An Optimal Algorithm for Reconstructing Point Set Order Types from Radial Orderings

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Abstract. Given a set P of n labeled points in the plane, the *radial system* of P describes, for each $p \in P$, the radial ordering of the other points around p. This notion is related to the *order type* of P, which describes the orientation (clockwise or counterclockwise) of every ordered triple of P. Given only the order type of P, it is easy to reconstruct the radial system of P, but the converse is not true. Aichholzer et al. (*Reconstructing Point Set Order Types from Radial Orderings*, in Proc. ISAAC 2014) defined $T(R)$ to be the set of order types with radial system R and showed that sometimes $|T(R)| = n - 1$. They give polynomial-time algorithms to compute $T(R)$ when only given R.
We describe an optimal $O(n^2)$ time algorithm for computing $T(R)$. The algorithm constructs the convex hulls of all possible point sets with the given radial system, after which sidedness queries on point triples can be answered in constant time. This set of convex hulls can be found in $O(n)$ time. Our results generalize to *abstract order types*.

1 Introduction

Let P be a set of n labeled points in the plane. The *chirotope* of P is a function that indicates the orientation of each triple of P (clockwise, counterclockwise, or collinear). Throughout this paper, we consider only point sets in *general position*, that is, without collinear triples. Two labeled point sets have the same *order type* if they have the same chirotope or if one chirotope is the negation of the other. Many problems on planar point sets do not depend on the exact coordinates of the points but only on their order type. Examples include computing the convex hull and determining whether two segments with endpoints in the point set intersect. A *generalized configuration of points* is a labeled point setand an arrangement of pseudo-lines such that each pair of points is on a pseudo-line and each pseudo-line contains exactly two points [5]. By the containment in semispaces defined by these supporting pseudo-lines, orientations of point triples are defined analogously to point sets: if a point c is to the left of the pseudo-line through a and b when going from a to b, then the triple (a, b, c) is oriented *counterclockwise*. *Abstract order types* are the generalization of point set order types to generalized configurations of points. For most combinatorial purposes, generalized configurations of points behave like point sets; their convex hull is the intersection of those halfspaces bounded by the pseudolines that contain all the points and determines a cycle of directed arcs. Their chirotope determines whether two arcs defined by pairs of points cross. We refer to the work of Goodman and Pollack (see, e.g., [6]) and to a book by Knuth [7] (who calls abstract order types "CC systems") for more details. In this paper, we will be solely concerned with abstract order types. As opposed to many other publications on the subject, we stress that we consider *labeled* abstract order types here (and not abstract order type isomorphism classes). That is, we say that two abstract order types are *equivalent* when the bijection between them is fixed and they have the same chirotope, or one chirotope is the negation of the other.

Radial systems. The *counterclockwise radial system* R_χ of an abstract order type χ on a set P defines, for each $p \in P$, the counterclockwise order $R_\chi(p)$ of the elements in $P \setminus \{p\}$ around p. We call each $R_\chi(p)$ a *counterclockwise radial ordering*. When χ is realizable as a point set, then $R_\chi(p)$ can be found by sweeping a ray around p in counterclockwise direction. Given a function U, we write $U \sim R_\chi$ when, for all $p \in P$, it holds that $U(p)$ is equal to $R_\chi(p)$ or the reverse of $R_\chi(p)$. Thus, in a sense, the relation $\sim$ "forgets" the clockwise/counterclockwise direction of each individual $R_\chi(p)$. We call U an *undirected radial system* and each $U(p)$ an *undirected radial ordering*. When we say *radial system*, we always mean counterclockwise radial system. It is possible to recover R_χ from U (all omitted proofs can be found in the full version of the paper):

Theorem 1.1. *Let χ be an abstract order type on V with $|V| = n$ and let $U \sim R_\chi$. Then U uniquely determines R_χ (up to complete reversal) and we can recover R_χ from U by reporting the direction of every $U(v)$ in $O(n)$ time.*

Aichholzer et al. [1] investigated under which circumstances the undirected radial system U of a generalized configuration of points P uniquely determines the abstract order type χ. They show that if P has a convex hull with at least four points, then U uniquely determines χ. More precisely: let $T(U)$ be the set of abstract order types with undirected radial system U (i.e., the sequences in U are known to originate from an abstract order type). We have

Theorem 1.2 ([1, Theorem 1 and Theorem 2]). *Consider an abstract order type χ on a set V with $n = |V| \geq 5$ and let $U \sim R_\chi$. Let $H \subseteq V$ be the points of the convex hull of χ. Then we can compute $|H|$ from U in polynomial time. Further, (i) if $|H| \neq 3$, then $T(U) = \{\chi\}$ and we can compute χ from U in polynomial time; and (ii) if $|H| = 3$, then $|T(U)| \leq n - 1$; all elements of $T(U)$ have convex hull size 3; and we can compute $T(U)$ from U in polynomial time.*

In the full version of [1] it is shown that (i) can be implemented in $O(n^3)$ time. There exist counterclockwise radial systems R with $|T(R)| = n - 1$. Hence, it is not possible to improve the bound on $|T(U)|$ in (ii), even if we consider counterclockwise radial systems instead of undirected radial systems [1].

Although U does not always uniquely determine χ, the pair (U, H), where H is the set of points on the convex hull, always suffices [1]. Thus, the abstract order types in $T(U)$ all have different convex hulls. Given an undirected radial system U on a set V, we say that a subset $H \subseteq V$ is *important* if H is the convex hull of some abstract order type in $T(U)$. An *important triangle* is an important set of size 3. Important sets are interrelated as follows.

Theorem 1.3 ([1, Propositions 1–4]). *Consider a radial system R on a set V with $n = |V| \geq 5$. If V has more than two important triangles, then all important triangles must have an element $v^* \in V$ in common. Thus, in general, exactly one of the following cases applies:*

(1) there is exactly one important set, and it has size at least four; or
(2) all important sets are triangles, there are at most $n-1$ of them, and they all share an element $v^* \in V$; or
(3) there are exactly two important sets, and they are disjoint triangles.

For cases (2) and (3), there exists actually a complete characterization of the important triangles. For an abstract order type $\chi \in T(U)$, an *inner* important triangle of χ is an important triangle of U that is not equal to the convex hull of χ. The following lemma reformulates the fact that an inner important triangle is not contained in a convex quadrilateral [1, 9].

Lemma 1.4 ([1, 9]). *Let χ be an abstract order type on a set P. A triangle $\langle a, b, c\rangle$ of χ is an inner important triangle iff the following conditions hold.*

(1) It is empty of points of P.
(2) It partitions $P \setminus \{a, b, c\}$ into three subsets P_a, P_b, and P_c, such that P_a is to the left of the directed line ba and to the right of ca, and P_b and P_c are defined analogously.
(3) For any two points $v, w \in P_a$, the pseudo-line vw intersects the edge bc; and similarly for points in P_b and P_c.

In this context, we mention that, if R is the radial system of some point set order type, then every abstract order type with radial system R can be realized as a point set [9, Theorem 27]. We do not consider realizability of abstract order types as point sets in this work. In the following, with a *realization of a radial system R*, we mean an abstract order type whose radial system is R.

Interestingly, realizability of radial systems cannot be decided by checking realizability of all induced radial systems up to any fixed constant size. Fig. 1 shows a construction which is not realizable as an abstract order type, while every radial system induced by any strict subset of the vertices can be realized, even as a point set order type.

Theorem 1.5. *For any $k \geq 3$, there exists a radial system R_k over $n = 2k + 1$ vertices that is not realizable as an abstract order type, but that becomes realizable as a point set order type when removing any point.*

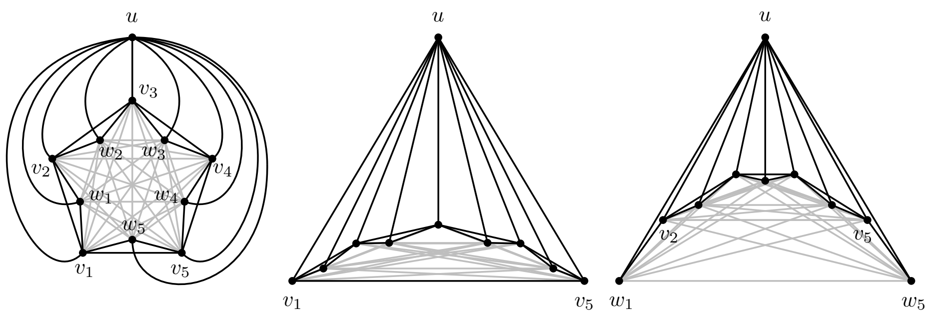

Fig. 1. The construction of R_5 on the left, and point set order type realizations of two induced radial systems after removing either w_5 or v_1 on the right.

Good drawings. A *good drawing* (sometimes also called *simple topological graph*) of a graph is a drawing in the plane or on the sphere where each vertex is represented by a distinct point, and each edge is represented by a Jordan arc between its two vertices; any two such arcs intersect in at most one point, which is either a common endpoint or a proper crossing. The *rotation* of a vertex v in a good drawing is the cyclic order of the edges incident to v. The *rotation system* of a good drawing is the set of the rotations of its vertices. The radial system of a point set P is equivalent to the rotation system of the complete geometric graph on P. A generalized configuration of points Q defines a good drawing of K_n where the vertices are embedded on the points of Q and every edge is a segment of a pseudo-line in Q. The radial system of Q is equivalent to the rotation system of this good drawing. In a good drawing of K_n, the rotation system determines which edges cross. Therefore, it fixes the drawing up to the ordering of the crossings; in particular, we can find out whether two edges cross by locally inspecting the rotations for the four vertices involved [8]. We will use good drawings as a tool to maintain important sets in our algorithm.

Related work. Variations on the notion of radial systems have been studied in many contexts. A prime example are *local sequences*, which are obtained by sweeping a line (instead of a ray) around each point. Goodman and Pollack [6] show that they determine the order type of P. Pilz and Welzl [9] describe a hierarchy on order types based on crossing edges in which two order types are considered equivalent iff they have the same radial system. We refer to Aichholzer et al. [1] for a more complete list of related work.

Our results. For a given undirected radial system U on n vertices (which has size $\Theta(n^2)$), we provide an algorithm to direct the n radial orderings in $O(n)$ time (Theorem 1.1). Our main algorithm identifies the set of convex hulls of all abstract order types consistent with the given radial system in $O(n)$ time (provided that the input is the radial system of an abstract order type). This set allows for constant-time queries to the chirotope for any of these abstract order types. Hence, this is a means of reporting an explicit representation of $T(U)$

in $O(n)$ time, significantly improving Theorem 1.2. We remark that this can be shown to be optimal, as an adversary can use any unconsidered point in a suitable example to alter $|T(n)|$ (e.g., by using it to "destroy" a top triangle as defined in Section 2.1). If we do not know that the set of permutations provided as input is indeed the radial system of an abstract order type, we show how to verify this in $O(n^2)$ time. A straight-forward adversary argument shows that $\Omega(n^2)$ time (i.e., reading practically the whole input) is necessary to verify whether $|T(n)| = 0$. In this sense, our algorithm is optimal.

For radial systems as a data structure, we require that we can obtain the relative order of three elements in a radial ordering in constant time. This can be done by storing not only the radial ordering, but also the rank of each element within some linear order defined by the radial ordering around each vertex, when considering the n elements to be identified by their index in $\{1, \ldots, n\}$.

2 Obtaining Chirotopes from Radial Systems

Let R be the radial system for which we want to obtain the set $T(R)$ of abstract order types that realize it. (This set may be empty.) Our algorithm for computing $T(R)$ (conceptually) constructs a good drawing of a plane graph on the sphere by adding the vertices one-by-one and maintaining the faces that are candidates for the convex hull. We will see later that this actually boils down to maintaining at most two sequences of vertices plus one special vertex. Throughout the description, we assume that the radial orderings indeed correspond to the radial system of an abstract order type. If any of the assumptions is not fulfilled, we know that there is no abstract order type for the given set of radial orderings. If R can be realized as an abstract order type, then the plane graph is the subdrawing of a drawing weakly isomorphic (cf. [8]) to the complete graph on any generalized configuration of points that realizes that abstract order type.

For a plane cycle $C = \langle c_0, \ldots, c_{m-1} \rangle$ of m vertices (which we think of as counterclockwise with its *interior* to its left) in a good drawing of the complete graph, we say that an edge $c_i v$ *emanates to the outside* of the cycle at c_i if we encounter v in a counterclockwise sweep in $R(c_i)$ from c_{i-1} to c_{i+1}.[4] Otherwise, $c_i v$ *emanates to the inside*. If cv emanates to the outside for all $c \in C$, then v *covers* the cycle. If cv emanates to the inside for all $c \in C$, then we say that v is *inside* the cycle, and *outside* otherwise. If v neither is inside C nor covers C, then the good drawing restricted to C plus all edges from vertices of C to v is not plane. We call a cycle $\langle c_0, \ldots, c_{m-1} \rangle, m \geq 4$, *compact* if it is plane and, for each c_i, the edges $c_i c_{i+2}, c_i c_{i+3}, \ldots, c_i c_{i-2}$ all emanate to the inside (i.e., its rotation system corresponds to the radial system of m points in convex position).

Observation 2.1. *In any realization of a radial system, a compact cycle corresponds to a set of points in convex position.*

Lemma 2.2. *Consider a radial system R. If Γ is a good drawing of the complete graph whose rotation system corresponds to R, then no element of an important*

[4] We consider all indices modulo the length of the corresponding sequence.

set is inside of a compact cycle in Γ. In particular, no edge crosses an edge of the cell in Γ that defines the convex hull of a realization.

Lemma 2.2 is closely related to Lemma 1.4 (see also [2, Theorem 3.2]). Consider a radial system R and a directed edge ab. Assume that ab is an edge of the convex hull of an abstract order type χ with $R_\chi \sim R$ (i.e., a *realization*) so that all other points of χ are to the left of ab. It is easy to see that the edge ab and R together uniquely determine the convex hull of our abstract order type. Hence, there is only one abstract order type realizing R with such an edge. We re-state the following well-known fact.

Lemma 2.3. *Given the radial system and a directed convex hull edge of an abstract order type, the orientation of a triple can be reported in constant time.*

2.1 Obtaining Hull Edges

Let P be a set of n points (or a generalized configuration of points), and let R be the radial system of the abstract order type χ of P. We assume that there is at least one abstract order type realizing R. The goal is to find a set of $O(n)$ *candidate edges* that may appear on the convex hull of a realization (i.e., the edges of the convex hull of P if there is no other realization of R or the union of the edges of all important triangles). Our algorithm incrementally builds a "hull structure" (defined below) for P. Before step k, we have a current set $P_{k-1} \subseteq P$ of $k-1$ points and a hull structure Z_{k-1} that represents the candidate edges for P_{k-1}. The algorithm selects a point $p_k \in P \setminus P_{k-1}$, adds it to P_{k-1}, and updates Z_{k-1}. A careful choice of p_k allows for updates in constant amortized time.

We begin with the description of the hull structure. Let $P_k \subseteq P$ be a set of k points ($k \geq 4$). The kth *hull structure* Z_k is an abstract representation of a graph with vertex set $V_k \subseteq P_k$ that is embedded on the sphere. That is, Z_k stores the incidences between the vertices, edges, and faces, but it does not assign coordinates to the points. Hull structures come in three types (see Fig. 2), which correspond in one-to-one-fashion to the three possible configurations of important sets in Theorem 1.3:

Type 1: Z_k is a compact cycle (recall that therefore, R restricted to V_k represents a convex $|V_k|$-gon with $|V_k| \geq 4$).

Type 2: Z_k consists of a compact cycle C and a *top vertex* t that covers C. The 3-cycles incident to t are called *top triangles*. A top triangle τ is marked either *unexamined*, *dirty*, or *empty*. Initially, τ is unexamined. Later, τ is marked either dirty or empty. "Dirty" indicates that τ cannot contain a convex hull vertex in its interior. "Empty" means that τ is a candidate for an important triangle. We orient each top triangle so that all other vertices of Z_k are to the exterior.

Type 3: Z_k is the union of two vertex-disjoint 3-cycles T_1 and T_2, called *independent triangles*. T_1 and T_2 are directed so that each has all of P_k to the interior. Moreover, the edges between the vertices of T_1 and T_2 appear as in Fig. 2.

Let R_k be the restriction of R to P_k. We maintain the following invariant: (a) if R_k has exactly one important set of size at least four, Z_k is of Type 1

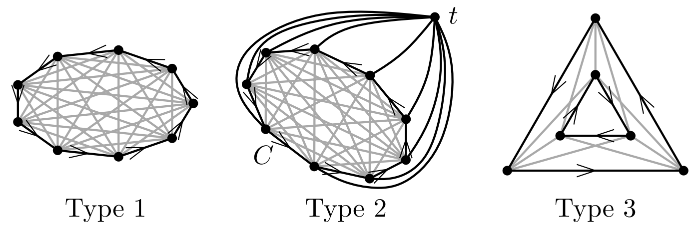

Fig. 2. The three different types of hull structures.

and represents the counterclockwise convex hull boundary; (b) if R_k has two disjoint important triangles, Z_k is of Type 3, and the important triangles are exactly the independent triangles; (c) if R_k has several important triangles with a common vertex, Z_k is of Type 2 and all important triangles appear as top triangles; (d) if R_k has exactly one important triangle, Z_k is of Type 2 or 3, with the important triangle as a top triangle (Type 2) or as an independent triangle (Type 3). Furthermore, if Z_k is of Type 2, no convex hull vertex for P lies inside a dirty triangle, and each point of P_k lies either in C or in a dirty triangle.

Initially, we pick 5 arbitrary points from P. Among those, there must be a compact 4-cycle Z_4 (e.g., [1, Figure 4]), which can be found in constant time. Our initial hull structure Z_4 is of Type 1, with vertex set $V_4 = P_4$. We next describe the insertion step for each possible type. For the running time analysis, we subdivide the algorithm into *phases*. Each phase is of Type 1, 2, or 3, and a new phase begins each time the type of the hull structure changes.

Type 1. We take an arbitrary vertex c of Z_{k-1} and check in constant time whether c has an incident edge in R emanating to the outside of Z_{k-1}. If not, the edges incident to c in Z_{k-1} are on the convex hull of P, and we are done; see below. Otherwise, let $p_k \in P \setminus P_{k-1}$ be the endpoint of such an edge. We set $P_k = P_{k-1} \cup \{p_k\}$, and we walk along Z_{k-1} (starting at c) to find the interval I of vertices for which the edge to p_k emanates to the outside. There are two cases: (i) if $I = Z_{k-1}$ (i.e., p_k covers Z_{k-1}), then Z_k is the hull structure of Type 2 with compact cycle Z_{k-1}, top vertex p_k, and all top triangles marked unexamined; (ii) if $I = \langle c_i, \dots, c_j \rangle$ is a proper subinterval of Z_{k-1}, the next hull structure Z_k is of Type 1 with vertex sequence $\langle p_k, c_j, \dots, c_i \rangle$ (R is realizable, so $c_j \neq c_i$).

Lemma 2.4. *We either obtain an edge from which the convex hull can be determined uniquely, or Z_k is a valid hull structure for P_k.*

Lemma 2.5. *A Type 1 phase that begins with a hull structure of size m and lasts for ℓ insertions takes $O(m + \ell)$ time. Furthermore, the next phase (if any) is of Type 2, beginning with a hull structure of size at most $m + \ell$.*

Type 2. We begin with a simple observation.

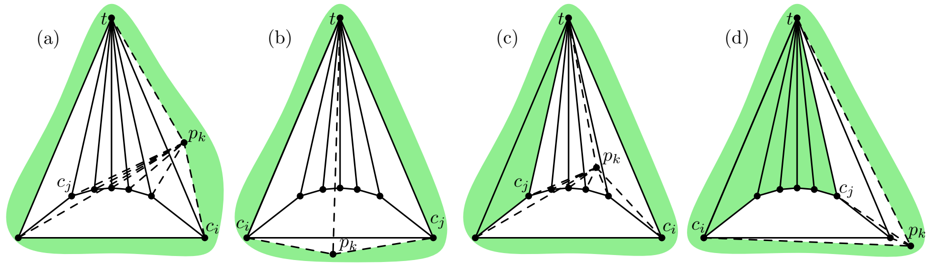

Fig. 3. Z_{k-1} is of Type 2 and p_k is not covering: if p_k forms a non-crossed 4-cycle, Z_k is of Type 1 (a, b); if not, Z_k is of Type 2 with p_k on the compact cycle (c, d). The algorithm will later discover that the triangle $\langle t, c_{j+1}, c_j \rangle$ in (c) is not important since it is inside a convex quadrilateral.

Observation 2.6. *Let Z_{k-1} be a Type 2 hull structure with compact cycle C and top vertex t. The vertices of C appear in their circular order in the clockwise radial ordering around t.*

We need to identify a suitable vertex p_k to insert. For this, we select an unexamined top triangle $\tau = \langle t, c_{i+1}, c_i \rangle$ and test whether c_i has an incident edge that emanates to the inside of τ. If yes, let $v \in P \setminus P_{k-1}$ be an endpoint of such an edge and check whether $c_i v$ crosses the edge tc_{i+1}. If so, then by Lemma 2.2 the vertices of τ lie inside a convex quadrilateral and there is no convex hull vertex inside τ. We mark τ dirty and proceed to the next unexamined triangle. If not, we set $p_k = v$ and $P_k = P_{k-1} \cup \{p_k\}$. If c_i has no incident edge emanating to the inside of τ, we perform the analogous steps on c_{i+1}. If c_{i+1} also has no such incident edge, we mark τ empty and proceed to the next unexamined triangle. (The empty triangle τ might still be crossed by an edge incident to t.)

Lemma 2.7. *We either find a new vertex p_k, or all candidate edges for P lie in Z_k. Furthermore, no dirty triangle contains a possible convex hull vertex of P.*

With p_k at hand, we inspect the boundary of C to find the interval I of vertices for which the edge to p_k emanates to the outside of C. First, if p_k does not cover C, i.e., $I = \langle c_i, \ldots, c_j \rangle$ is a proper subinterval of C, then p_k must lie between c_{i-1} and c_{j+1} in the clockwise order around t, as in any realization one of the cases in Fig. 3 applies. If p_k is between c_{i-1} and c_i or between c_j and c_{j+1}, then either $\langle p_k, t, c_{i-1}, c_i \rangle$ or $\langle t, p_k, c_j, c_{j+1} \rangle$ is a compact 4-cycle containing P_k, and we make it the next hull structure Z_k

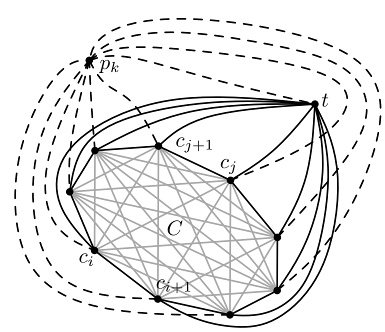

Fig. 4. Z_{k-1} is of Type 2 and p_k is covering.

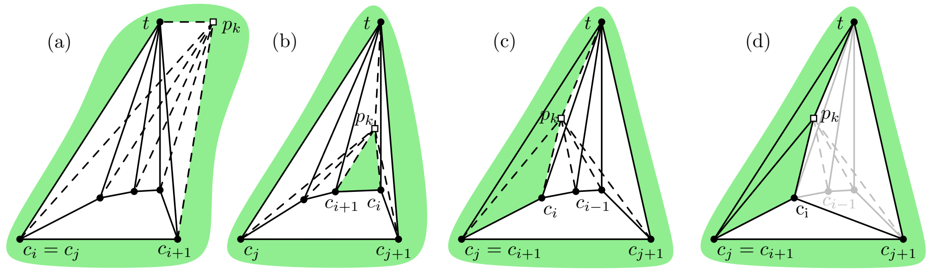

Fig. 5. Z_{k-1} is of Type 2 and p_k (box) is covering: if t and p_k are between the same vertices in each other's rotation, Z_k is of Type 1 (a); if these vertices are disjoint, Z_k is of Type 3 (b); if t and p_k have a common neighbor c_j in the other's rotation (c), the new top vertex c_j of Z_k structure requires the construction of a new compact cycle (d).

of Type 1; see Fig. 3 (a). The green areas in the figures are the only regions where we might still find candidate edges. Otherwise, if $i+1=j$ and the edge tp_k crosses c_ic_{i+1}, the compact 4-cycle $\langle t, c_j, p_k, c_i\rangle$ contains P_k and becomes the next Type 1 hull structure Z_k; see Fig. 3 (b). In any other case (i.e., p_k lies between c_i and c_j in clockwise order around t and if $i+1=j$ then tp_k does not cross c_ic_{i+1}), Z_k is of Type 2 and obtained from Z_{k-1} by removing the top triangles between c_i and c_j and adding the top triangles $\langle t, p_k, c_i\rangle$ and $\langle t, c_j, p_k\rangle$; see Fig. 3 (c) and (d). If c_ip_k intersects an edge of Z_{k-1}, then $\langle t, p_k, c_i\rangle$ lies in a compact 4-cycle and is marked dirty. Otherwise, it is marked unexamined. We handle $\langle t, c_j, p_k\rangle$ similarly.

Second, suppose p_k covers C and let i, j be so that p_k is between c_i and c_{i+1} in clockwise order around t and t lies between c_j and c_{j+1} in clockwise order around p_k. Observation 2.6 ensures that these edges are well-defined; see Fig. 4. Now there are three cases. First, if $i=j$, then one of $\langle c_i, c_{i+1}, t, p_k\rangle$ or $\langle c_i, c_{i+1}, p_k, t\rangle$ defines a compact 4-cycle containing P_k, so Z_k is of Type 1 and consists of this cycle; see Fig. 5 (a). Second, if $\{i, i+1\} \cap \{j, j+1\} = \emptyset$, then Z_k is of Type 3, with independent triangles $\langle p_k, c_i, c_{i+1}\rangle$ and $\langle t, c_j, c_{j+1}\rangle$; see Fig. 5 (b). Third, suppose that $j = i+1$ or $i = j+1$, say, $j = i+1$. Then Z_k is of Type 2, with top vertex c_j and compact cycle $\langle t, p_k, c_i, c_{j+1}\rangle$. The top triangle $\langle c_j, c_{j+1}, c_i\rangle$ is dirty, the other top triangles are unexamined; see Fig. 5 (c–d).

Lemma 2.8. *The resulting hull structure is valid for P_k.*

Lemma 2.9. *A Type 2 phase that begins with a hull structure of size m and lasts for ℓ insertions takes $O(\ell + m)$ time. Furthermore, if the next phase (if any) is of Type 1, it begins with a hull structure of size at most 4.*

Type 3. Let $T_1 = \langle a, b, c\rangle$ and $T_2 = \langle a', c', b'\rangle$ be the two independent triangles of Z_{k-1}, and let p_k be an arbitrary vertex of $P \setminus P_{k-1}$. We set $P_k = P_{k-1} \cup \{p_k\}$, and we distinguish three cases. First, if p_k is inside both T_1 and T_2, then $Z_k =$

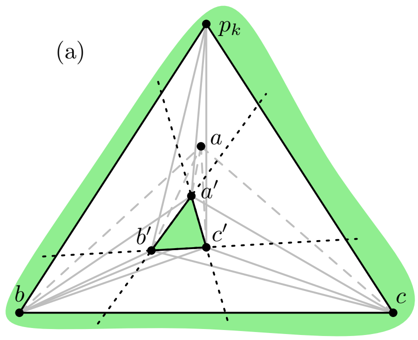

Fig. 6. If a vertex of an independent triangle is in a compact 4-cycle (e.g., $\langle p_k, a', c', c \rangle$), then Z_k if of Type 3 (a). Otherwise, Z_k is of Type 2 with top vertex c (b).

Z_{k-1}. Second, suppose that p_k is outside, say, T_1, and that $\{p_k, a, b, c\}$ forms a compact 4-cycle C. (Hence, p_k is inside T_2; recall that "inside" and "outside" is defined by the cycle's orientation.) Then $Z_k = C$ is of Type 1. Third, suppose that p_k is outside T_1 but $\{p_k, a, b, c\}$ does not form a compact 4-cycle. W.l.o.g., suppose further that a is inside the triangle $\langle p_k, b, c \rangle$. There are two subcases (see Fig. 6): (a) if a lies inside a compact 4-cycle, we replace a by p_k in T_1 to obtain an independent 3-cycle that, together with T_2, defines Z_k, again of Type 3; (b) otherwise, a is an element of a compact 4-cycle C that involves p_k, one vertex of T_2 and one other vertex of T_1. Then, Z_k is a Type 2 hull structure with compact cycle C whose top vertex is the vertex of T_1 that is not an element of C. The top triangles incident to the vertex of T_2 are marked dirty, the remaining top triangles are marked unexamined.

Lemma 2.10. *The resulting structure Z_k is a valid hull structure for P_k.*

Observation 2.11. *A Type 3 phase with ℓ insertions takes $O(\ell)$ time. If the next phase (if any) is of Type 2, it begins with a hull structure with at most 5 vertices, if it is of Type 1, it begins with a hull structure of size 4.*

To wrap up, we get the following lemma:

Lemma 2.12. *The final hull structure Z_n contains all candidate edges for R, and it can be obtained in $O(n)$ time.*

2.2 Obtaining the Actual Hulls from a Hull Structure

After having obtained Z_n, it remains to identify the faces that are important sets. If Z_n is of Type 1, then it is the only important set of R. If this is not the case, we want to obtain all the important triangles of R, i.e., all convex hulls of abstract order types realizing the radial system.

Lemma 2.13. *Given a Type 2 hull structure, we can decide in linear time which top triangles are important triangles of R.*

Lemma 2.14. *For a Type 3 hull structure, we can decide in linear time which of the two independent triangles are important triangles of R.*

For each important set we obtained for the radial system R, its chirotope is now given by Lemma 2.3.

Theorem 2.15. *Given a radial system R of an abstract order type, we can answer queries to the chirotopes of $T(R)$ in constant time, after $O(n)$ preprocessing time.*

Recall that we assumed that there is at least one realization of R. We can now check this assumption in the following way. We build the dual pseudo-line arrangement using an arbitrary chirotope we obtained for R using Lemma 2.3. This whole process takes $O(n^2)$ time [3, 4]. If it fails then R has no realization. Otherwise, the dual pseudo-line arrangement explicitly gives the rotation system of the corresponding abstract order type, which we now compare to R.

Corollary 2.16. *Testing whether a set of radial orderings is the radial system of an abstract order type can be done in $O(n^2)$ time.*

We can apply our insights to obtain all important sets of a given chirotope.

Theorem 2.17. *Given an abstract order type, a hull structure of its radial system can be found in $O(n \log n)$ time. Further, the faces in the hull structure that can become convex hulls can be reported in the same time.*

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