:

Theorem 1.1 (Soberón-Strausz-Theorem [14]). *Let $P \subset \mathbb{R}$ be a set of size $(t+1)(m-1)(d+1)+1$. Then, there exists a t-tolerant Tverberg m-partition for P.*

Equivalently, for all n-point sets there exists a t-tolerant Tverberg $\lceil n/(d+1)(t+1) \rceil$-partition.

Soberón and Strausz [14] also conjectured this bound to be tight. A lower bound was recently proven by Soberón [13]: at least $m(\lfloor d/2 \rfloor + t + 1)$ points are necessary to guarantee the existence of a t-tolerant Tverberg m-partition.

So far, no exact or approximation algorithms for tolerant Tverberg partitions appear in the literature.

Our contribution. In Section 2, we consider the problem of computing tolerant Tverberg partitions in low dimensions. We present an algorithm for the one-dimensional case and use a dimension-reduction argument to extend the algorithm to multidimensional input:

Theorem 1.2. *Given a set $P \subset \mathbb{R}^d$ of size $2^{d-1}(m(t+2)-1)$, a t-tolerant Tverberg m-partition for P can be computed in time $\mathcal{O}(2^{d-1}dmt + mt \log t)$.*

For $d = 1$, the bound on the number of points is tight and improves the Soberón-Strausz bound from Theorem 1.1 by $t(m-2)$. For $d = 2$, the new bound improves the bound by Soberón and Strausz for large enough m and t.

For higher dimensions, we describe in Section 3 an approximation-preserving reduction to the regular Tverberg problem based on a lemma by García-Colín. Thus, we can apply existing and possible future algorithms for the regular Tverberg problem in the tolerant setting:

Proposition 1.3. *Let $P \subset \mathbb{R}^d$ and let $\mathcal{A}$ be an algorithm that computes a regular Tverberg m-partition for any point set of size $n_{\mathcal{A}}(m)$ in time $T_{\mathcal{A}}(m)$. Then, a*

$(\lfloor |P|/n_{\mathcal{A}}(m) \rfloor - 1)$*-tolerant Tverberg* m*-partition for* P *can be computed in time* $\mathcal{O}\left(T_{\mathcal{A}}(m) \cdot |P|/n_{\mathcal{A}}(m)\right)$.

Finally, we show in Section 4 that it is coNP-complete to determine whether a given Tverberg partition has tolerance t if the dimension is part of the input:

Theorem 1.4. TESTINGTOLERANTTVERBERG *is coNP-complete.*

This holds even if we restrict the input to Tverberg partitions of size 2.

2 Low Dimensions

We start with an algorithm for the one-dimensional case that yields a tight bound. This can be bootstrapped to higher dimensions with a lifting approach similar to the algorithm by Mulzer and Werner [9]. In two dimensions, we also get an improved bound if the size of the desired partition and the tolerance is large enough.

2.1 One Dimension

Let $P \subset \mathbb{R}$ with $|P| = n$, and let $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ be a t-tolerant Tverberg m-partition of P. By definition, there is no subset $R \subset P, |R| = t$ whose removal separates the convex hulls of the sets in $\mathcal{T}$. Bounding the size of the sets in $\mathcal{T}$ gives us more insight into the structure.

Lemma 2.1. *Let* $P \subset \mathbb{R}$ *with* $|P| = n$ *and let* $\mathcal{T} = \{T_1, T_2, \ldots, T_m\}$ *be a* t*-tolerant Tverberg* m*-partition of* P*. Then,*

(i) for $i = 1, \ldots, m$*, we have* $|T_i| \geq t + 1$*; and*
(ii) for $i, j = 1, \ldots, m$*,* $i \neq j$*, we have* $|T_i \cup T_j| \geq 2t + 3$*.*

Proof. (i) Suppose $|T_i| \leq t$. After removing T_i from P, the intersection of the convex hulls of the sets in $\mathcal{T}$ becomes empty, and $\mathcal{T}$ would not be t-tolerant.

(ii) Suppose there are $T_i, T_j \in \mathcal{T}$ with $|T_i \cup T_j| \leq 2t + 2$. By (i), we have $|T_i| = |T_j| = t+1$. Let $p_{\min} = \min(T_i \cup T_j)$ and assume w.l.o.g. that $p_{\min} \in T_i$ (see Figure 2). Then $|T_i \setminus \{p_{\min}\}| = t$, and removing the set $T_i \setminus \{p_{\min}\}$ separates the convex hulls of T_i and T_j. This again contradicts $\mathcal{T}$ being t-tolerant.

Lemma 2.1 immediately implies a lower bound on the size of any point set that admits a t-tolerant Tverberg m-partition.

Corollary 2.2. *Let* $P \subset \mathbb{R}$ *with* $|P| < m(t + 2) - 1$*. Then* P *has no* t*-tolerant Tverberg* m*-partition.*

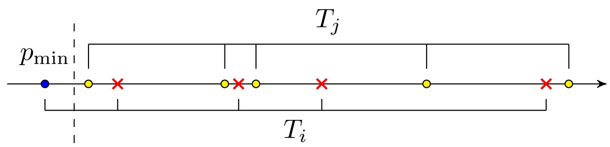


Fig. 2. The convex hulls of two sets of size $t+1$ can be separated by removing t points.

Now what happens for $|P| = m(t+2)-1$? Note that for $t > 0$ and $m > 2$, we have $m(t+2)-1 < 2(t+1)(m-1)+1$, the bound by Soberón and Strausz. Thus, proving that a t-tolerant Tverberg m-partition exists for any one-dimensional point set of size $m(t+2)-1$ would disprove the conjecture by Soberón and Strausz.

Let $P \subset \mathbb{R}$ be of size $m(t+2)-1$. By Lemma 2.1, in any t-tolerant Tverberg partition of P, one set has to be of size $t+1$ and all other sets have to be of size $t+2$. Let $\mathcal{T} = \{T_1, \ldots, T_m\}$ be a Tverberg m-partition of P such that T_1 contains every mth point of P and each other set T_i $(i \geq 2)$ has one point in each interval defined by the points of T_1; see Fig. 3 for $m = 3$ and $t = 2$. Note that $|T_1| = t+1$ and $|T_i| = t+2$ for $i \geq 2$. We will show that $\mathcal{T}$ is t-tolerant. Intuitively, $\mathcal{T}$ maximizes the interleaving of the sets, making the convex hulls more robust to removals.

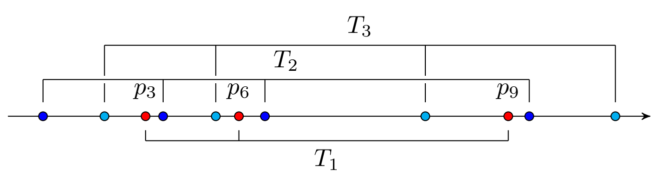


Fig. 3. A 2-tolerant Tverberg 3-partition for 11 $(= 3(2+2)-1)$ points.

Lemma 2.3. *Let $P \subset \mathbb{R}$ with $|P| = m(t+2)-1$, and let $\mathcal{T} = \{T_1, \ldots, T_m\}$ be an m-partition of P. Suppose that $|T_1| = t+1$, and write $T_1 = (p_1, p_2, \ldots, p_{t+1})$, sorted from left to right. Suppose that each interval $\mathcal{I} \in \{(-\infty, p_1), (p_1, p_2), \ldots, (p_{t+1}, \infty)\}$ contains one point from each T_i, for $i = 2, \ldots, m$. Then $\mathcal{T}$ is a t-tolerant Tverberg m-partition for P.*

Proof. Suppose there exist $T_i, T_j \in \mathcal{T}$, $i \neq j$, and a subset $R \subset P$ of size t such that removing R from P separates the convex hulls of T_i and T_j. Let h be a point that separates $\mathsf{conv}(T_i \setminus R)$ and $\mathsf{conv}(T_j \setminus R)$. Define $a = \max\{p \in T_1 \mid p \leq h\}$ and $b = \min\{p \in T_1 \mid p > h\}$, where $a = -\infty$ if all points in T_1 are greater than h and $b = +\infty$ if all points in T_1 are less than h. Let

$T_i^{\leq a} = \{p \in T_i \mid p \leq a\}$ and $T_i^{\geq b} = \{p \in T_i \mid p \geq b\}$, and define $T_j^{\leq a}$, $T_j^{\geq b}$ similarly. Since removing R separates the convex hulls of T_i and T_j at h, R must contain either $T_i^{\leq a} \cup T_j^{\geq b}$ or $T_i^{\geq b} \cup T_j^{\leq a}$. Figure 4 shows the situation. By construction, we know that $|T_i^{\leq a}| = |T_j^{\leq a}| = |T_1^{\leq a}|$ and $|T_i^{\geq b}| = |T_j^{\geq b}| = |T_1^{\geq b}|$. We thus have $t \geq |R| \geq |T_i^{\leq a} \cup T_j^{\geq b}| = |T_j^{\leq a} \cup T_i^{\geq b}| = |T_1^{\leq a} \cup T_1^{\geq b}|$. However, since $|T_1^{\leq a} \cup T_1^{\geq b}| = |T_1| = t + 1$, this is a contradiction.

Thus, even after removing t points, the convex hulls of the sets in $\mathcal{T}$ intersect pairwise. Helly's theorem [6] now guarantees that the convex hulls of all sets in $\mathcal{T}$ have a common intersection point. Hence, $\mathcal{T}$ is t-tolerant.

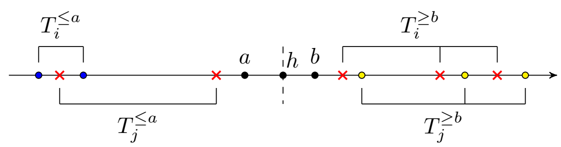


Fig. 4. The convex hulls of two elements in $\mathcal{T}$ are separated after the removal of R. Crosses mark removed points (i.e., points in R). Points not used in the proof of Lemma 2.3 are left out for clarification.

Lemma 2.3 immediately gives a way to compute a t-tolerant Tverberg m-partition in $\mathcal{O}(mt \log mt)$ time for $|P| = m(t+2) - 1$ by sorting P. However, it is not necessary to know the order of all of P. Algorithm 1 exploits this fact to improve the running time. It repeatedly partitions the point set until it has selected all points whose ranks are multiples of m. These points form the set T_1. Initially, the set Q contains only the input P (line 4). In lines 6–10, we select from each set in Q an element whose rank is a multiple of m (line 8) and we split the set at this element. Here, $\texttt{select}(P, k)$ is a procedure that returns the element with rank k of P. After termination of both loops in lines 5–11, all remaining sets in Q correspond to points in P between two consecutive points in T_1. In lines 12–14, the points in the sets in Q are distributed equally among the elements T_i ($i \geq 2$) of the returned partition.

Theorem 2.4. *Let $P \subset \mathbb{R}$ be a set of size $m(t+2) - 1$. On input (P, m), Algorithm 1 returns a t-tolerant Tverberg partition for P in time $\mathcal{O}(mt \log t)$.*

Proof. After each complete run of the inner for-loop (lines 6–10), each element $P' \in Q$ has size strictly less than r: initially, Q contains only P and r is strictly greater than $|P|/2$. Hence, both new sets added to Q in line 9 are of size strictly less than r. Since r is halved after each run (line 11), the invariant is maintained.

We will now check that Lemma 2.3 applies. We only split the sets in Q at elements whose rank is a multiple of m, so the ranks do not change modulo m. By the invariant, after the termination of the outer while-loop in lines 5–11, each

Algorithm 1: 1d-tolerant-Tverberg

```
   input  : P ⊂ ℝ, size of partition m
1  r ← m;
2  while r ≤ |P|/2 do
3  |  r ← 2 · r;
4  Q ← {P}; T_1, T_2, ..., T_m ← ∅, ∅, ..., ∅;
5  while r ≥ m do
6  |  foreach P' ∈ Q with |P'| ≥ r do
7  |  |  remove P' from Q;
8  |  |  p_r ← select(P', r);
9  |  |  Q ← Q ∪ {{p' ∈ P' | p' < p_r}, {p' ∈ P' | p' > p_r}};
10 |  |  T_1 ← T_1 ∪ {p_r};
11 |  r ← r/2;
12 foreach P' ∈ Q do
13 |  foreach j ∈ {2, 3, ..., m} do
14 |  |  remove any point from P' and add it to T_j;
15 return {T_1, T_2, ..., T_m};
```

set in Q has size strictly less than m. Thus, T_1 contains the points in P whose rank is a multiple of m and each set $P' \subset P$ in Q contains all points between two consecutive points in T_1. Since these are distributed equally among $T_2, \dots, T_m$, Lemma 2.3 now shows the correctness of the algorithm.

Let us consider the running time. Computing the initial r in lines 1–3 requires $\mathcal{O}(\log(|P|/m)) = \mathcal{O}(t)$ time. The split-element in line 8 can be found in time $\mathcal{O}(|P'|)$ [3]. Thus, since the sets are disjoint, one iteration of the outer while-loop requires $\mathcal{O}(|P|)$ time, for a total of $\mathcal{O}(\log(|P|/m)|P|) = \mathcal{O}(\log(t)mt)$. By the same argument, both for-loops in lines 12–14 require linear time in the size of P. This results in a total time complexity of $\mathcal{O}(mt \log t)$, as claimed.

2.2 Higher Dimensions

We use a lifting argument [9] to extend Algorithm 1 to higher-dimensional input. Given a point set $P \subseteq \mathbb{R}^d$ of size n, let h be a hyperplane that splits P evenly (if n is odd, h contains exactly one point of P). We then partition P into $\lfloor n/2 \rfloor$ pairs (p_i^-, p_i^+), where $p_i^- \in h^-$ and $p_i^+ \in h^+$. We obtain a $(d-1)$-dimensional point set with $\lfloor n/2 \rfloor$ elements by mapping each pair to the intersection of the connecting line segment with h.

Let $q_i = p_i^+ p_i^- \cap h$ be the mapped point for (p_i^-, p_i^+) and $\mathcal{T}' = \{T_1', \dots, T_m'\}$ a t-tolerant Tverberg m-partition of $Q = \{q_1, \dots, q_{\lfloor n/2 \rfloor}\}$. We obtain a Tverberg m-partition $\mathcal{T}$ with tolerance t for P by replacing each q_i in $\mathcal{T}'$ by its corresponding pair (p_i^-, p_i^+). Thus, we can repeatedly project the set P until Algorithm 1 is applicable. Then, we lift the one-dimensional solution back to higher dimensions.

Algorithm 2 follows this approach. For $d = 1$, Algorithm 1 is applied (lines 1–2). Otherwise, we take an appropriate hyperplane orthogonal to the x_d-axis and compute the lower-dimensional point set (lines 3–7). This is always possible since we can assume w.l.o.g. that all points have distinct x_d coordinates by a simple rotation argument. Finally, the result for $d-1$ dimensions is lifted back to d dimensions (lines 10–11). Using Theorem 2.4, it is easy to show that

Algorithm 2: DimReduct-Tolerant-Tverberg

input : point set $P \subset \mathbb{R}^d$, tolerance parameter t, size of partition m
output: t-tolerant Tverberg partition for P of size m

1 **if** $d = 1$ **then**
2 | **return** `1d-tolerant-Tverberg` (P, m)
3 $h \leftarrow$ hyperplane that halves P according to the x_d-coordinate;
4 **foreach** $i \in \{1, 2, \ldots, |P \cap h^-|\}$ **do**
5 | $p_i^- \leftarrow$ remove any point from P that belongs to $P \cap h^-$;
6 | $p_i^+ \leftarrow$ remove any point from P that belongs to $P \cap h^+$;
7 | $q_i \leftarrow$ first $d-1$ coordinates of $p_i^- p_i^+ \cap h$;
8 $Q \leftarrow \{q_1, q_2, \ldots, q_{|P \cap h^-|}\}$;
9 $\{T_1', T_2', \ldots, T_m'\} \leftarrow$ `DimReduct-Tolerant-Tverberg`(Q, t, m);
10 **foreach** $j \in \{1, 2, \ldots, m\}$ **do**
11 | $T_j \leftarrow \{p_i^-, p_i^+ \mid q_i \in T_j'\}$;
12 **return** $\{T_1, T_2, \ldots, T_m\}$;

Algorithm 2 achieves the bounds claimed in Theorem 1.2:

Theorem 2.5 (Theorem 1.2 restated). *Given a set $P \subset \mathbb{R}^d$ of size $2^{d-1}(m(t+2)-1)$, a t-tolerant Tverberg m-partition for P can be computed in time $\mathcal{O}(2^{d-1}dmt + mt \log t)$.*

Proof. Since the size of P halves in each recursion step, $2^{d-1}(m(t+2)-1)$ points suffice to ensure that Algorithm 1 can be applied to $m(t+2)-1$ points in the base case. Each projection and lifting step can be performed in linear time, using a median computation. Since the size of the point set decreases geometrically, the total time for projection and lifting is thus $\mathcal{O}(2^{d-1}dmt)$. Since Algorithm 1 has running time $O(mt \log t)$, the result follows.

For $d \geq 3$, the bound from Proposition 1.2 is worse than the Soberón-Strausz bound. However, in two dimensions, we have

$$2^{2-1}(m(t+2)-1) < (2+1)(m-1)(t+1)+1 \Leftrightarrow m/(m-3) < t$$

This holds for instance if $m \geq 4 \wedge t \geq 5$ or $m \geq 7 \wedge t \geq 2$. Thus, Algorithm 2 gives a strict improvement over the Soberón-Strausz bound for large enough m and t.

3 Reduction to the Regular Tverberg Problem

We now show how to use any algorithm that computes approximate regular Tverberg partitions in order to find tolerant Tverberg partitions. For this, we must increase the tolerance of a Tverberg partition. In the following, we show that one can merge elements of several Tverberg partitions for disjoint subsets of P to obtain a Tverberg partition with higher tolerance for the whole set P. The following lemma is also implicit in the Ph.D. thesis of García-Colín [4].

Lemma 3.1. *Let $\mathcal{T}_1, \dots, \mathcal{T}_k$ be Tverberg m-partitions for disjoint point sets P_1, $\dots$,$P_k \subset \mathbb{R}^d$. Let $T_{i,j}$ be the jth element of $\mathcal{T}_i$ and $t_i \geq 0$ the tolerance of $\mathcal{T}_i$. Then, $\mathcal{T} = \{T_j = \bigcup_{i=1}^k T_{i,j} \mid j \in \{1, 2, \dots, m\}\}$ is a Tverberg m-partition of $P = \bigcup_{j=1}^k P_i$ with tolerance $t = \sum_{i=1}^k t_i + k - 1$.*

Proof. Take $R \subseteq P$ with $|R| = t$. As $t = \sum_{i=1}^k t_i + k - 1 < \sum_{i=1}^k (t_i + 1)$, there is an i with $|P_i \cap R| \leq t_i$. Since $\mathcal{T}_i$ is t_i-tolerant, we have $\bigcap_{j=1}^m \mathsf{conv}(T_{i,j} \setminus R) \neq \emptyset$. Because each $T_{i,j}$ is contained in the corresponding set T_j of $\mathcal{T}$, the convex hulls of the elements in $\mathcal{T}$ still intersect after the removal of R.

From a mathematical perspective, the main motivation for introducing tolerance to Tverberg partitions is the possibility to achieve better bounds than by just combining regular Tverberg partitions. This provides deeper insight in the intersection pattern of convex sets. Nevertheless, Lemma 3.1 is interesting from an algorithmic viewpoint as it enables us to benefit from existing approximation algorithms for regular Tverberg partitions by implying a simple algorithm: compute regular Tverberg partitions for disjoint subsets of P and then merge them using Lemma 3.1. This proves Proposition 1.3:

Proposition 3.2 (Proposition 1.3 restated). *Let $P \subset \mathbb{R}^d$ and let $\mathcal{A}$ be an algorithm that computes a regular Tverberg m-partition for any point set of size $n_{\mathcal{A}}(m)$ in time $T_{\mathcal{A}}(m)$. Then, a $(\lfloor |P|/n_{\mathcal{A}}(m) \rfloor - 1)$-tolerant Tverberg m-partition for P can be computed in time $\mathcal{O}(T_{\mathcal{A}}(m) \cdot |P|/n_{\mathcal{A}}(m))$.*

Proof. We split P into $\lfloor |P|/n_{\mathcal{A}} \rfloor$ disjoint sets and use $\mathcal{A}$ to obtain for each subset a regular Tverberg partition. Applying Lemma 3.1, we obtain a $(\lfloor |P|/n_{\mathcal{A}} \rfloor - 1)$-tolerant Tverberg m-partition. Since the merging step in Lemma 3.1 takes linear time in $|P|$, the total running time is $\mathcal{O}(T_{\mathcal{A}}(m) \cdot |P|/n_{\mathcal{A}}(m))$, as claimed.

Table 1 shows specific values for Proposition 1.3 applied to Miller & Sheehy's and Mulzer & Werner's algorithm.

Algorithm	Tolerance	Running time
Proposition 1.3 with Miller-Sheehy	$\lfloor \|P\|/2m(d+1)^2 \rfloor - 1$	$m^{\mathcal{O}(\log d)} d^{\mathcal{O}(\log d)} \|P\|$
Proposition 1.3 with Mulzer-Werner	$\lfloor \|P\|/4m(d+1)^3 \rfloor - 1$	$d^{\mathcal{O}(\log d)} \|P\|$

Table 1. Proposition 1.3 applied to existing approximation algorithms for the regular Tverberg problem.

Remark 3.3. Lemma 3.1 gives a quick proof of a slightly weaker version of the Soberón-Strausz bound: partition P into $t+1$ disjoint sets of size at least $\lfloor |P|/(t+1) \rfloor$. By Tverberg's theorem, for each subset there exists a Tverberg partition with no tolerance of size $\lceil \lfloor |P|/(t+1) \rfloor/(d+1) \rceil$. Using Lemma 3.1, we obtain a t-tolerant Tverberg partition of size $\lceil \lfloor |P|/(t+1) \rfloor/(d+1) \rceil \geq \lceil |P|/(t+1)(d+1) \rceil - 1$ of P, which is at most one less than the Soberón-Strausz bound. This weaker bound was also stated by García-Colín [4]. Again, as already mentioned after Lemma 3.1, this is interesting mostly from an algorithmic perspective since it implies that computing slightly worse tolerant Tverberg partitions than guaranteed by the Soberón-Strausz bound is polynomial-time equivalent to computing regular Tverberg partitions.

4 Hardness of Tolerance Testing

Teng [15] proved that deciding whether a point is a centerpoint (TestingCenter) is coNP-complete. We show the same for deciding whether a Tverberg partition is t-tolerant (TestingTolerantTverberg) by a reduction from TestingCenter. The problems are formally defined as follows:

*Problem 4.1 (*TestingCenter*).*

Given a point set $P \subset \mathbb{R}^d$, and a centerpoint candidate $c \in \mathbb{R}^d$, where d is part of the input.

Decide whether c is a centerpoint of P.

*Problem 4.2 (*TestingTolerantTverberg*).*

Given a point set $P \subset \mathbb{R}^d$, a partition $\mathcal{T}$ of P, and a conjectured tolerance $t \in \mathbb{N}$, where d is part of the input.

Decide whether $\mathcal{T}$ is a t-tolerant Tverberg partition of P.

Note that the size of the partition $\mathcal{T}$ in the definition of TestingTolerantTverberg can be constant.

The following lemma is folklore. We include the proof for completeness. It is used in the reduction to connect the tolerance of a Tverberg partition with the depth of points in the intersection of the convex hulls.

Lemma 4.3. *Let $P \subset \mathbb{R}^d$ and let $c \in \mathbb{R}^d$. Then c has depth $t+1$ w.r.t. P if and only if for all subsets $R \subset P$ with $|R| \leq t$, we have $c \in \mathsf{conv}(P \setminus R)$.*

Proof. We prove both directions by showing the contrapositive.

"$\Rightarrow$" Suppose there is some $R \subset P, |R| \leq t$ with $c \notin \mathsf{conv}(P \setminus R)$. Then, there is a half-space h^+ that contains c but no points from $\mathsf{conv}(P \setminus R)$. Thus, $c \in h^+$ and $|P \cap h^+| \leq |R| \leq t$, and hence c has depth at most t w.r.t. P.

"$\Leftarrow$" Assume c has depth $t' \leq t$ w.r.t. P. Let h^+ be a half-space that contains c and t' points from P. Set $R = h^+ \cap P$. Then, $|R| \leq t$ and $c \notin \mathsf{conv}(P \setminus R)$.

We are now ready to prove Theorem 1.4:

Theorem 4.4 (Theorem 1.4 restated). TESTINGTOLERANTTVERBERG *is coNP-complete.*

Proof. We first check that TESTINGTOLERANTTVERBERG is indeed contained in coNP. Let $\mathcal{T}$ be a Tverberg partition of $P \subset \mathbb{R}^d$ that is claimed to have tolerance t. A witness to $\mathcal{T}$ not being a t-tolerant Tverberg partition is a subset $R \subseteq P$ of size at most t such that $\bigcap_{T_i \in \mathcal{T}} \mathsf{conv}(T_i \setminus R) = \emptyset$. Checking if R is a witness reduces to testing the feasibility of the linear program defined by the following constraints for each element T_i in $\mathcal{T}$:

$$\begin{aligned} \alpha_{i,1}\, p_{i,1} + \alpha_{i,2}\, p_{i,2} + \cdots + \alpha_{i,|T_i \setminus R|}\, p_{i,|T_i \setminus R|} - x &= 0 \\ \alpha_{i,1} + \alpha_{i,2} + \cdots + \alpha_{i,|T_i \setminus R|} &= 1 \\ \forall j \in \{1, 2, \ldots, |T_i \setminus R|\} : \alpha_{i,j} &\geq 0 \qquad , \end{aligned}$$

where $p_{i,j}$ denotes the jth point in $T_i \setminus R$. The linear program is feasible if and only if $\bigcap_{T_i \in \mathcal{T}} \mathsf{conv}(T_i \setminus R) \neq \emptyset$, i.e., if R is not a witness. Since the number of constraints and variables is polynomial in the input size, feasibility checking of the above linear program can be carried out in polynomial time.

Let $(P \subset \mathbb{R}^d, c \in \mathbb{R}^d)$ be an input to TESTINGCENTER. We embed the vector space $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ by identifying it with the hyperplane $h : x_{d+1} = 0$. Define $t = \lceil |P|/(d+1) \rceil - 1$ and let ℓ be the line that is orthogonal to h and passes through c. Furthermore, let T^- and T^+ be sets of $t+1$ arbitrary points in $\ell \cap h^-$ and $\ell \cap h^+$, respectively. Set $T = T^- \cup T^+$. We claim that $\{P, T\}$ is a Tverberg 2-partition for $P \cup T$ with tolerance t if and only if c is a centerpoint of P; see Figure 5.

"$\Rightarrow$" Assume $\{P, T\}$ is a t-tolerant Tverberg 2-partition. By construction of T, we have $\mathsf{conv}(P) \cap \mathsf{conv}(T) = \{c\}$. Thus, c lies in the intersection of both convex hulls even if an arbitrary subset of size at most t is removed. Lemma 4.3 implies that c has depth $t + 1 = \lceil |P|/(d+1) \rceil$ w.r.t. P, so c is a centerpoint for P.

"$\Leftarrow$" Assume c is a centerpoint for P. By definition, c has depth at least $\lceil |P|/(d+1) \rceil = t+1$ w.r.t. P. Lemma 4.3 then implies that c is contained in the convex hull of P even if any t points from P are removed. Since T contains $t+1$ points on both sides of a line through c, c is also contained in $\mathsf{conv}(T)$ if any t points from T are removed. Thus, $\{P, T\}$ is a t-tolerant Tverberg 2-partition for $P \cup T$.

5 Conclusion

We have shown that for each set $P \subset \mathbb{R}$ of size $m(t+2)-1$, a t-tolerant Tverberg partition of size m can be found in time $\mathcal{O}(mt \log t)$. The bound on the size of P is tight, and it improves the Soberón-Strausz bound in one dimension. Combining this with a lifting method, we could also get improved bounds in two dimensions and an efficient algorithm for tolerant Tverberg partitions in any fixed dimension. However, the running time is exponential in the dimension.

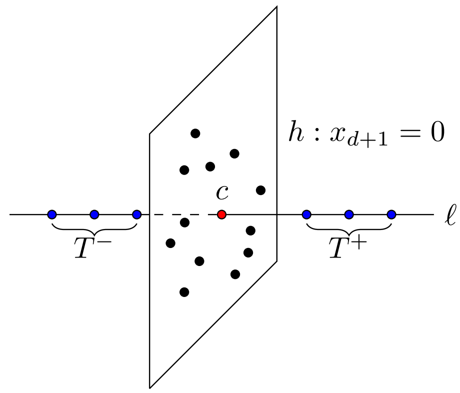


Fig. 5. Reduction of TestingCenter to TestingTolerantTverberg

This motivated us to look for a way of reusing the existing technology for the regular Tverberg problem. We have presented a reduction to the regular Tverberg problem that enables us to reuse the approximation algorithms by Miller & Sheehy and Mulzer & Werner.

Finally, we proved that testing whether a given Tverberg partition is of some tolerance t is coNP-complete. Unfortunately, this does not imply anything about the complexity of finding tolerant Tverberg partitions. It is not even clear whether computing tolerant Tverberg partitions is harder than computing regular Tverberg partitions. However, we could show that computing tolerant Tverberg partitions with smaller tolerance than guaranteed by the Soberón-Strausz bound is polynomial-time equivalent to computing regular Tverberg partitions.

It remains open whether the bound by Soberón and Strausz is tight for $d > 2$. We believe that our results in one and two dimensions indicate that the bound can be improved also in general dimension. Another open problem is finding a *pruning strategy* for tolerant Tverberg partitions. By this, we mean an algorithm that efficiently reduces the sizes of the sets in a t-tolerant Tverberg partition without deteriorating the tolerance. Such a pruning strategy could be used to improve the quality of our algorithms. In Miller & Sheehy's and Mulzer & Werner's algorithms, Carathéodory's theorem was used for this task. Unfortunately, this result does not preserve the tolerance of the pruned partitions. The generalized tolerant Carathéodory theorem [8] also does not seem to help. It remains an interesting problem to develop criteria for superfluous points in tolerant Tverberg partitions.

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