Spanners for Directed Transmission Graphs*

Haim Kaplan† *Wolfgang Mulzer*‡ *Liam Roditty*§ *Paul Seiferth*‡

Abstract

Let $P \subset \mathbb{R}^2$ be a planar n-point set such that each point $p \in P$ has an *associated radius* $r_p > 0$. The *transmission graph* G for P is the directed graph with vertex set P such that for any $p, q \in P$, there is an edge from p to q if and only if $d(p, q) \leq r_p$.

Let $t > 1$ be a constant. A *t-spanner* for G is a subgraph $H \subseteq G$ with vertex set P so that for any two vertices $p, q \in P$, we have $d_H(p, q) \leq t d_G(p, q)$, where d_H and d_G denote the shortest path distance in H and G, respectively (with Euclidean edge lengths). We show how to compute a t-spanner for G with $O(n)$ edges in $O(n(\log n + \log \Psi))$ time, where Ψ is the ratio of the largest and smallest radius of a point in P. Using more advanced data structures, we obtain a construction that runs in $O(n \log^6 n)$ time, independent of Ψ.

We give two applications for our spanners. First, we show how to use our spanner to find a BFS tree from any given start vertex in $O(n \log n)$ time (in addition to the time it takes to build the spanner). Second, we show how to use our spanner to extend a reachability oracle to answer geometric reachability queries. In a *geometric reachability query* we ask whether a vertex p in G can "reach" a target q which is an arbitrary point the plane (rather than restricted to be another vertex q of G in a standard reachability query). Our spanner allows the reachability oracle to answer geometric reachability queries with an additive overhead of $O(\log n \log \Psi)$ to the query time and $O(n \log \Psi)$ to the space.

1 Introduction

A common model for wireless sensor networks is the *unit-disk graph*: each sensor p is modeled by a unit disk centered at p, and there is an edge between two sensors if and only if their disks intersect [10]. Intersection graphs of disks with arbitrary radii have also been used to model sensors with different transmission strengths [3, Chapter 4]. Intersection graphs of disks are undirected. However, for some networks we may want a directed model. In such networks, a sensor p that can transmit information to a sensor q may not be able to receive information from q. This motivated various researchers to consider what we call here *transmission graphs* [19,23]. A transmission graph G is defined

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†School of Computer Science, Tel Aviv University, Israel, haimk@post.tau.ac.il

‡Institut für Informatik, Freie Universität Berlin, Germany {mulzer,pseiferth}@inf.fu-berlin.de

§Department of Computer Science, Bar Ilan University, Israel liamr@macs.biu.ac.il

for a set of points P where each point $p \in P$ has a (transmission) radius r_p associated with it. Each vertex of G corresponds to a point of P, and there is a directed edge from p to q if and only if q lies in the disk $D(p)$ of radius r_p around p. We weight each edge pq of G by the distance between p and q, denoted by $|pq|$.

As many other kinds of geometric intersection graphs, a transmission graph may be dense and may contain $\Theta(n^2)$ edges. Thus, if one applies a standard graph algorithm, like breadth first search (BFS), to a dense transmission graph, it runs slowly, since it requires an explicit representation of all the edges in the graph. For some applications a sparse approximation of G that preserves distances suffices. Therefore, given a transmission graph G, implicitly represented by a list of points and their associated radii, it is desirable to construct a sparse approximation of G that preserves its connectivity and proximity properties. We want to construct this approximation efficiently, without generating an explicit representation of G.

For any $t > 1$, a subgraph H of G is a *t-spanner* for G if the distance between any pair of vertices p and q in H is at most t times the distance between p and q in G, i.e., $d_H(p, q) \leq t \cdot d_G(p, q)$ for any pair p, q (see [18] for an overview of spanners for geometric graphs). Fürer and Kasivisawnathan show how to compute a t-spanner for unit- and general disk graphs that are variations of the Yao graph [12, 24]. Peleg and Roditty [19] give a construction for t-spanners in transmission graphs in any metric space with bounded doubling dimension. We continue these studies by giving an almost linear time algorithm that constructs a t-spanner of a transmission graph of a planar set of points ($P \subset \mathbb{R}^2$) with respect to the Euclidean metric (i.e. $|pq|$ is the Euclidean distance between p and q).

Our construction is also based on the *Yao graph* [24]. The basic Yao graph is a t-spanner for the complete graph defined by n points in the plane (with Euclidean distances as the weights of the edges). To determine the points adjacent to a particular point q, we divide the plane by equally spaced rays emanating from q and connect q to its closest point in each wedge (the number of wedges increases as t gets smaller). Adapting this construction to transmission graphs poses a severe computational difficulty, as we want to consider, in each wedge, only the points p with $q \in D(p)$ and to pick the closest point to q only among those. Since finding the exact closest point turns out to be difficult, we need to relax this requirement in a subtle way, without hurting the approximation too much. This makes it possible to construct the spanner efficiently.

Even with a good t-spanner at hand, we sometimes wish to obtain exact solutions for certain problems on disk graphs. Working in this direction, Cabello and Jejčič gave an $O(n \log n)$ time algorithm for computing a BFS tree in a unit-disk graph, rooted at any given vertex [4]. For this, they exploited the special structure of the Delaunay triangulation of the disk centers. We show that our spanner admits similar properties for transmission graphs. As a first application of our spanner, we get an efficient algorithm to compute a BFS tree in a transmission graph rooted at any given vertex.

For another application, we consider *reachability oracles*. A reachability oracle is a data structure that can answer *reachability queries*: given two vertices s and t determine if there is a directed path from s to t. The quality of a reachability oracle is measured

by its query time, its space requirement, and its preprocessing time. For transmission graphs, we can ask for a more general *geometric* reachability query: given a vertex s and *any* point $q \in \mathbb{R}^2$, determine if there is vertex t such that there is a directed path from s to t in G, and q lies in the disk of t. We show how to extend any given reachability oracle to answer geometric queries with a small *additive* increase in space and query time.

Our Contribution and the Organization of the Paper. In Section 3, we show how to compute, for every fixed $t > 1$, a t-spanner H of G. Our construction is quite generic and can be adapted to several situations. In the simplest case, if the *spread* Φ (i.e., the ratio between the largest and the smallest distance in P) is bounded, we can obtain a t-spanner in time $O(n(\log n + \log \Phi))$ (Section 3.1). With a little more work, we can weaken the assumption to a bounded *radius ratio* Ψ (the ratio between the largest and smallest radius in P), giving a running time of $O(n(\log n + \log \Psi))$ (Section 3.2). Note that a bound on Φ implies a bound on Ψ: let $d_{\max}$ be the largest distance and $d_{\min}$ be the smallest distance between any pair of distinct points in P. We can set all radii larger than $d_{\max}$ to be $d_{\max}$ and all radii smaller than $d_{\min}$ to $d_{\min}/2$. This does not change the transmission graph and we have $\Psi \leq 2\Phi$. Using even more advanced data structures, we can compute a t-spanner in time $O(n \log^6 n)$, without any dependence on Φ or Ψ (Section 3.3).

In Section 4.1 we show how to adapt a result by Cabello and Jejčič [4] to compute a BFS-tree in a transmission graph, from any given vertex $p \in P$, in $O(n \log n)$ time, once we have the spanner ready.

In Section 4.2 we show how to use a spanner to extend a reachability oracle to answer geometric reachability queries. Specifically, we show that any reachability oracle for a transmission graph with radius ratio Ψ, that requires $S(n)$ space, and answers a query in $Q(n)$ time, can be extended in $O(n \log n \log \Psi)$ time, to an oracle that can answer geometric reachability queries, requires $S(n) + O(n \log \Phi)$ space, and answers a query in $Q(n) + O(\log n \log \Phi)$ time.

2 Preliminaries and Notation

We let $P \subset \mathbb{R}^2$ denote a set of n points in the plane. Each point $p \in P$ has a *radius* $r_p > 0$ associated with it. The elements in P are called *sites*. The *spread* of P, Φ, is defined as $\Phi = \max_{p,q \in P} |pq| / \min_{p \neq q \in P} |pq|$, and the *radius ratio* Ψ of P is defined as $\Psi = \max_{p,q \in P}(r_p/r_q)$. A simple volume argument shows that $\Phi = \Omega(n^{1/2})$. Furthermore, as stated in the introduction, we can always assume that $\Psi \leq 2\Phi$. Given a point $p \in \mathbb{R}^2$ and a radius r, we denote by $D(p, r)$ the closed disk with center p and radius r. If $p \in P$, we use $D(p)$ as a shorthand for $D(p, r_p)$. We write $C(p, r)$ for the boundary circle of $D(p, r)$.

Our constructions make extensive use of planar grids. For $i \in \{0, 1, \dots\}$, we define $\mathcal{Q}_i$ to be the *grid at level* i. It consists of axis-parallel squares with diameter 2^i that partition the plane in grid-like fashion (the *cells*). We write $\mathrm{diam}(\sigma)$ for the diameter of a grid cell σ. Each grid $\mathcal{Q}_i$ is aligned so that the origin lies at the corner of a cell. The

distance $d(\sigma, \tau)$ between two grid cells σ, τ is the smallest distance between any pair of points in $\sigma \times \tau$, see Figure 1. We assume that our model of computation allows us to find in constant time for any given point the grid cell containing it.

3 Spanners for Directed Transmission Graphs

3.1 Efficient Spanner Construction for a Set of Points with Bounded Spread

First, we give a spanner construction for the transmission graph whose running time depends on the spread. Later, in Section 3.2, we will tune this construction so that the running time depends on the radius ratio. The main result which we prove in this section is as follows.

Theorem 3.1. *Let G be the transmission graph for a set P of n points in the plane with spread Φ. For any fixed $t > 1$, we can compute a t-spanner for G in $O(n \log \Phi)$ time. The construction needs $O(n \log \Phi)$ space.*

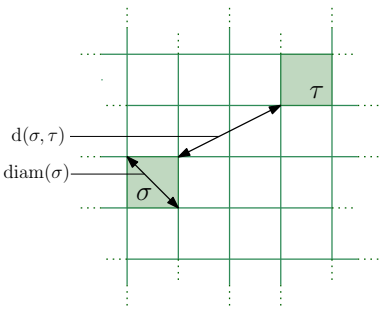


Fig. 1: The grid (green) and two cells σ and τ.

Let ρ be a ray originating from the origin and let $0 < \alpha < 2\pi$. A *cone* with *opening angle* α and *middle axis* ρ is the closed region containing ρ and bounded by the two rays obtained by rotating ρ clockwise and counterclockwise by $\alpha/2$.

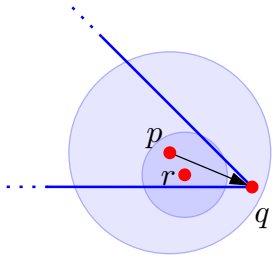


Fig. 2: A cone C_q (blue) at a site q. Since $q \notin D(r)$, we pick the edge pq.

Given a cone C and a point $q \in \mathbb{R}^2$, we write C_q for the copy of C obtained by translating the origin to q. We call q the *apex* of C_q. Ideally, our spanner should look as follows. Let $\mathcal{C}$ be a set of k cones with opening angles $2\pi/k$ that partition the plane. For each site $q \in P$ and each cone $C \in \mathcal{C}$, we pick the site $p \in P \cap C_q$ with $q \in D(p)$ that is closest to q (see Figure 2). We add the edge pq to H. The resulting graph has $O(kn)$ edges. Using standard techniques, one can show that H is a t-spanner, if k is large enough as a function of t. This construction has been reported before and seems to be folklore [6, 19].

Unfortunately, the standard algorithms for computing the Yao graph do not seem to adapt easily to our setting without a penalty in their running times [9]. The problem is that for each site q and each cone C_q, we need to search for a nearest neighbor of q only among those sites $p \in C_q$ such that $q \in D(p)$. This seems to be hard to do with the standard approaches. Thus, we modify the construction to search only for an *approximate*

nearest neighbor of q and argue that picking an approximately shortest edge in each cone suffices to obtain a spanner.

We partition each cone C_q into "intervals" obtained by intersecting C_q with annuli around q whose inner and outer radii grow exponentially; see Figure 3. There can be only $O(\log \Phi)$ non-empty intervals. We cover each such interval by $O(1)$ grid cells whose diameter is "small" compared to the width of the interval. This gives two useful properties. (i) We only need to consider edges from the interval closest to q that contains sites with outgoing edges to q; all other edges to q will be longer. (ii) If there are multiple edges from the same grid cell, their endpoints are close together, and it suffices to consider only one of them.

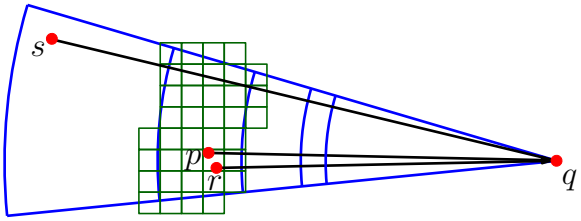


Fig. 3: A cone C_q covered by discretized intervals. We only need one of the edges pq, rq for H.

To make this approach more concrete, we define a decomposition of P into pairs of subsets of P contained in certain grid cells. These pairs represent a discretized version of the intervals (see Definition 3.2 below). This is motivated by another spanner construction based on the *well-separated pair decomposition* (WSPD). Let $c > 1$ be a parameter. A c-WSPD for P is a set of pairs $(A_i, B_i), \ldots, (A_m, B_m)$ such that $A_i, B_i \subseteq P$, and for each pair a, b of points of P there is a single index j such that $a \in A_j$ and $b \in B_j$ or vice versa. Furthermore, for any $1 \leq i \leq m$ we have that $c \max\{\mathrm{diam}(A_i), \mathrm{diam}(B_i)\} \leq d(A_i, B_i)$. Here $\mathrm{diam}(A_i)$ is the diameter of A_i and $d(A_i, B_i)$ is the minimum distance between any pair a, b with $a \in A_i$ and $b \in B_i$. Callahan and Kosaraju show that there always exists a WSPD with $m = O(n)$ pairs which can be computed efficiently [5].

It is well known [18] that one can obtain a t-spanner for the complete (undirected) Euclidean graph with vertex set P from a c-WSPD, for a large enough $c = c(t)$, by putting in the spanner an edge ab for each pair (A_i, B_i) in the WSPD, where a is an arbitrary point in A_i and b is an arbitrary point in B_i. It turns out that a similar approach works for transmission graphs. However, since they are directed, we need to find for *each* site in B_i an incoming edge from a site in A_i, if such an edge exists, and vice versa. This causes two difficulties: we cannot afford to check all possible edges in $A_i \times B_i$, since this would lead to a quadratic running time, and we cannot control the indegree of a point p since it may belong to many sets A_i and B_i. We address the second problem by taking only $O(1)$ edges into a particular point q, within each of the k cones of the Yao construction described above. For the first problem, we identify in each A_i a special subset that "covers" all edges from a site in A_i to a site in B_i, such that each site appears in a constant number of such subsets.

The concrete implementation of this idea is captured by Definition 3.2. A pair (A_i, B_i) corresponds to sets $P \cap \sigma$ and $P \cap \tau$ for two grid cell σ, τ that have the same diameter and that are well separated (Property (i)). For a grid cell τ, we denote by m_τ the site of largest radius in $P \cap \tau$ and we define a particular subset $R_\tau \subseteq P \cap \tau$ to be the set of sites *assigned* to τ. Property (ii) in Definition 3.2 guarantees that each edge pq of G with $q \in \sigma$ and $p \in \tau$ is either "represented" in the decomposition by an edge originating in m_τ or we have that $p \in R_\tau$. Specifically, edges pq with $q \in P \cap \sigma$ and $p \in P \cap \tau$ such that the disk $D(p)$ is "large" relative to $|pq|$ are represented by the edge $m_\sigma q$. This allows us to define the sets R_σ such that each site appears in $O(1)$ such sets, see Figure 4.

Definition 3.2. *Let $c > 2$ and let G be the transmission graph of a planar point set P. A* c-separated annulus decomposition *for G consists of a finite set $\mathcal{Q} \subset \bigcup_{i=0}^{\infty} \mathcal{Q}_i$ of* grid cells*, a symmetric* neighborhood relation *$N \subseteq \mathcal{Q} \times \mathcal{Q}$ between these cells, and a subset of* assigned sites *$R_\sigma \subseteq P \cap \sigma$ for each grid cell $\sigma \in \mathcal{Q}$. A* c-separated annulus decomposition *for G has the following properties:*

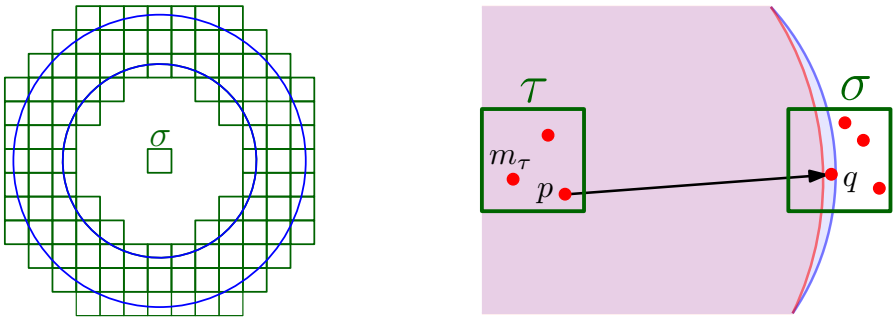
(i) For every $(\sigma, \tau) \in N$, $\operatorname{diam}(\sigma) = \operatorname{diam}(\tau)$, and $d(\sigma, \tau) = \gamma \operatorname{diam}(\sigma)$, for some $\gamma \in [c-2, 2c)$.

(ii) for every edge pq of G, there is a pair $(\sigma, \tau) \in N$ with $q \in \sigma$, $p \in \tau$, and either $p \in R_\tau$ or $q \in D(m_\tau)$.

The following fact is direct consequence of Definition 3.2. For each cell $\sigma \in \mathcal{Q}$, we define its *neighborhood* as $N(\sigma) = \{\tau \mid (\sigma, \tau) \in N\}$.

Lemma 3.3. *For each cell $\sigma \in \mathcal{Q}$, we have $|N(\sigma)| = O(c^2)$, and for each cell $\tau \in \mathcal{Q}$ the number of cells $\sigma \in \mathcal{Q}$ such that $\tau \in N(\sigma)$ is $O(c^2)$.*

Proof. This follows from Definition 3.2(i) via a standard volume argument. □



(a) By Property (i) in Definition 3.2 $N(\sigma)$ covers an annulus.

(b) Since $D(m_\tau)$ (red) does not contain q, we need to put p in R_τ to cover the edge pq (Property (ii)).

Fig. 4: Illustration of Definition 3.2

Given this decomposition, we first present a simple (and rather inefficient) rule for picking incoming edges such that the resulting graph is a t-spanner. Then we explain how to compute the decomposition using a *quadtree*. Finally, we exploit the quadtree to make the spanner construction efficient.

Obtaining a Spanner. Let $t > 1$ be the desired stretch. We pick a suitable separation parameter c and a number of cones k that depend on t, as specified later. Let $(\mathcal{Q}, N, R_\sigma)$ be a c-separated annulus decomposition for G. For a cone $C \in \mathcal{C}$ and an integer $\ell \in \mathbb{N}$, we define C^ℓ as the cone with the same middle axis as C but with an opening angle ℓ times larger than the opening angle of C. For $\sigma \in \mathcal{Q}$, let C_σ be the copy of C with the center of σ as the apex.

To obtain a t-spanner $H \subseteq G$, we pick the incoming edges for each site $q \in P$ and each cone $C \in \mathcal{C}$ as follows (see Algorithm 1). We consider the cells of $\mathcal{Q}$ containing q in increasing order of diameter. Let σ be one such cell containing q that we process. We traverse all neighboring cells τ of σ, that are contained in C^2_σ. For each such neighboring cell τ, we check if there exists a site $r \in R_\tau \cup \{m_\tau\}$ that has an outgoing edge to q. If such a site exists, we add to H an edge to q from a single, arbitrary, such site r. After considering *all* neighbors τ of σ we terminate the processing of q and C if we added at least one edge incoming to q. If we have not added any edge into q while processing all neighbors τ of σ we continue to the next largest cell containing q. We use here the extended cones C^2_σ (instead of the cone C_q) to gain certain flexibility that will be useful for later extensions of Algorithm 1.

```
1 Q_q ← cells of Q that contain q
2 Sort the cells in Q_q in increasing order by diameter
3 Make q active
4 while q is active do
5 |   σ ← next largest cell in Q_q
6 |   foreach cell τ ∈ N(σ) that is contained in C²_σ do
7 |   |   if there is a r ∈ R_τ ∪ {m_τ} with q ∈ D(r) then
8 |   |   |   Take an arbitrary such r, add the edge rq to H, and set q to inactive
```

Algorithm 1: Selecting the incoming edges for q and the cone C.

For each cone $C \in \mathcal{C}$ and each site $q \in P$ there is only one cell $\sigma \in \mathcal{Q}_q$ that produces incoming edges for q. We have k cones and $|N(\sigma)| = O(c^2)$ by Lemma 3.3, so q has $O(c^2k)$ incoming edges. It follows that the size of H is $O(n)$ since c and k are constants.

Next we show that H is a t-spanner. For this, we show that every edge pq of G is represented in H by an approximate path. We prove this by induction on the ranks of the edge lengths. This is done in a similar manner as for the standard Yao graphs, but with a few twists that require three additional technical lemmas. Lemma 3.4 deals with the imprecision introduced by taking the cone C^2_σ instead of C_q. It follows from this lemma that if pq is contained in the cone C_q then Algorithm 1 picks at least one edge rq with $r \in C^4_q$. Lemma 3.5 and Lemma 3.6 encapsulate geometric facts that are used to bound the distance between the endpoints r and p depending on whether $|rq|$ is larger or smaller than $|pq|$. Lemma 3.6 is due to Bose et al. [2] and for completeness we include their proof.

Lemma 3.4. *Let $c > 3 + \frac{2}{\sin(\pi/k)}$ and let $\ell \in \{1, \ldots, \lfloor k/2 \rfloor\}$. Consider a cell $\sigma \in \mathcal{Q}_i$ and a cone $C \in \mathcal{C}$. Fix two points $q, s \in \sigma$. Every cell $\tau \in \mathcal{Q}_i$ with $d(\sigma, \tau) \geq (c-2)2^i$ that*

intersects the cone C_q^ℓ is contained in the cone $C_s^{2\ell}$. In particular, any point $p \in C_q^\ell$ with $|pq| \geq (c-2)2^i$ lies in a cell that is fully contained in $C_s^{2\ell}$.

Proof. Let x be a point in $\tau \cap C_q^\ell$. By assumption, $|xq| \geq (c-2)2^i$. Let $D = D(x, 2^i)$ be the disk with center x and radius 2^i. Then, $\tau \subseteq D$. We show that $C_s^{2\ell}$ contains D and thus τ. Since σ has diameter 2^i, and C_q^ℓ contains x, the translated copy C_s^ℓ must intersect D. If $D \subset C_s^\ell$, we are done. Otherwise, there is a boundary ray ρ of C_s^ℓ that intersects the boundary of D. Let y be the first intersection of ρ with the boundary of D. See Figure 5.

Since $s \in \sigma$ and $x \in \tau$, the triangle inequality gives that $|ys| \geq |xs| - |xy| \geq (c-3)2^i$. Let ρ' be the boundary ray of $C_s^{2\ell}$ corresponding to ρ and let y' be the orthogonal projection of y onto ρ'. Since $|ys| \geq (c-3)2^i$ and since the angle between ρ and ρ' is $\pi\ell/k$, we get that $|yy'| \geq (c-3)2^i \sin(\pi\ell/k)$. It follows that $|yy'| \geq 2 \cdot 2^i$ for $c > 3 + \frac{2}{\sin(\pi\ell/k)}$. This holds for any $\ell \in \{1, \ldots, \lfloor k/2 \rfloor\}$ if $c \geq 3 + \frac{2}{\sin(\pi/k)}$. Thus, $\tau \subset D \subset C_s^{2\ell}$. □

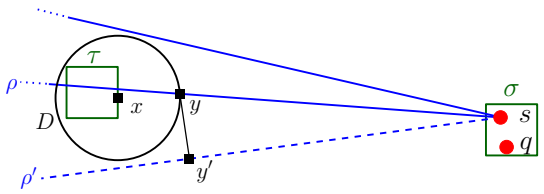


Fig. 5: The boundary ray ρ of C_s^ℓ intersects the boundary of D in y.

Let p be a point in C_q such that pq is an edge of G, and $p \in \tau \in N(\sigma)$ where σ is a cell containing q. Then by Lemma 3.4, τ is contained in C_σ^2. It follows that Algorithm 1 either finds an edge rq before processing σ, or finds an edge rq with $r \in \tau$ while processing σ. By applying Lemma 3.4 again we get that $r \in C_q^4$. This fact is described in greater detail and is being used in the proof of Lemma 3.7 below.

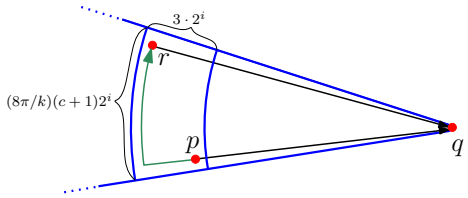
Lemma 3.5. *Let $C \in \mathcal{C}$, and let $q \in \mathbb{R}^2$. Suppose there are two points $p, r \in C_q^4$ with $(c-2)2^i \leq |pq| \leq |rq| \leq (c+1)2^i$. Then $|pr| \leq ((8\pi/k)(c+1)+3)2^i$.*

Proof. The points p and r lie in an annulus around q with inner radius $(c-2)2^i$ and outer radius $(c+1)2^i$. Since $p, r \in C_q^4$, when going from p to r, we must travel at most $(8\pi/k)(c+1)2^i$ units along the circle around q with p on the boundary, then at most $3 \cdot 2^i$ units radially towards r. Thus, $|pr| \leq (8\pi/k)(c+1)2^i + 3 \cdot 2^i$. □

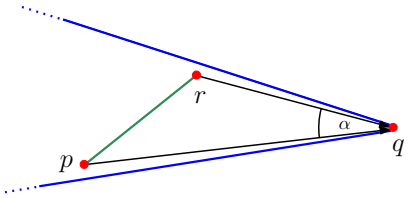
Lemma 3.6 (Lemma 10 in [2])**.** *Let $k \geq 25$ be large enough such that*

$$\frac{1 + \sqrt{2 - 2\cos(8\pi/k)}}{2\cos(8\pi/k) - 1} = 1 + \Theta(1/k) \leq t$$

for our desired stretch factor t. For any three distinct points $p, q, r \in \mathbb{R}^2$ such that $|rq| \leq |pq|$ and $\alpha = \angle pqr$ is between 0 and $8\pi/k$, we have $|pr| \leq |pq| - |rq|/t$.



(a) Lemma 3.5. Two sites in an annulus are close to each other.



(b) Lemma 3.6. If α is small and $|rq| \leq |pq|$, then $|pr| < |pq|$.

Proof. By the law of cosines and since $0 \leq \alpha \leq 8k/\pi$ we have that

$$|pr|^2 = |pq|^2 + |rq|^2 - 2|pq| \cdot |rq| \cos\alpha \leq |pq|^2 + |rq|^2 - 2|pq| \cdot |rq| \cos(8\pi/k)$$

Introducing t by adding and subtracting equal terms, this is

$$\begin{aligned} &= |pq|^2 - \frac{2}{t}|pq| \cdot |rq| + \frac{1}{t^2}|rq|^2 + \frac{t^2-1}{t^2}|rq|^2 - \frac{2(t\cos(8\pi/k)-1)}{t}|pq| \cdot |rq| \\ &= \Big(|pq| - \frac{|rq|}{t}\Big)^2 + \frac{t^2-1}{t^2}|rq|^2 - \frac{2(t\cos(8\pi/k)-1)}{t}|pq| \cdot |rq|. \end{aligned}$$

We complete the proof by showing that under the assumptions of the lemma $\frac{t^2-1}{t^2}|rq|^2 - \frac{2(t\cos(8\pi/k)-1)}{t}|pq| \cdot |rq| \leq 0$. We have that

$$\begin{aligned} \frac{t^2-1}{t^2}|rq|^2 - \frac{2(t\cos(8\pi/k)-1)}{t}|pq| \cdot |rq| &= \frac{|rq|^2}{t^2}\Big(t^2 - 1 - 2(t^2\cos(8\pi/k) - t)\frac{|pq|}{|rq|}\Big) \\ &\leq \frac{|rq|^2}{t^2}\Big(t^2 - 1 - 2(t^2\cos(8\pi/k) - t)\Big), \end{aligned}$$

where the last inequality follows since $|pq| \geq |rq|$ and

$$t \geq \frac{1+\sqrt{2-2\cos(8\pi/k)}}{2\cos(8\pi/k)-1} \geq \frac{1}{2\cos(8\pi/k)-1} \geq \frac{1}{\cos(8\pi/k)},$$

so $t\cos(8\pi/k) \geq 1$. Now we have that

$$t^2 - 1 - 2(t^2\cos(8\pi/k) - t) = (1 - 2\cos(8\pi/k))t^2 + 2t - 1 \leq 0$$

if $\cos(8\pi/k) > 1/2$ and

$$\frac{1+\sqrt{2-2\cos(8\pi/k)}}{2\cos(8\pi/k)-1} \leq t\ .$$

The latter inequality holds by assumption and $\cos(8\pi/k) > 1/2$ for $k \geq 25$. □

We are now ready to bound the stretch of the spanner H. This is done in two steps. In the first step (Lemma 3.7) we prove that for any edge pq of G which is not in H, there exists a shorter edge rq in H, such that r is "close" to p. This fact allows us to prove, via a fairly standard inductive argument, that H is indeed a spanner of G.

Lemma 3.7. *Let c and k be such that $c > 3 + \frac{2}{\sin(\pi/k)}$ as required by Lemma 3.4, k satisfies the conditions of Lemma 3.6 and, in addition, $c \geq 2 + \frac{2t}{t-1}$ and $k \geq \frac{16\pi t}{t-1}$. Let pq be an edge of G. Then either pq is in H or there is an edge rq in H such that $|pr| \leq |pq| - |rq|/t$.*

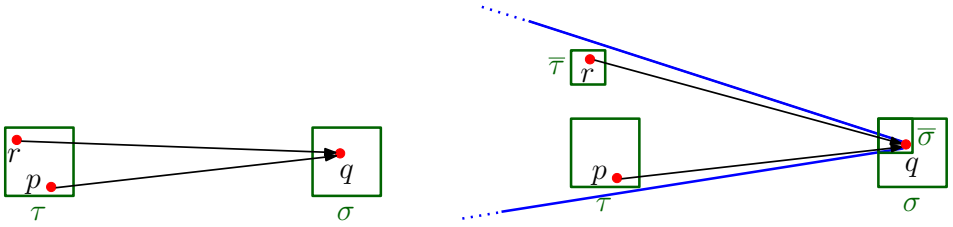
Proof. Let N be the neighborhood relation of the *c-separated annulus decomposition* used by Algorithm 1. Let $(\sigma, \tau) \in N$ be a pair of neighboring cells satisfying requirement (ii) of Definition 3.2 with respect to pq. In particular we have that $q \in \sigma$ and $p \in \tau$. If there is more than one such pair $(\sigma, \tau) \in N$, we consider the pair with minimum diameter. Let $\text{diam}(\sigma) = 2^i$, that is $\sigma, \tau \in \mathcal{Q}_i$.

Let $C \in \mathcal{C}$ be the cone such that $p \in C_q$. Since $p \in C_q \cap \tau$ and since $d(\sigma, \tau) \geq (c-2)2^i$, Lemma 3.4 implies that $\tau \subset C^2_\sigma$. Hence, τ is considered for incoming edges for q (line 6 in Algorithm 1). We split the rest of the proof into two cases.

Case 1: q remains active until (σ, τ) is considered. Requirement (ii) of Definition 3.2 guarantees that Algorithm 1 finds an incoming edge rq for q with $r \in \tau$. If $r = p$, we are done, so suppose that $r \neq p$. Since $\text{diam}(\sigma) = 2^i$ and $|rq| \geq d(\sigma, \tau) \geq (c-2)2^i$ we have

$$\begin{aligned} |pr| &\leq 2^i = |pq| - (|pq| - 2^i) \leq |pq| - (|rq| - 2 \cdot 2^i) \\ &\leq |pq| - (|rq| - 2|rq|/(c-2)) \leq |pq| - |rq|(1 - 2/(c-2)) \leq |pq| - |rq|/t, \end{aligned}$$

for $c \geq 2 + \frac{2t}{t-1}$.



(a) Case 1: p and r are in the same cell σ. (b) Case 2: p and r are in different cells with different levels but in the same cone C^4_q.

Case 2: q becomes inactive before (σ, τ) is considered. Then Algorithm 1 has selected an edge rq while considering a pair $(\bar{\sigma}, \bar{\tau}) \in N$ with $q \in \bar{\sigma}$, $r \in \bar{\tau}$ and $\text{diam}(\bar{\sigma}) \leq 2^{i-1}$. We now distinguish two subcases.

Subcase 2a $|rq| \geq |pq|$. From Property (i) of Definition 3.2 follows that $d(\sigma, \tau) \geq (c-2)2^i$ and therefore $|pq| \geq (c-2)2^i$. It also follows from the same property that $d(\bar{\sigma}, \bar{\tau}) \leq 2c2^{i-1}$, so $|rq| \leq 2c2^{i-1} + 2 \cdot 2^{i-1} = (c+1)2^i$. Combining these inequalities we obtain that $(c-2)2^i \leq |pq| \leq |rq| \leq (c+1)2^i$ and therefore $|pq| \geq |rq| - 3 \cdot 2^i$. Lemma 3.5

implies that $|pr| \leq ((8\pi/k)(c+1)+3)2^i$, and thus we have

$$\begin{aligned}
|pr| &\leq ((8\pi/k)(c+1)+3)2^i \\
&\leq |pq| - |pq| + ((8\pi/k)(c+1)+3)2^i \\
&\leq |pq| - \big(|rq| - 3\cdot 2^i - ((8\pi/k)(c+1)+3)2^i\big) \\
&\leq |pq| - \Big(|rq| - \frac{(8\pi(c+1)-6)2^i}{k}\Big) \\
&\leq |pq| - |rq|\Big(1 - \frac{(8\pi(c+1)-6)}{k(c-2)}\Big) \\
&\leq |pq| - |rq|\Big(1 - \frac{16\pi}{k}\Big)\ .
\end{aligned}$$

The third inequality follows since $|pq| \geq |rq| - 3\cdot 2^i$ as we argued above, and the fifth inequality follows since $2^i \leq |rq|/(c-2)$. The last inequality holds for $c \geq 5$ (which follows from our assumptions). Now we clearly have that

$$|pq| - |rq|\Big(1 - \frac{16\pi}{k}\Big) \leq |pq| - |rq|/t,$$

for $k \geq \frac{16\pi t}{t-1}$.

Subcase 2b $|rq| < |pq|$. By assumption, we have $p \in C_q \subset C_q^4$. Furthermore, by applying Lemma 3.4 with the midpoint of $\bar{\sigma}$ as q, r as p, and q as s, in the statement of the lemma, we get that $r \in C_q^4$. Since $p, r \in C_q^4$ and since the opening angle of C_q^4 is $8\pi/k$, it follows from Lemma 3.6 that $|pr| \leq |pq| - |rq|/t$. □

Lemma 3.8. *For any $t > 1$, there are constants c and k such that H is a t-spanner for the transmission graph G.*

Proof. We pick the constants c and k so that Lemma 3.7 holds. We prove by induction on the indices of edges when ordered by their lengths, that for each edge pq of G, there is a path from p to q in H of length at most $t|pq|$. For the base case, consider the shortest edge pq in G. By Lemma 3.7, if pq is not in H then there is an edge rq in H such that $|pr| \leq |pq| - |rq|/t$. Since pq is an edge of G, it follows that $r_p \geq |pq|$ and therefore pr must also be an edge of G, and it is shorter than pq. This gives a contradiction and therefore pq must be in H.

For the induction step, consider an edge pq of G. If pq is in H we are done. Otherwise by Lemma 3.7 there is an edge rq in H such that $|pr| \leq |pq| - |rq|/t$. As argued above, pr is an edge of G shorter than $|pq|$ so by the induction hypothesis, there is a path from p to r in H of length no larger than $t|pr|$. It follows that

$$d_H(p,q) \leq d_H(p,r) + |rq| \leq t|pr| + |rq| \leq t(|pq| - |rq|/t) + |rq| \leq t|pq|,$$

as required. □

Finding the Decomposition. We use a quadtree to define the cells of the decomposition. We recall that a *quadtree* is a rooted tree T in which each internal node has degree four. Each node v of T is associated with a cell σ_v of some grid $\mathcal{Q}_i$, $i \geq 0$, and if v is an internal node, the cells associated with its children partition σ_v into four congruent squares, each with diameter $\operatorname{diam}(\sigma_v)/2$. If σ_v is from $\mathcal{Q}_i$ then we say that v is of *level* i. Note that all nodes of T at the same distance from the root are of the same level.

Let c be the required parameter for the annulus decomposition. We scale P such that the closest pair in P has distance c. (We use P to denote also the scaled point set). Let L be the smallest integer such that we can translate P so that it fits in a single cell σ of $\mathcal{Q}_L$. Since c is constant and P has spread Φ, the diameter of P (after scaling) is $c\Phi$ and therefore $L = O(\log \Phi)$. We translate P so that it fits in σ and we associate the root r of our quadtree T with this cell σ, i.e. $\sigma_r = \sigma$. By the definition of a level, r is of level L.

We continue constructing T top down as follows. We construct level $i-1$ of T, given level i, by splitting the cell σ_v of each node v, whose cell σ_v is not empty, into four congruent squares, and associate each of these squares with a child of v. We stop the construction of T after generating the cells of level 0. The scaling which we did to P ensures that each cell of a leaf node at level 0 contains at most one point.

We now set $\mathcal{Q} = \{\sigma_v \mid v \in T\}$. We define N as the set of all pairs $(\sigma_v, \sigma_w) \in \mathcal{Q} \times \mathcal{Q}$ such that v and w are at the same level in T and $d(\sigma_v, \sigma_w) \in [c-2, 2c) \operatorname{diam}(\sigma_v)$.[1] For $\sigma \in \mathcal{Q}$, we define R_σ to be the set of all sites $p \in \sigma \cap P$ with $r_p \in [c, 2(c+1)) \operatorname{diam}(\sigma_v)$.

Lemma 3.9. *$(\mathcal{Q}, N, R_\sigma)$ is a c-separated annulus decomposition for G.*

Proof. Property (i) of Definition 3.2 follows by construction. To prove that Property (ii) holds consider an edge pq of G. Let i be the integer such that $|pq| \in [c, 2c)2^i$. Let σ, τ be the cells of $\mathcal{Q}_i$ with $p \in \sigma$ and $q \in \tau$. By construction, σ and τ are assigned to nodes of the quadtree and thus contained in $\mathcal{Q}$. Since $\operatorname{diam}(\sigma) = \operatorname{diam}(\tau) = 2^i$, we have

$$(c-2)2^i \leq |pq| - 2\operatorname{diam}(\sigma) \leq d(\sigma, \tau) \leq |pq| < c2^{i+1},$$

and therefore $(\sigma, \tau) \in N$ by our definition of N. Since pq is an edge of G, it follows that $r_p \geq |pq| \geq c2^i$. If $r_p < (c+1)2^{i+1}$, then $p \in R_\sigma$. Otherwise, $r_{m_\sigma} \geq r_p \geq (c+1)2^{i+1}$, and $q \in \tau \subset D(m_\sigma)$. □

Computing the Edges of H. We find edges for each cone $C \in \mathcal{C}$ separately as follows. For each pair of neighboring cells σ and $\tau \in N(\sigma)$ such that τ is contained in C^2_σ we find all incoming edges to points in σ from points in τ simultaneously. To do this efficiently, we need to sort the points in σ along the x and y directions. Therefore, we process the cells bottom-up along T in order of increasing levels. This way we can obtain a sorted list of the points in each cell σ by merging the sorted lists of its children. See Algorithm 2.

Note that the edges selected by Algorithm 2 have the same properties as the edges selected by Algorithm 1. Thus, by Lemma 3.8, the resulting graph is a t-spanner. Let Q be the set of active sites in σ_v when processing v. Let $\tau \in N(\sigma_v)$ such that τ is

[1] We denote the interval $[a \operatorname{diam}(\sigma_v), b \operatorname{diam}(\sigma_v))$ by $[a, b) \operatorname{diam}(\sigma_v)$.

```
1 for i = 0, ..., L do
2     foreach v ∈ T of level i do
3         Q ← active sites in σ_v ∩ P
          // preproccesing
4         Sort Q in x and y-direction by merging the sorted lists of the children of v
          foreach τ ∈ N(σ_v) contained in C^2_{σ_v} do
5             R ← R_τ ∪ {m_τ}
              // edge selection
6             For each site q ∈ Q, find a r ∈ R with q ∈ D(r), if it exists; add the
              edge rq to H
7         Set all q ∈ Q for which at least one incoming edge was found to inactive
```

Algorithm 2: Selecting the edges for H for a fixed cone C.

contained in $C^2_{\sigma_v}$ and let $R = R_\tau \cup \{m_\tau\}$. Assume $|Q| = n$ and $|R| = m$. To find the edges from points in R to points in Q efficiently, we use the fact that these sets of points are separated by a line parallel to either the x- or the y-axis.

Assume without loss of generality that ℓ is the x-axis, the points of R are above ℓ and the points of Q are below ℓ, and assume that Q is sorted along ℓ. For each point $p \in R$ we take the part of $D(p)$ which lies below ℓ and compute the union of these "caps". This union is bounded from above by ℓ and from below by the lower envelope of the arcs of the boundaries of the caps. The complexity of the boundary of this union is $O(m)$ and it can be computed in $O(m \log m)$ time [21]. See Figure 8.

Once we have computed this union we check for each $q \in Q$ whether q lies inside it. This can be done by checking whether the intersection, z, of a vertical line through q with the union is above or below q. If q is above z then we add the edge rq to H where r is the point such that $z \in \partial D(r)$. We perform this computation for all points in Q together by a simple sweep in x-direction while traversing in parallel the lower envelope of the caps and the points of Q. This clearly takes $O(m + n)$ time.

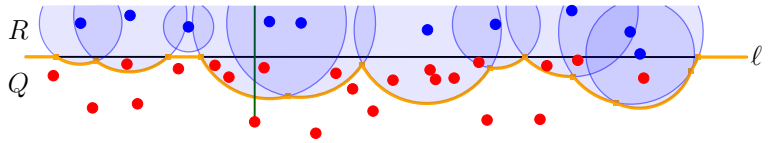


Fig. 8: The lower envelope (orange), the sites Q (red) and R (blue), and the sweepline (green).

We thus proved the following lemma.

Lemma 3.10. *Let Q, R, and ℓ be as above with $|Q| = n$ and $|R| = m$. Suppose that Q is sorted along ℓ and that ℓ separates Q and R. We can compute in $O(m \log m + n)$ time for each $q \in Q$ one disk from R that contains it, provided that such a disk exists.*

Analysis. We prove that Algorithm 2 runs in $O(n \log \Phi)$ time and uses $O(n \log \Phi)$ space. The running time is dominated by the edge selection step described in Lemma 3.10. We argue that each site participates in $O(1)$ edge selection steps as a disk center (in R) and in $O(\log \Phi)$ edge selection steps as a vertex looking for incoming edges. From these observations (and the fact that $\Phi = \Omega(n^{1/2})$) the stated time bound essentially follows.

Lemma 3.11. *We construct the spanner H of the transmission graph G in $O(n \log \Phi)$ time and space.*

Proof. The quadtree T can be computed in $O(n \log \Phi)$ time and space [11], and within this time bound we can also compute $N(\sigma_v)$, R_{σ_v}, and m_{σ_v} for each node $v \in T$.

Merging the sorted lists of the points in σ_w for each child w of v to obtain the sorted list of the points in σ_v (line 4 in Algorithm 2) takes time linear in the number of points in σ_v. Summing up over all nodes v in a single level of T we get that the total merging time per level is $O(n)$, and $O(n \log \Phi)$ for all levels.

To analyze the time taken by the edge selection steps (line 6 in Algorithm 2), consider a particular pair $(\sigma, \tau) \in N$ for which the algorithm runs the edge selection step. By Lemma 3.10, if we charge m_τ by $O(1)$, each disk center in R_τ by $O(\log n)$ and each active site in $\sigma \cap P$ by $O(1)$ then the total charges cover the cost of the edge selection step for (σ, τ). There are $O(n \log \Phi)$ nodes in T and therefore $O(n \log \Phi)$ cells τ in $\mathcal{Q}$. By Lemma 3.3 each such cell τ participates in an edge selection step of $O(c^2) = O(1)$ pairs. So the total charges to the points m_τ over all cells τ, is $O(n \log \Phi)$.

By construction, each $p \in P$ is assigned to $O(1)$ sets R_τ and by Lemma 3.3 each τ participates in an edge selection steps of $O(c^2) = O(1)$ pairs. It follows that the total charges to a point p from edge selections steps of pairs (σ, τ) such that $p \in R_\tau$ is $O(\log n)$.

Finally, each site is active for $O(c^2) = O(1)$ pairs in N at each of $O(\log \Phi)$ levels. So the total charges to a point p from edge selections steps of pairs (σ, τ) such that p is active in $\sigma \cap P$ is $O(n \log \Phi)$. We conclude that the total running time of all edge selection steps is $O(n \log n + n \log \Phi) = O(n \log \Phi)$, since $\log \Phi = \Omega(\log n)$. □

Theorem 3.1 follows by combining Lemmas 3.8 and 3.11.

HAIM SAYS: inconsistent terminology: point vs. site

3.2 From Bounded Spread to Bounded Radius Ratio

Let $P \subset \mathbb{R}^2$ be a set of sites with radius ratio Ψ. We extend our spanner construction from Section 3.1 such that the running time depends on Ψ, the ratio between the largest to smallest radii, rather than on the spread Φ. This is a more general result as we may assume that $\Psi \leq 2\Phi$ (see Section 2). We prove the following theorem.

Theorem 3.12. *Let G be the transmission graph for a set P of n sites in the plane with radius ratio Ψ. For any fixed $t > 1$, we can compute a t-spanner for G in $O(n(\log n + \log \Psi))$ time and $O(n \log \Psi)$ space.*

The main observation which we use is that sites that are close together form a clique in G and can be handled using classic spanner constructions, while sites that are far away from each other belong to distinct components of G and can be dealt with independently.

Given t, we pick sufficiently large constants $k = k(t)$ and $c = c(t)$ as specified in Section 3.1. We scale the input such that the *smallest radius* is c. Let $M = c\Psi$ be the largest radius after we did the scaling. First, we partition P into sets that are far apart and can be handled separately.

Lemma 3.13. *We can partition P into sets $P_1, \ldots, P_\ell$, such that each set P_i has diameter $O(n\Psi)$ and for any $i \neq j$, no site of P_i can reach a site of P_j in G. Computing the partition takes $O(n \log n)$ time and $O(n)$ space.*

Proof. We assign to each site $p \in P$ an axis-parallel square S_p that is centered at p and has side-length $2M$. We define the intersection graph G_S that has a vertex for each point in P, and an edge between two vertices p and q if and only if $S_p \cap S_q \neq \emptyset$. ($G_S$ is undirected.)

If follows that if there is no (undirected) path from p to q in G_S, then there is no (directed) path from p to q in G. We can compute the connected components of G_S in $O(n \log n)$ time by sweeping the plane using a binary search tree [20]. Let $P_1, \ldots, P_\ell$ be the vertex sets of these connected components. By construction, each set of sites P_i has diameter $O(nM)$ and for any $i \neq j$, no site in P_i can reach a site in P_j in G. □

By Lemma 3.13, we may assume that the diameter of our input set P is $O(n\Psi)$. We compute a hierarchical decomposition T for P as in Section 3.1, with a little twist as follows. We translate P so that it fits in a single grid cell σ of diameter $O(n\Psi)$. Starting from σ, we recursively subdivide each non-empty cell into four congruent cells of half the diameter. We do not subdivide cells of level 0 whose diameter is 1. We partition all cells of a particular level in $O(n)$ time and $O(n)$ space.

We construct a quadforest T such that the roots of its trees correspond to the non-empty cells of level $L = \lceil \log \Psi \rceil$ in our decomposition. Each internal node of T corresponds to a non-empty cell obtained when subdividing the cell of its parent. It suffices to store only the lowest L levels, since larger cells cannot contribute any edges to the spanner (as we will argue below). The forest T requires $O(n \log \Psi)$ space and we compute it in $O(n(\log n + \log \Psi))$ time.

We cannot derive from T a c-separated annulus decomposition for G as we did in Section 3.1. In particular a cell corresponding to a leaf of T may now contain many points, that are adjacent in G. For edges induced by such pairs of points we cannot satisfy Property (ii) of Definition 3.2.

We can (and do) derive from T a *partial c-separated annulus decomposition* $(\mathcal{Q}, N, R_\sigma)$ exactly as described in Section 3.1 before Lemma 3.9. This decomposition satisfies Property (ii) of Definition 3.2 for all edges pq with $d(\sigma, \tau) \geq (c-2)$, where σ and τ are the level 0 cells of T containing q and p, respectively. The proof that Property (ii) of Definition 3.2 holds for these edges is the same as the proof of Lemma 3.9. In particular, in the proof of Lemma 3.9, we argue that pairs of cells at level i guarantee Property (ii) of Definition 3.2 for edges of length in $[c, 2c)2^i$. Since the edges of G are of length

at most $M = c\Psi$, the cells up to level $L = \lceil \log \Psi \rceil$ suffice to guarantee Property (ii) of Definition 3.2 for all edges pq with $d(\sigma, \tau) \geq (c-2)$.

We mark all sites of P as active, and we run Algorithm 2 of Section 3.1 using T and the partial c-separated annulus decomposition that we derived from it. The resulting graph H is not yet a t-spanner since the decomposition was only partial.

To make H a spanner we add to it more edges that "take care" of the edges not "covered" by the c-separated annulus decomposition. We consider each pair of level 0 cells σ and τ with $d(\sigma, \tau) < c-2$. The set of points $Q = (P \cap \sigma) \cup (P \cap \tau)$ form a clique, since the distance between each pair of points in Q is no larger than c. We compute a Euclidean t-spanner for Q of size $O(|Q|)$ in $O(|Q| \log |Q|)$ time [18] and for each (undirected) edge pq of this spanner we add pq and qp to H. As each site $p \in P$ participates in $O(c^2)$ such spanners, we generate in total $O(n)$ edges in $O(n \log n)$ time.

We now prove that H is indeed a t-spanner. The proof is analogous to the proof of Lemma 3.8.

Lemma 3.14. *For any $t > 1$, there are constants $c = c(t)$ and $k = k(t)$ such that H is a t-spanner for the transmission graph G.*

Proof. By construction, H is a subgraph of G. Let pq be an edge of G, and let σ and τ be the level 0 cells with $q \in \sigma$ and $p \in \tau$. If $d(\sigma, \tau) < c-2$, then the Euclidean t-spanner for σ and τ contains a path from p to q of length at most $t|pq|$.

For the remaining edges, the lemma is proved by induction on the rank of the edges when we sort them by length, as in Lemma 3.8. The proof is almost verbatim as before; we only comment on the base case. Let pq be the shortest edge in G. If the endpoints p and q lie in level 0 cells whose distance is less than $c-2$, we have already argued that H contains an approximate path from p to q. Otherwise, the same argument as in Lemma 3.8 applies, and the algorithm includes pq in H. □

Using Lemma 3.14, Theorem 3.12 follows just as Theorem 3.1 in Section 3.1. The analysis of the space and time required by our construction is exactly as in Lemma 3.11, but now T has $O(\log \Psi)$ levels.

3.3 Spanners for Unbounded Spread and Radius Ratio

We eliminate the dependency of our bounds on the radius ratio at the expense of a more involved data structure and an additional polylogarithmic factor in the running time. Given $P \subset \mathbb{R}^2$ and the desired stretch factor $t > 1$, we choose appropriate parameters $c = c(t)$ and $k(t)$ as in Section 3.2 and rescale P such that the distance between the closest pair of points in P is $c + 2$.

First, we compute a compressed quadtree T for P. A *compressed quadtree* is a rooted tree in which each internal node has degree 1 or 4. Each node v is associated with a cell σ_v of a grid $\mathcal{Q}_i$. If v has degree 4, then the cells associated of its children partition σ_v into 4 congruent squares of half the diameter, and at least two of them must be non-empty. If v has degree 1, then the cell associated with the only child w of v has diameter at most $\operatorname{diam}(v)/4$ and $(\sigma_v \setminus \sigma_w) \cap P = \emptyset$. Each internal node of T contains at least two sites

in its cell and each leaf at most one site. Note that, in contrast with (uncompressed) quadtrees, the diameter of σ_v may be smaller than 2^{L-i}, where i is the the distance of v to the root and 2^L is the diameter of the root. A compressed quadtree for P with $O(n)$ nodes can be computed in $O(n \log n)$ time [13].

To simplify the notation in the rest of this section, we write $\operatorname{diam}(v)$ instead of $\operatorname{diam}(\sigma_v)$, and for two nodes v, w, we write $d(v, w)$ for $d(\sigma_v, \sigma_w)$.

Our approach is to use the algorithm from Section 3.1 on the compressed quadtree T. There are two problems with this: for one, the depth of T may be linear, so considering all sites for incoming edges at each level, as in Algorithm 2 would be too expensive. Instead we use Chan's dynamic nearest neighbor data structure to quickly identify the relevant sites. It has the following properties.

Theorem 3.15 (Chan, Afshani and Chan, Chan and Tsakalidis [1, 7, 8])**.** *There exists a dynamic data structure that maintains a planar point set S such that*

(i) we can insert a point into S in amortized time $O(\log^3 n)$;

(ii) we can delete a point from S in amortized time $O(\log^6 n)$; and

(iii) given a query point q, we can find the nearest neighbor for q in S in worst-case time $O(\log^2 n)$.

The space requirement is $O(n)$.

We note that the history of Theorem 3.15 is a bit comlicated: Chan's original paper [7] describes a *randomized* data structure with $O(n \log \log n)$ space. Afshahni and Chan [1] describe a *randomized* three-dimensional range reporting structure that improves the space to $O(n)$. Chan and Tsakalidis [8] show how to make both the dynamic nearest neighbor structure and the range reporting structure deterministic, which gives the current form of Theorem 3.15.

The second problem is to define an appropriate neighborhood relation. As in Section 3.1, the neighborhood relation N should consist of pairs (σ_v, σ_w) whose nodes v and w have the same level in T and satisfy $d(v, w) \in [c-2, 2c) \operatorname{diam}(v)$. The set R_{σ_v} should consist of all sites in $\sigma_v \cap P$ whose radius is in $[c-2, 2(c+1)) \operatorname{diam}(v)$, a slightly larger interval than in the previous sections. To make sure that N and R_σ fulfill Definition 3.2(ii), we insert $O(n)$ additional nodes into T so that $\mathcal{Q}$ contains the appropriate cells. To find these nodes, we adapt the WSPD algorithm of Callahan and Kosaraju [5].

Lemma 3.16. *Given a constant $c > 5$, we can in $O(n \log n)$ time insert $O(n)$ nodes into T so that $\mathcal{Q} = \{\sigma_v \mid v \in T\}$ with N and R_σ as above is a c-separated annulus decomposition for G. In the same time, we can compute N and all sets R_σ.*

Proof. First, we run the usual algorithm for finding a c-well-separated pair decomposition on T [5]; see Algorithm 3 for pseudocode. It is well known [17] that the algorithm runs in $O(n)$ time and returns a set W of $O(n)$ pairs $\{v, w\}$ of nodes in T such that

```
   call wspd1(r) on the root of T
1  wspd1(v) :
2  if v is a leaf then
3  |  return ∅
4  else
5  |  Return the union of wspd1(w) and wspd2({w_1, w_2}) for all children w and
   |  pairs of distinct children w_1, w_2 of v
1  wspd2({v, w}) :
2  if d(v, w) ≥ c max{diam(v), diam(w)} then
3  |  return {v, w}
4  else if  diam(v) ≤ diam(w) then
5  |  return the union of wspd2({v, u}) for all children u of w.
6  else
7  |  return the union of wspd2({u, w}) for all children u of v
```

Algorithm 3: Computing a well-separated pair decomposition from a compressed quadtree T

(a) for each two distinct sites p, q, there is exactly one $\{v, w\} \in W$ with $q \in \sigma_v$, $p \in \sigma_w$;

(b) for each $\{v, w\} \in W$, we have $c \cdot \max\{\text{diam}(v), \text{diam}(w)\} \leq d(v, w)$;

(c) for every call wspd2($\{v, w\}$), $\max\{\text{diam}(v), \text{diam}(w)\} \leq \min\{\text{diam}(\overline{v}), \text{diam}(\overline{w})\}$, where $\overline{v}$, $\overline{w}$ are the parents of v and w in T;

In particular, note that since we scaled P such that the closest pair has distance $c + 2$, (b) can be satisfied by cells from $\mathcal{Q}_0$ or above. For each pair $\{v, w\} \in W$, we insert two nodes v' and w' into T such that $\text{diam}(v') = \text{diam}(w')$ and such that $d(v', w')$ is approximately $c \cdot \text{diam}(v')$. Suppose that $\{v, w\}$ was generated through a call from $\{v, \overline{w}\}$ in Algorithm 3. Set $r = \min\{d(v, w)/c, \text{diam}(\overline{w})\}$, rounded down to the next power of 2. First, observe that

$$r \leq \text{diam}(\overline{w}) \leq \text{diam}(\overline{v}), \tag{1}$$

because $r \leq \text{diam}(\overline{w})$ by definition and $\text{diam}(\overline{w}) \leq \text{diam}(\overline{v})$ by (c) and our assumption that wspd2($\{v, \overline{w}\}$) was called. Furthermore, we have

$$r \geq \max\{\text{diam}(v), \text{diam}(w)\}, \tag{2}$$

because $\text{diam}(v)$, $\text{diam}(w)$ are powers of two, so that if r comes from $\text{diam}(\overline{w})$, (2) holds by (c) and if r comes from $d(v, w)/c$, (2) holds by (b). By (1), (2), we can insert nodes v', w' into T between v and $\overline{v}$ and between w and $\overline{w}$ such that $\text{diam}(v') = \text{diam}(w') = r$ and such that $\sigma_v \subseteq \sigma_{v'} \subseteq \sigma_{\overline{v}}$ and $\sigma_w \subseteq \sigma_{w'} \subseteq \sigma_{\overline{w}}$. The case that $\{v, w\}$ was generated through a call from $\{\overline{v}, w\}$ is similar.

To insert all the new nodes into T efficiently, we consider each parent-child pair $\overline{v}, v$ in T and collect the new nodes that were created between $\overline{v}$ and v. We insert them into T between $\overline{v}$ and v by decreasing diameter. If the same cell appears several times, we

merge the corresponding nodes. This takes $O(n \log n)$ time. To find the sets R_σ, we consider each site $p \in P$ and we identify the nodes v in T with $p \in R_{\sigma_v}$ in $O(\log n)$ time: there are at most two integers i with $r_p \in [c-2, 2(c+1))2^i$ (if $c > 5$). For each such i, we find the cell $\sigma \in \mathcal{Q}_i$ containing p. This takes $O(1)$ time. Next, we determine whether σ appears in T. This needs $O(\log n)$ time with an appropriate data structure. Thus, the total time to find all sets R_σ is $O(n \log n)$. In the same way we can compute N in time $O(n \log n)$.

We now argue that this construction yields a c-separated annulus decomposition for P. Definition 3.2(i) holds by construction. For 3.2(ii), fix an edge pq in G. Since W is a c-WSPD, by (a) there is a pair $\{v, w\} \in W$ with $q \in \sigma_v$ and $p \in \sigma_w$. Suppose that $\{v, w\}$ was generated by a call from $\{v, \overline{w}\}$. Thus, we have inserted nodes v' and w' into T with $\sigma_v \subseteq \sigma_{v'} \subseteq \sigma_{\overline{v}}$, $\sigma_w \subseteq \sigma_{w'} \subseteq \sigma_{\overline{w}}$, and with $\operatorname{diam}(v') = \operatorname{diam}(w') = r$. Hence, $q \in \sigma_{v'}$ and $p \in \sigma_{w'}$. We claim that $(\sigma_{v'}, \sigma_{w'}) \in N$: first observe that

$$d(v', w') \geq d(v, w) - 2r \geq cr - 2r = (c-2)\operatorname{diam}(v'), \tag{3}$$

since $r \leq d(v, w)/c$ by definition. Second, if r comes from $d(v, w)/c$, we have $d(v, w)/2c < r \leq d(v, w)/c$, so

$$d(v', w') \leq d(v, w) \leq 2cr. \tag{4}$$

If $r = \operatorname{diam}(\overline{w})$, then

$$d(v', w') \leq d(v, \overline{w}) + \operatorname{diam}(\overline{w}) \leq (c+1)\operatorname{diam}(\overline{w}) \leq 2cr, \tag{5}$$

because $\{v, w\}$ was generated through a call from $\{v, \overline{w}\}$, so $\{v, \overline{w}\}$ is not well-separated, and because $\operatorname{diam}(v) \leq \operatorname{diam}(\overline{w})$ by (c). By (3),(4) and (5), we get $(\sigma_{v'}, \sigma_{w'}) \in N$. Finally, since pq is an edge of G, we have $r_p \geq d(v', w') \geq (c-2)\operatorname{diam}(w')$, by (3). If $r_p < (c+1)\operatorname{diam}(w')$, then $p \in R_{\sigma_{w'}}$. Otherwise let m be the site in $\sigma_{w'} \cap P$ with the largest radius. Then, $r_m \geq r_p \geq (c+1)\operatorname{diam}(w')$, so $D(m)$ contains $\sigma_{v'}$ and thus q. This establishes Definition 3.2(ii). □

Finding the Edges. To construct the spanner $H \subseteq G$ for a stretch factor $t > 1$, we choose appropriate constants $k = k(t)$ and $c = c(t)$. The algorithm proceeds as follows: we scale P such that the closest pair has distance c, and we compute a compressed quadtree T for P. To obtain a c-separated annulus decomposition $(\mathcal{Q}, N, R_\sigma)$ for G, we augment T with $O(n)$ nodes as in Lemma 3.16. For each leaf v of T with σ_v non-empty, we create a dynamic nearest neighbor (NN) data structure S_v as in Theorem 3.15. We sort the cells of T by increasing diameter. A site p is called *active* if $p \in S_v$ for some node v in T. Initially, all sites of P are active. Fix a cone $C \in \mathcal{C}$. To select the spanner edges for C, we consider the nodes of T in increasing order and perform two steps for each node v, similar to Algorithm 2 of Section 3.1: let w be the child of v such that $|S_w|$ is largest. To get S_v, we insert all active sites of the remaining children into S_w (preproccesing). Then we use S_v to do the edge selection for all $\tau \in N(\sigma_v)$ contained in $C^2_{\sigma_v}$; see Algorithm 4. We take a site $r \in R = R_\tau \cup \{m_\tau\}$ and repeatedly query S_v. Let q be the result. If rq constitutes an edge in G, we add rq to H, delete q, and do another

query with r. Otherwise, we continue with the next site of R, until all of r is processed structure.

```
   // preproccesing
 1 Let w be the child of v whose S_w contains the most sites
 2 Insert all active sites of each child w' ≠ w of v into S_w
 3 Set S_v ← S_w
 4 foreach τ ∈ N(σ_v) contained in C^2_{σ_v} do
 5 |   foreach r ∈ R = R_τ ∪ {m_τ} do
   |   |   // edge selection
 6 |   |   q ← NN(v, r) // query S_v with r
 7 |   |   while q ∈ D(r) and q ≠ ∅ do
 8 |   |   |   add the edge rq to H; delete q from S_v; q ← NN(v, r)
 9 |   reinsert all deleted sites into S_v
10 delete all q from S_v for which at least one edge rq was found
```

Algorithm 4: Selecting incoming edges for the sites of a node v and a cone C.

Next, we analyze the running time.

Lemma 3.17. *Algorithm 4 has a total running time of $O(n \log^6 n)$ and uses $O(n \log\log n)$ space.*

Proof. It takes $O(n \log n)$ to compute the compressed quadtree and to find the neighboring pairs as in Lemma 3.16. The total time for creating and merging the dynamic nearest neighbor structures is $O(n \log^4 n)$: every time a site is inserted, the size of the NN structure that contains it increases by a factor of at least two. Thus, each site is inserted $O(\log n)$ times at an amortized cost of $O(\log^3 n)$.

For the edge selection, consider two nodes v, w in T whose cells are neighbors. For each site in $R = R_{\sigma_w} \cup m_{\sigma_w}$, we perform one nearest neighbor query (line 6, Algorithm 4). Since T has $O(n)$ nodes, we have $O(n)$ sites m_{σ_w}. By construction, each site is assigned to R_{σ_w} for at most two nodes w. Thus, since each cell has $O(c^2)$ neighbors by Lemma 3.3 and Lemma 3.16, the time spend for these queries is $O(n \log n)$. Furthermore, for each edge we create in the while loop of line 7, we perform at most two deletions, one insertion and one additional NN query in v. Since H has $O(n)$ edges, the total work is $O(n \log^6 n)$. The total size of the compressed quadtree and of the associated data structures is $O(n)$. Furthermore, a dynamic nearest neighbor structure with m elements requires $O(m \log\log m)$ space [7]. Thus, since at any time each site lies in at most one dynamic nearest neighbor structure, the total space requirement is $O(n \log\log n)$. □

The edges selected by Algorithm 4 have the same properties as the edges selected by Algorithm 1. Thus, by Lemma 3.8 we obtain a t-spanner H. Together with Lemma 3.17 this establishes the following theorem.

Theorem 3.18. *Let G be the transmission graph for a n-point set $P \subset \mathbb{R}^2$. For any $t > 1$, we can compute a t-spanner for G in time $O(n \log^6 n)$ and space $O(n)$.*

4 Applications

We present two applications of our spanner construction: first, we use it to extract *exact* BFS trees from a transmission graph. Second, we show how to extend a given reachability data structure for additional queries specific to transmission graphs. Both applications rely on the *power diagram*, a weighted version of the Voronoi Diagram. It represents the union of a set of disks (e.g., the disks corresponding to a weighted n-point set $P \subset \mathbb{R}^2$) as a planar subdivision. It has construction time $O(n \log n)$, complexity $O(n)$, and—augmented with a point location structure—it supports the following query in time $O(\log n)$: given a point q, find a site in P whose disk contains q, if it exists [15,16].

4.1 From Spanners to BFS Trees

We show how to compute the BFS tree in a transmission graph G for a given root $s \in P$ using the spanner constructions from the previous section. We adapt a technique that Cabello and Jejčič developed for unit-disk graphs [4]. Denote by $d_h(s,p)$ the BFS distance (also known as hop distance) from s to p in G. For $i \in \mathbb{N}_0$ let $W_i \subseteq P$ be the sites $p \in P$ with $d_h(s,p) = i$. Cabello and Jejčič used the Delaunay triangulation (DT) to efficiently identify W_{i+1}, given $W_0, \ldots, W_i$. Our t-spanner provides similar properties for transmission graphs as the DT does for unit-disk graphs.

Lemma 4.1. *Let t be small enough, and let H be the t-spanner for G as in Theorem 3.1, 3.12 or 3.18. Let $v \in W_{i+1}$, for some $i \geq 1$. Then, there is a site $u \in W_i$ and a path $u = q_l, \ldots, q_1 = v$ in H with $d_h(s, q_j) = i + 1$ for $j = l - 1, \ldots, 1$.*

Proof. We focus on the spanner from Theorem 3.12, since it has the most complicated structure and subsumes all other cases.

Since $v \in W_{i+1}$, there is a $w \in W_i$ with $v \in D(w)$. If H contains the edge wv, the claim follows by setting $u = w$ and $q_2, q_1 = u, v$. Otherwise, we construct the path backwards from v (see Figure 9). Suppose we have already constructed a sequence $v = q_1, q_2, \ldots, q_k$ of sites in P such that (i) for $j = 1, \ldots, k-1$, $q_{j+1}q_j$ is an edge of H; (ii) for $j = 1, \ldots k$, we have $q_j \in D(w)$ and $d_h(s, q_j) = i + 1$; and (iii) for $j = 1, \ldots, k-1$, $|wq_{j+1}| < |wq_j|$. We begin with the sequence $q_1 = v$, fulfilling the invariant.

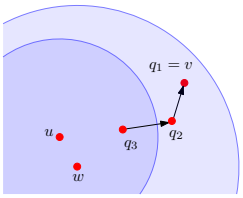


Fig. 9: The partial path constructed backwards from v. Setting $q_4 = u$ will complete it.

Let c be the constant from the spanner construction of Section 3.2, and recall that we scale P such that the smallest radius is c. Suppose that we have $q_1, \ldots, q_k$ and that

wq_k is not an edge of H (otherwise we could finish by setting $u = w$). Let $\sigma, \tau \in \mathcal{Q}_0$ be the cells with $w \in \tau$ and $q_k \in \sigma$. We distinguish two cases, depending on $d(\sigma, \tau)$, and we either show how to find u to complete the path from u to v or how to choose q_{k+1}.

Case 1: $d(\sigma, \tau) < c - 2$. Let $Q = (P \cap \sigma) \cup (P \cap \tau)$. We have that $w, q_k \in Q$. The algorithm of Section 3.2 constructs a Euclidean spanner for Q and adds its edges to H. In particular, there is a directed path π from w to q_k that uses only sites of Q. By construction, all sites of Q have pairwise distance at most c. Thus, for each $p \in Q$ we have $p \in D(w)$ and $q_k \in D(p)$, and therefore $i \leq d_h(s, p) \leq i + 1$. Let u be the last site of π with $d_h(s, u) = i$. To obtain the desired path from u to v we take the subpath of π starting at u and append the partial path $q_k, \ldots, q_1 = v$.

Case 2: $d(\sigma, \tau) \geq c - 2$. Since wq_k is not an edge of H, by Lemma 3.7 there exists an edge rq_k in H with $|wr| < |wq_k|$. We set $q_{k+1} = r$. Since $q_k \in D(w)$, we have $q_{k+1} \in D(w)$ and $i \leq d_h(s, q_{k+1}) \leq i + 1$. If $d_h(s, q_k) = i$, we set $u = q_{k+1}$ and are done. Otherwise, q_{k+1} fulfills properties (i)–(iii).

Since the distance to w decreases in each step and since P is finite, this process eventually stops and yields the lemma. □

```
1  W_0 ← {s}; d[s] = 0; π[s] = s; i = 0; and, for p ∈ P \ {s}, d[p] = ∞ and π[p] = NIL
2  while W_i ≠ ∅ do
3  |  compute power diagram with point location structure PD_i of W_i
4  |  queue Q ← W_i ; W_{i+1} ← ∅
5  |  while Q ≠ ∅ do
6  |  |  p ← dequeue(Q)
7  |  |  foreach edge pq of H do
8  |  |  |  u ← PD_i(q) // query PD_i with q
9  |  |  |  if q ∈ D(u) and d[q] = ∞ then
10 |  |  |  |  enqueue(Q, q); d[q] = i + 1; π[q] = u; add q to W_{i+1}
11 |  i ← i + 1
```

Algorithm 5: Computing the BFS tree for G with root s using the spanner H.

The BFS tree for s is computed iteratively; see Algorithm 5 for pseudocode. Initially, we set $W_0 = \{s\}$. Now assume we computed everything up to W_i. By Lemma 4.1, all sites in W_{i+1} can be reached from W_i in the subgraph of H induced by $W_i \cup W_{i+1}$. Thus, we can compute W_{i+1} as follows: for each $u \in W_i$, start a BFS search in H from u. Every time we encounter a new vertex q, we check if it lies in a disk around a site of W_i, but not in W_i. If so, we add q to W_{i+1} and add the new neighbors of q to the queue. Otherwise, we discard q for now. To test whether q lies in a disk of W_i, we compute a power diagram for W_i in time $O(|W_i| \log |W_i|)$ and query it with t.

Each edge pq of H is considered at most twice by Algorithm 5, and each time we query a power diagram with q in $O(\log n)$ time. Since H is sparse, the total time required is $O(n \log n)$. This establishes the following theorem.

Theorem 4.2. *Let G be the transmission graph of a planar n-point set $P \subset \mathbb{R}^2$. Given*

a spanner H for G as in Theorem 3.1, Theorem 3.12, or Theorem 3.18, we can compute in $O(n \log n)$ additional time a BFS tree rooted at any given site $s \in P$.

4.2 Geometric Reachability Oracles

Let G be a directed graph. If there is a directed path from a vertex s to a vertex t in G, we say *s can reach t* (in G). A *reachability oracle* for a graph G is a data structure that can answer efficiently for any given pair s, t of vertices of G whether s can reach t. Reachability oracles have been studied extensively over the last decades (see, e.g., [14, 22] and the references therein).

When G is a transmission graphs we are interested in a more general type of reachability query where the target t is not necessarily a vertex of G, but an arbitrary point in the plane. We say that a site s can reach a point $t \in \mathbb{R}^2$ if there is site q in G such that $t \in D(q)$ and such that s can reach q in G. We call a data structure that supports this type of queries a *geometric reachability oracle*. We can use our spanner construction from Theorem 3.12 to extend any reachability oracle for a transmission graph to a geometric reachability oracle with a small overhead in space and query time. More precisely, we prove the following theorem.

Theorem 4.3. *Let G be the transmission graph for a set P of n points in the plane with radius ratio Ψ. Given a reachability oracle for G that requires $S(n)$ space and has query time $Q(n)$, we can obtain in $O(n \log n \log \Psi)$ time a geometric reachability oracle that requires $S(n) + O(n \log \Psi)$ space and can answer a query in $O(Q(n) + \log n \log \Psi)$ time.*

Given a query s, t with a target $t \in \mathbb{R}^2$, our strategy is to find a small subset $Q \subseteq P$ such that for each $q \in Q$, $t \in D(q)$, and Q "covers the space around t" in the following sense. For any disk $D(p)$ such that $t \in D(p)$ there is a site $q \in Q$ with $q \in D(p)$. In particular the edge pq is in G.

Such a set Q satisfies that s can reach t if and only if s can reach some point $q \in Q$. Once we have computed Q we decide whether s can reach t by querying the given reachability oracle with s, q for all $q \in Q$. The answer is positive if and only if it is positive for at least one point $q \in Q$.

In what follows, we construct a data structure of size $O(n \log \Psi)$ that allows to find such a set Q of size $O(1)$ in $O(\log n \log \Psi)$ time. Theorem 4.3 is then immediate.

The Data Structure. We compute a 2-spanner H for G as in Theorem 3.12. Let k (the number of cones) and c (the separation parameter) be the two constants used by the construction of H, and recall that we scaled P such that the smallest radius of a site in P is c. Let T be the quadforest used by the construction of H. The trees in T have depth $O(\log \Psi)$ and each node $v \in T$ corresponds to a grid cell σ_v from some grid $\mathcal{Q}_i$, $i \geq 0$. Our data structure is obtained by augmenting each node $v \in T$ by a power diagram PD_{σ_v} for the sites in $\sigma_v \cap P$, together with a point location data structure. This requires $O(|\sigma_v \cap P|)$ space and $O(|\sigma_v \cap P| \log |\sigma_v \cap P|)$ time [15, 16] for each v. Since any site of P is in $O(\log \Psi)$ cells of T, we need $O(n \log \Psi)$ space and $O(n \log n \log \Psi)$ time in total.

```
1 L ← depth of T
2 for i = 0, ..., L do
3 |   σ ← cell of Q_i with t ∈ σ
4 |   foreach τ ∈ N(σ) contained in C_σ^2 do
5 |   |   q ← PD_τ(t) // query PD_τ with t
6 |   |   if t ∈ D(q), add q to Q
7 |   Stop if at least one q was added to Q
```

Algorithm 6: Query Algorithm for a cone C and a point t.

Performing a Query. Let a query point $t \in \mathbb{R}^2$ be given. Let σ be the cell in $\mathcal{Q}_0$ that contains t. To find Q, we first go through all non-empty cells $\tau \in \mathcal{Q}_0$ with $d(\sigma, \tau) \leq c-2$, and we take an arbitrary site $q \in \tau \cap P$ with $t \in D(q)$ and add it to Q, if such a site exists. For this, we query PD_τ with t. Second, we go through all cones $C \in \mathcal{C}$, and we run Algorithm 6 with C and t to find the remaining sites for Q. Algorithm 6 is similar to Algorithms 1 & 2, and it simulates inserting t into the spanner: we go through the grids at all levels of T, and we consider the cell σ that contains t. We find the neighborhood $N(\sigma)$ of all relevant non-empty cells in T that are at the same level as σ. Then we select sites that would form incoming edges for t in the spanner. In particular, if we consider the transmission graph where t is added as a sink (i.e., with infinitesimally small radius), Lemma 3.7 holds for the incoming edges of t. Using this fact, we can prove that our data structure has the desired properties.

Lemma 4.4. *Let P be a planar n-point set with radius ratio Ψ. We can construct in $O(n \log n \log \Psi)$ time a data structure that finds for any given query point $t \in \mathbb{R}^2$ a set $Q \subseteq P$ with $|Q| = O(1)$ such that for any site $p \in P$, if $t \in D(p)$ we have that $D(p) \cap Q \neq \emptyset$. The query time is $O(\log n \log \Psi)$ and the space requirement is $O(n \log \Psi)$.*

Proof. The construction time and space requirement are immediate. For the query time we use that T has depth $O(\log \Psi)$ and thus inserting t and computing the new neighborhoods can be done in time $O(\log n \log \Psi)$. Since querying the point location structure of a power diagram requires $O(\log n)$ time, the total query time is $O(\log n \log \Psi)$.

By construction, Q has size $O(1)$: in the first step, we add at most one site for every cell with distance $c-2$ from σ, and there are $O(c^2)$ such cells. In the second step, for each cone we only add sites for one level of T. By Lemma 3.3, there are $O(c^2)$ such sites.

Now let $p \in P$ be a site with $t \in D(p)$. It remains to show that $D(p) \cap Q \neq \emptyset$. If $p \in Q$, we are done. If not, we let σ and τ be the cells in $\mathcal{Q}_0$ with $t \in \sigma$ and $p \in \tau$. If $d(\sigma, \tau) \leq c-2$, there is a site $q \in \tau \cap Q$. Since $\mathrm{diam}(\tau) = 1$ and $r_p \geq c$, we have $q \in D(p)$. If $d(\sigma, \tau) > c-2$, we consider pt as an edge that was not selected by Algorithm 6. By Lemma 3.7, there is an edge qt with $q \in Q$ and $|pq| < |pt|$. Since $t \in D(p)$ we also have $q \in D(p)$. This finishes the proof. □

5 Conclusion

We have described the first construction of spanners for transmissions graphs in near-linear time, and we have given several applications that demonstrate its usefulness. Our techniques are quite general, and we expect that they will be applicable in similar settings. For example, in ongoing work we consider how to extend our results to (undirected) disk intersection graphs. This would significantly improve the bounds of Fürer and Kasiviswanathan [12].

Our most general spanner construction requires quite heavy machinery, namely a dynamic data structure for planar Euclidean nearest neighbors. It remains an interesting question whether this data structure can be simplified or avoided completely for the spanner construction. In ongoing work, we explore variants of dynamic nearest neighbors that may lead to improved theoretical and practical bounds.

Finally, we believe that our work indicates that transmission graphs constitute an interesting and fruitful model of geometric graphs worthy of further investigation. In a companion paper, we consider several questions concerning reachability in transmission graphs. In particular, we describe several constructions of reachability oracles for them (see Section 4.2), providing many opportunities to apply Theorem 4.3. Also, in this context our spanner construction plays a crucial role in obtaining fast preprocessing algorithms.

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