An Optimal Algorithm for Reconstructing Point Set Order Types from Radial Orderings

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August 12, 2016

Abstract

Let P be a set of n labeled points in the plane. The *radial system* of P describes, for each $p \in P$, the order in which a ray that rotates around p encounters the points in $P \setminus \{p\}$. This notion is related to the *order type* of P, which describes the orientation (clockwise or counterclockwise) of every ordered triple in P. Given only the order type, the radial system is uniquely determined and can easily be obtained. The converse, however, is not true. Indeed, let R be the radial system of P, and let $T(R)$ be the set of all order types with radial system R (we define $T(R) = \emptyset$ for the case that R is not a valid radial system). Aichholzer et al. (*Reconstructing Point Set Order Types from Radial Orderings*, in Proc. ISAAC 2014) show that $T(R)$ may contain up to $n-1$ order types. They also provide polynomial-time algorithms to compute $T(R)$ when only R is given.

We describe a new algorithm for finding $T(R)$. The algorithm constructs the convex hulls of all possible point sets with the radial system R. After that, orientation queries on point triples can be answered in constant time. A representation of this set of convex hulls can be found in $O(n)$ queries to the radial system, using $O(n)$ additional processing time. This is optimal. Our results also generalize to *abstract order types*.

1 Introduction

Let $P = \{p_1, \dots, p_n\}$ be a set of n labeled points in the plane, where the point p_i is given the label i. The *chirotope* $\chi : [n]^3 \to \{-1, +1, 0\}$ of P is a function that assigns to each triple $(i, j, k) \in [n]^3$ the orientation $\chi(i, j, k)$ of the corresponding point triple $(p_i, p_j, p_k) \in P^3$ (clockwise (-1), counterclockwise $(+1)$, or collinear (0)). If the elements of (i, j, k) are not pairwise distinct, then we set $\chi(i, j, k) = 0$. Throughout this paper, we assume that P is in *general position*, i.e., its chirotope χ has $\chi(i, j, k) \neq 0$, for all $(i, j, k) \in [n]^3$ with pairwise distinct elements.

Let P and P' be two sets of n labeled points in the plane, and let χ and χ' be their chirotopes. We say that χ and χ' are *equivalent* if either $\chi(i, j, k) = \chi'(i, j, k)$, for all $(i, j, k) \in [n]^3$, or $\chi(i, j, k) = -\chi'(i, j, k)$, for all $(i, j, k) \in [n]^3$. This defines an equivalence relation on the chirotopes. An equivalence class in this relation is called *order type*. Many problems on planar point sets do not depend on the exact coordinates of the points but only on their order type. Examples include computing the convex hull and determining whether two segments with endpoints in the point set intersect. As far as algorithms are concerned, it often turns out that access to the order type suffices in order to obtain efficient results. For example, Knuth[13] shows that the convex hull of a point set can be computed in $O(n \log n)$ time, even if one can only access its order type.[1]

Given a function $\chi : [n]^3 \to \{-1, +1, 0\}$, it is a hard problem to determine whether χ is a chirotope for a labeled planar point set.[2] To get around this difficulty, one uses the notion of *abstract order types*. Recall that an arrangement of *pseudo-lines* in the plane is a set of x-monotone planar curves such that each pair of curves intersects in exactly one point and such that this intersection is crossing. A *generalized configuration of points* consists of a labeled point set P in the plane and an arrangement of pseudo-lines such that each pseudo-line contains exactly two points from P and such that each pair of points from P lies on a pseudo-line.[11] Now we can define a chirotope on P as follows: if a point $p_k \in P$ is to the left of the pseudo-line through $p_i, p_j \in P$, directed from p_i to p_j, then the triple (i, j, k) is oriented *counterclockwise*. Otherwise, it is oriented *clockwise*. An equivalence class of chirotopes obtained in this way is called an *abstract order type*. Abstract order types can be characterized by a simple set of axioms.[13] For most combinatorial purposes, generalized configurations of points behave like point sets: their convex hull is the intersection of those halfspaces bounded by the pseudolines that contain all the points, and it determines a cycle of directed arcs. The chirotope determines whether two arcs defined by pairs of points cross. We refer to the work of Goodman and Pollack (see, e.g., their work on semispaces of configurations[12]) and to a book by Knuth[13] (who calls abstract order types "CC systems") for more details.

In this paper, we will be solely concerned with abstract order types. We stress that, as opposed to many other publications on the subject, we consider *labeled* abstract order types (and do not consider chirotopes equivalent if they can be obtained by a permutation of their arguments). In the following, we will not distinguish between an abstract order type and a chirotope that represents it.

Radial systems. We now define the main notion studied in this paper. Let $P = \{p_1, \dots, p_n\}$ be a generalized configuration of points, and let χ be the abstract order type of P. The *counterclockwise radial system* of χ, denoted R_χ, assigns to each $i \in [n]$ the cyclic permutation $R_\chi(i)$ of $[n] \setminus \{i\}$ that is given by the labels of the points in $P \setminus \{p_i\}$ in counterclockwise order around p_i. We call each $R_\chi(i)$ a *counterclockwise radial ordering*. If χ is realizable as a point set, then $R_\chi(i)$ equals the order of point labels found by sweeping a ray around p_i in counterclockwise direction. Given a function U that assigns to each $i \in [n]$ a cyclic permutation $U(i)$ of $[n] \setminus \{i\}$, we write $U \sim R_\chi$ if, for all $i \in [n]$, it holds that $U(i)$ equals $R_\chi(i)$ or the reverse of $R_\chi(i)$. Thus, the relation $\sim$ "forgets" the clockwise/counterclockwise direction of each individual $R_\chi(i)$. We call an equivalence class under $\sim$ an *undirected radial system*. When we say *radial system*, we always mean counterclockwise radial system. Radial systems were studied systematically by Aichholzer et al.[1] Before we describe their results, let us first review some related notions that have appeared in the literature.

Related work. Variants of the notion of radial systems have been studied in many contexts. First and foremost, there is the concept of *local sequences*. Whereas our radial orderings are obtained by sweeping a ray around each point, local sequences are obtained by sweeping a line.

[1]Actually, Knuth considers the generalized setting of abstract order types (to be defined later); many algorithms, as, e.g., Graham's scan, can also be slightly adapted to work by accessing only order type information.

[2]To be precise, this problem is complete for the existential theory of the reals $\exists\mathbb{R}$.[16]

More precisely, let P be a finite point set in the plane. For a point $p \in P$, the *local sequence of ordered switches* of p is the cyclic sequence in which the points of P are encountered when rotating a directed line through p. Additionally, we record whether a point appears before or after p on the directed line. Without this additional information, we get the *local sequence of unordered switches.* Goodman and Pollack[12] show that both concepts determine the order type of P, and thus carry the same information. Wismath[22] describes a method to reconstruct a point set (up to vertical translation and scaling) from its local sequences of ordered switches if, in addition, the x-coordinates of the points and the local sequences of directed switches are given. He also mentions that the radial system does not always determine the order type. Felsner and Weil[8] (Theorem 8) and Streinu[20] independently obtain a necessary and sufficient condition for sequences to be local sequences of unordered switches of an abstract order type. This condition allows for testing their realizability in polynomial time.

Another variation on radial systems was studied by Tovar, Freda, and LaValle[21] in the context of a robot that can sense landmarks around it. Disser et al.[6] and Chen and Wang[5] consider the *polygon reconstruction problem from angles*, where the objective is to reconstruct a polygon when given, for each vertex v, the angles with the other vertices of the polygon visible from v. Pilz and Welzl[18] describe a hierarchy on order types based on crossing edges; two order types are equivalent in their partial order if and only if they have the same radial system. We refer to the work by Aichholzer et al.[1] for a more complete list of related work.

Good drawings. Radial systems are also closely related to *good drawings.* Let G be a graph. A *drawing* of G is a representation of G with vertices as distinct points in the plane or on the sphere and edges as Jordan arcs whose endpoints are the corresponding vertices. It is usually assumed that no edges pass through vertices and two edges intersect only in a finite number of points. A *good drawing* (sometimes also called a *simple topological graph*) of G is a drawing of G in the plane or on the sphere where each vertex is represented by a distinct point, and each edge is represented by a Jordan arc between its two vertices; any two such arcs intersect in at most one point, which is either a common endpoint or a proper crossing. Two drawings on the sphere are *isomorphic* if they are equivalent under a homeomorphism of the sphere. We consider two drawings in the plane isomorphic if they are isomorphic after a stereographic projection to the sphere.

The *rotation* of a vertex v in a drawing is the cyclic order of the edges incident to v. The *rotation system* of a drawing is the set of the rotations of its vertices. Clearly, good drawings are a generalization of geometric graphs. The radial system of a point set P is equivalent to the rotation system of the complete geometric graph on P. A generalized configuration of points Q defines a good drawing of K_n where the vertices are embedded on the points of Q and every edge is a segment of a pseudo-line in Q.

A good drawing is *pseudo-linear* if its edges can be simultaneously extended to obtain a pseudo-line arrangement, or if it is isomorphic to such a drawing. The good drawings obtained from generalized configurations of points are exactly the pseudo-linear drawings (up to isomorphism). See Figure 1 for examples. The radial system of Q is equivalent to the rotation system of this good drawing. In a good drawing of K_n, the rotation system determines which edges cross. Therefore, it fixes the drawing up to the ordering of the crossings; in particular, we can see whether two edges cross by locally inspecting the rotations for the four vertices involved.[14] Later, we will use good drawings as an important tool in our reconstruction algorithm.

It is well-known that not every rotation system can be realized by a good drawing of the corresponding graph. Kynčl[15] showed that a rotation system of K_n is the rotation system of a good drawing if and only if this is true for every 5-vertex subset. He approaches the problem from the aspect of *abstract topological graphs*, where a graph is given together with a list of crossing edge pairs. For abstract topological graphs of K_n, he shows that from every 6-vertex subset one can obtain the unique rotation system of the corresponding good drawing, if it exists. For non-complete abstract topological graphs, the realizability problem is NP-complete.[14] Deciding whether there is a good drawing of a non-complete graph with a given rotation system seems to be an open problem. (There, the rotation system no longer determines the set of crossing edge pairs.)

Some good drawings are isomorphic to drawings where each edge is an x-monotone curve (after a projection to the plane). Such drawings are called *monotone*. Clearly, all pseudo-linear drawings are monotone. (Using Lemma 3.2, it is an easy exercise to provide an example showing that the converse is not true; Kynčl[14] provides all five non-isomorphic good drawings of K_5, of which only three are pseudo-linear.) Balko, Fulek, and Kynčl[3] characterize monotone good drawings of K_n, and Aichholzer et al.[2] provide an $O(n^5)$ time algorithm for deciding whether a given rotation system is the one of a monotone good drawing of K_n. For non-complete graphs, no similar results are known.

In terms of rotation systems of good drawings, our algorithm solves the problem for pseudo-linear drawings of K_n, that is, whether a given rotation system is the one of a pseudo-linear drawing of K_n. We are not aware of any related results in connection with non-complete graphs. Note that our problem is not concerned with finding a good drawing of a given rotation system, but with, in these terms, deciding whether the rotation system is the one of a pseudo-linear drawing, and determining the edges of all possible unbounded cells in all possible pseudo-linear drawings.

Also note that for any good drawing, the vertex triples can be oriented by defining the unbounded cell. However, this must not be confused with the order type, as only point sets have an order type. Finally, let us recall that there are good drawings of K_n that have the same rotation system, but are non-isomorphic to each other. In particular, even though the set of crossing edge pairs is determined, as well as the direction in which an edge crosses another one (the *extended rotation system*, the order in which an edge is crossed by other edges is, in general, not fixed). (This is not even the case for geometric graphs; e.g., slightly perturbing the vertices of an almost-regular hexagon influences the order in which its diagonals cross.) A detailed discussion of this can also be found in Kynčl[14]. However, due to a result by Gioan[10], if a drawing of K_n is pseudo-linear, then all good drawings with the same rotation system are pseudo-linear as well.

Properties of Radial Systems. Before we can describe our results, we provide a quick overview of the previous work. Aichholzer et al.[1] investigated under which circumstances the undirected radial system U of a generalized configuration of points P uniquely determines the abstract order type χ. They show that if P has a convex hull with at least four points, then U uniquely determines χ. In the following, let U be an undirected radial system that originates from an abstract order type, and let $T(U)$ be the set of abstract order types with undirected radial system U.

Theorem 1.1 (Theorem 1 and 2 in Aichholzer et al.[1])**.** *Let $n \geq 5$ and consider an abstract order type χ on $[n]$. Let U be the undirected radial system of χ, and let $H \subseteq [n]$ be the elements of the convex hull of χ. Then, we can compute $|H|$ from U in polynomial time. Furthermore,*

(i) if $|H| \neq 3$, then $T(U) = \{\chi\}$, and we can compute χ from U in polynomial time; and

(ii) if $|H| = 3$, then $|T(U)| \leq n-1$; all elements of $T(U)$ have a convex hull with exactly three elements; and we can compute $T(U)$ from U in polynomial time.

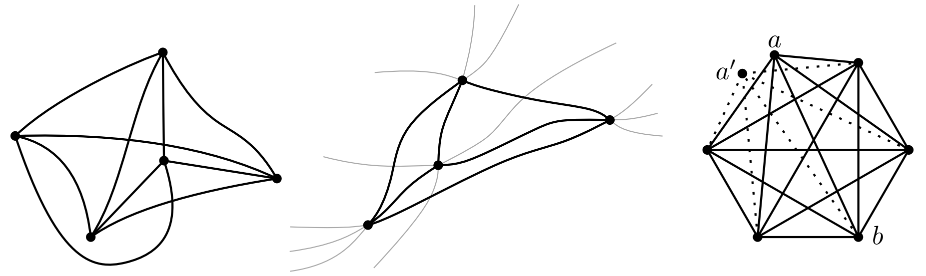

Figure 1: Left: A good drawing of K_5. It is monotone since there is a homeomorphism of the plane s.t. all edges are x-monotone. Middle: A generalized configuration of points with gray pseudo-lines. The induced pseudo-linear good drawing is shown by the black edges. Right: A geometric drawing of K_6 and a perturbation of vertex a to vertex a', that shows that, even though the original and the perturbed drawing have the same rotation system (i.e., the underlying point set has the same radial system), and the directions in which two edges cross each other match, the order in which other edges cross ab is different from the one of $a'b$ (giving two non-isomorphic labeled drawings).

In the full version of their paper[1], Aichholzer et al. show that (i) can be implemented in $O(n^3)$ time. Furthermore, they show that there exist counterclockwise radial systems R with $|T(R)| = n - 1$. Hence, it is not possible to improve the bound on $|T(U)|$ in (ii), even if we consider counterclockwise radial systems instead of undirected radial systems.[1]

Although U does not always uniquely determine χ, the pair (U, H), where H is the set of elements on the convex hull of χ, always suffices.[1] Thus, the abstract order types in $T(U)$ all have different convex hulls. Given an undirected radial system U on $[n]$, we say that a subset $H \subseteq [n]$ is *important* if H is the convex hull of some abstract order type in $T(U)$. An *important triangle* is an important set of size 3. Important sets are interrelated as follows.

Theorem 1.2 (Propositions 1–4 in Aichholzer et al.[1])**.** *Let $n \geq 5$ and consider a radial system R on $[n]$. If R has more than two important triangles, then all important triangles must have one element $i^* \in [n]$ in common. Thus, combining with Theorem 1.1, we can conclude that exactly one of the following cases applies:*

(1) There is an important set of size at least four, which is the only important set.

(2) There are between 1 and $n - 1$ important sets. All important sets are triangles, and if there is more than one important set, there is an element $i^* \in [n]$ that is contained in all of them.

(3) There are exactly two important sets, they are triangles, and they are disjoint.

For cases (2) and (3), there is actually a complete characterization of the important triangles. For an abstract order type $\chi \in T(U)$, an *inner* important triangle of χ is an important triangle of U that is not equal to the convex hull of χ. The following lemma reformulates the fact that an inner important triangle is not contained in a convex quadrilateral (see Figure 2).

Lemma 1.3 (Aichholzer et al.[1], Pilz and Welzl[18])**.** *Let P be a generalized configuration of n points, and let χ be the abstract order type of P. A triple $(a, b, c) \in [n]^3$ is an inner important triangle iff the following conditions hold.*

(1) The triangle $p_a p_b p_c$ is empty of points of P.

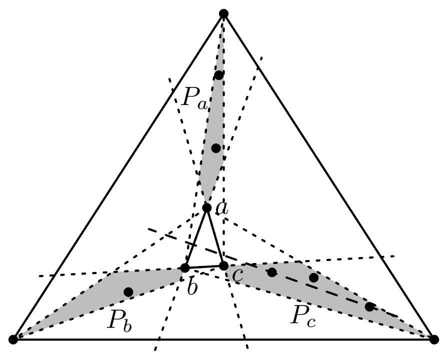

Figure 2: An inner important triangle $\langle a, b, c\rangle$ partitions the point set into three subsets, in which each pair of points intersects the opposite edge.

(2) The triangle $p_a p_b p_c$ partitions $P \setminus \{p_a, p_b, p_c\}$ into three subsets P_a, P_b, and P_c, such that P_a is to the left of the directed line $p_b p_a$ and to the right of the directed line $p_c p_a$, and similarly for P_b and P_c.

(3) For any two points $v, w \in P_a$, the pseudo-line vw intersects the edge $p_b p_c$; and similarly for P_b and P_c.

In this context, we mention that, if R is the radial system of some point set order type, then every abstract order type with radial system R can be realized as a point set (see Theorem 27 in Pilz and Welzl[18]). We do not deal with the realizability of abstract order types as point sets in this work. In the following, by a *realization of a radial system R*, we mean an abstract order type whose radial system is R.

Our results. We assume that the radial system is given in a data structure that lets us obtain the relative order of three elements in a radial ordering in constant time. We call such a query a *triple test*. For example, a triple test can be carried out in $O(1)$ time if we store not only the radial ordering, but also the rank of each element within some linear order defined by the radial ordering around each vertex. (If this structure is not provided, it can be obtained in $\Theta(n^2)$ time.)

For a given undirected radial system U on n elements (which has size $\Theta(n^2)$), we provide an algorithm to direct the n radial orderings in a consistent manner in $O(n)$ triple tests and $O(n)$ additional time (Theorem 2.3).

Our main algorithm identifies the convex hulls of all abstract order types consistent with a given radial system R in $O(n)$ time (provided that R is the radial system of an abstract order type). This set allows for constant-time queries to each chirotope in $T(R)$. Throughout the paper, when we speak of the "convex hull" of a set of vertices, we mean a combinatorial representation as the cyclic permutation of the vertices that appear on the convex hull, in this order.

Theorem 1.4. *Given a radial system R of an abstract order type, we can find in $O(n)$ triple tests and $O(n)$ additional processing a data structure that represents the convex hulls of all chirotopes in $T(R)$. With this data structure, we can answer queries to the chirotopes of $T(R)$ in constant time.*

Hence, this is a means of reporting an explicit representation of $T(R)$ in $O(n)$ time, significantly improving Theorem 1.1. We remark that we can show that $\Omega(n)$ triple tests are necessary, as an adversary can use any unconsidered point in a suitable example to alter $|T(R)|$

(e.g., by using it to "destroy" a top triangle as defined in Section 3.2, see Proposition 3.16). In this sense, our result is optimal.

We assume that the input consists of the permutations of an actual radial system. If this is not the case, our algorithm might fail, because it operates under an assumption that is not satisfied, or it may compute a structure that represents all chirotopes that are consistent with the triple tests performed by the algorithm. If we do not know that the set of permutations provided as input is indeed the radial system of an abstract order type, we show how to verify this in $O(n^2)$ time. For some input R, we define $T(R) = \emptyset$ if R is not a valid radial system. A straight-forward adversary argument shows that $\Omega(n^2)$ triple tests (i.e., reading practically the whole input) is necessary to verify whether $T(R) = \emptyset$. (The adversary can exchange two unread elements in the radial ordering around a point, cf. Proposition 3.17.)

Finally, when considering the algorithmic complexity of determining the realizability of a radial system, the question arises whether there are constant-size non-realizable subsets in any non-realizable radial system. If this were the case, one could check for realizability by examining the induced radial systems up to a certain constant size. Unfortunately, this is not the case. In Section 4, we show the following result.

Theorem 1.5. *For any $k \geq 3$, there exists a radial system R_k over $n = 2k + 1$ elements that is not realizable as an abstract order type, but that becomes realizable as a point set order type when removing any point.*

2 Directing Undirected Radial Systems

Let U be an undirected radial system on n elements. We show how to obtain a counterclockwise radial system R from U in $O(n)$ triple queries. Thus, for the remaining sections, we can assume that any radial system is oriented counterclockwise. We use two simple observations for 4-sets of points of a set with counterclockwise radial system R. A *swap* for an element i inverts the radial ordering $R(i)$ (resulting in a new radial system). Note that when restricting $R(i)$ to a 4-set containing i, since there are only two ways in which three vertices can be ordered around a fourth one, a swap corresponds to changing the order of two (arbitrary) neighboring elements in $R(i)$ restricted to this 4-set. We use two crucial observations.

Observation 2.1. *For a 4-set $\{i, j, k, l\}$, the counterclockwise radial ordering $R(l)$ is uniquely determined by the counterclockwise radial orderings $R(i)$, $R(j)$, and $R(k)$.*

Observation 2.2. *For a 4-set $\{i, j, k, l\}$, consider two counterclockwise radial systems R_A and R_B that are realized by abstract order types. Then the counterclockwise radial orderings of $R_A(i)$, $R_A(j)$, $R_A(k)$, and $R_A(l)$ differ from the counterclockwise radial orderings of $R_B(i)$, $R_B(j)$, $R_B(k)$, and $R_B(l)$ by an even number of swaps.*

Theorem 2.3. *Let $n \geq 5$, let χ be an abstract order type on n elements, and let $U \sim R_\chi$. Then U uniquely determines R_χ (up to complete reversal), and we can compute R_χ from U in $O(n)$ triple queries and $O(n)$ total time.*

Proof. We choose the direction of $R(1)$ arbitrarily. Now, there are four possible choices for the directions of $R(2)$ and $R(3)$. For each choice, we consider the resulting induced counterclockwise radial system on $\{1, \ldots, 5\}$, and we check whether it is realizable as an abstract order type. This can be done in constant time, as there is only a constant number of abstract order types on five elements. If no choice yields a realizable counterclockwise radial system, U cannot be realized, and we stop.

Next, we argue that at most one choice can lead to a realizable counterclockwise radial system. For the sake of contradiction, suppose that two different choices for $R(2)$ and $R(3)$ lead to realizable counterclockwise radial systems R_A and R_B on $\{1,\ldots,5\}$. Let us assume first that only $R(2)$ is inverted in R_B. Then, by applying Observation 2.2 on $\{1,2,3,4\}$ and on $\{1,2,3,5\}$, we see that also $R(4)$ and $R(5)$ are inverted. But then for the 4-set $\{1,2,4,5\}$, we have three swaps between R_A and R_B, a contradiction to Observation 2.2. The same argument rules out that only $R(3)$ is inverted, so assume that both $R(2)$ and $R(3)$ are inverted. Then, again by Observation 2.2, the directions for $R(4)$ and $R(5)$ remain unchanged. So, for the 4-set $\{1,2,4,5\}$, we have only one swap between R_A and R_B, which is a contradiction.

Since the directions of $\{1,\ldots,5\}$ are now fixed, we use Observation 2.1 to fix the direction of all $i=6,\ldots,n$ by considering the 4-set $\{1,2,3,i\}$. Thus, we conclude that if U is realized by an abstract order type, we can obtain the unique counterclockwise radial system R in $O(n)$ time. □

Note that Theorem 2.3 actually holds for good drawings and not only for radial systems of abstract order types.

3 Obtaining Chirotopes from Radial Systems

Let R be a given system of permutations. Our goal is to obtain the set $T(R)$ of abstract order types that realize R, if R represents a valid radial system. Otherwise, $T(R)$ is empty. Our algorithm (conceptually) constructs a good drawing of a plane graph on the sphere by adding vertices successively while maintaining the faces that are candidates for the convex hull. We will see that this actually boils down to maintaining at most two sequences of vertices plus one special vertex. Throughout, we assume that the radial orderings indeed correspond to the radial system of an abstract order type. If any of our assumptions does not hold, we know that there is no abstract order type for the given set of radial orderings. If R can be realized as an abstract order type, then the plane graph is the subdrawing of a drawing weakly isomorphic (cf. Kynčl[14]) to the complete graph on any generalized configuration of points that realizes that abstract order type.

Let $C=\langle c_0,\ldots,c_{m-1}\rangle$ be a plane cycle with m vertices contained in a good drawing Γ of the complete graph that realizes a radial system R. (We think of C as counterclockwise with the *interior* to its left.) Let c_i be a vertex of C, and let v be a vertex not in $\{c_{i-1},c_i,c_{i+1}\}$.[3] We say that the edge c_iv *emanates to the outside* of C if v lies between c_{i-1} and c_{i+1} in $R(c_i)$ in counterclockwise order. Otherwise, c_iv *emanates to the inside*. Let w be a vertex that does not belong to C. If cw emanates to the outside for all $c\in C$, then w *covers* C. If cw emanates to the inside for all $c\in C$, then w *lies inside* C; otherwise, w *lies outside* C. If w neither is inside C nor covers C, then Γ restricted to C plus all edges from vertices of C to w is not plane. A cycle $C=\langle c_0,\ldots,c_{m-1}\rangle, m\geq 4$, in Γ is *compact* if C is plane and, for each c_i, the edges $c_ic_{i+2}, c_ic_{i+3},\ldots,c_ic_{i-2}$ all emanate to the inside.

Observation 3.1. *Let R be a radial system and let C be a compact cycle in R. Let P be a generalized configuration of points realizing R. Then the vertices of C are in convex position in P.*

Lemma 3.2. *Consider a radial system R, and let Γ be a good drawing of the complete graph whose rotation system corresponds to R. Let S be an important set of R. Then, the vertices of S define a cell in Γ, and no edge of Γ crosses an edge of this cell. Furthermore, no element of S lies inside a compact cycle in Γ.*

[3] We consider all indices modulo the length of the corresponding sequence.

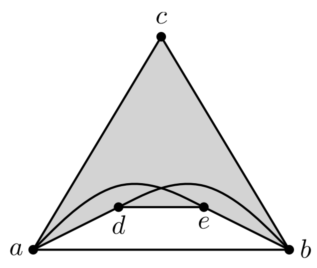

Figure 3: Illustration of the proof of Lemma 3.2 where $H_S = \langle a, b, c \rangle$ and $C = \langle a, d, e, b \rangle$. The shaded region is to the inside of both H_S and C. (Note the direction of the cycles by which, e.g., c is to the inside of C.) After removing c, the crossing will be on the unbounded face.

Proof. Let P be a generalized configuration of points that realizes R and whose convex hull H_S is a cycle with vertex set S. Let G be the embedding of the complete graph on n vertices that is obtained by taking the pseudoline arrangement for P and keeping only the parts of the pseudolines between the vertices. Two edges in the embedding G cross if and only if they cross in Γ. In particular, H_S is not crossed in Γ, and there is no edge in Γ that emanates to the outside of H_S.

Now, suppose Γ contains a compact cycle C such that some vertices of H_S are in the interior of C (C and H_S may share some vertices). Figure 3 shows an example where C and H_S share two vertices. Recall that we defined the "interior" of C via its direction, so a point inside C does not need to be drawn such that it is separated from the unbounded cell by C. We remove all vertices not in S or C from G. Note that the curves forming the diagonals of C cross in a bounded cell of the resulting embedded graph that is in the interior of both H_S and C, as all elements of the graph are in the interior of H_S and the crossings are in the interior of C by assumption. By removing some vertices from H_S that are not vertices of C, the cell outside of H_S (the unbounded face of G) grows. Eventually, the boundary of this cell contains a crossing p from the interior of C, as p is not separated from the growing cell by C (i.e., H_S is inside C). This means that the crossing is on the convex hull boundary of a subset of P containing C, a contradiction to Observation 3.1 and the assumption that C is a compact cycle in Γ.

□

Lemma 3.2 is closely related to Lemma 1.3 (see also Theorem 3.2 in Balko et al.[3]).

3.1 Abstract Order Types Determined by Convex Hull Edges

Consider a radial system R, and let χ be an abstract order type that realizes R. Let ab be a directed edge on the convex hull of χ so that all other points of χ are to the left of ab. It is easy to see that the edge ab and R together uniquely determine the convex hull of our abstract order type. Hence, there is only one abstract order type realizing R where ab has this property. Wismath's approach[22] obtains a point set from its local sequences and its x-coordinates. However, it can be observed that an abstract order type of a radial system is already determined by the relative horizontal order of the points, and the actual values of the x-coordinates are irrelevant (given that the abstract order type is the order type of a point set).[4] When thinking of a as a point at vertical infinity, the radial ordering around a gives a horizontal order of the remaining points. We re-state the following well-known fact.

[4]Wismath actually constructs a point set. Without being given the x-coordinates, deciding whether there exists such a point set would be equivalent to deciding stretchability of a pseudo-line arrangement[12], an $\exists\mathbb{R}$-complete problem[17] (see also Schaefer[19]). But recall that this work is concerned only with abstract order types.

Lemma 3.3. *Given a radial system R and a directed convex hull edge ab of an abstract order type χ realizing R, we can compute for each triple $(i, j, k) \in [n]^3$ the orientation $\chi(i, j, k)$ in $O(1)$ triple queries and hence in constant time.*

Proof. We know that every point except for a and b is to the left of ab. Let $\langle v_2 = b, \dots, v_n \rangle$ be the linear order obtained from $R(a)$, starting with b. If a is involved in a sidedness query, this order already determines the orientation of a triple. Otherwise, let v_i, v_j, v_k be a triple of points, with $i < j < k$. Then, if v_j is contained in the triangle av_iv_k, the triple (v_i, v_j, v_k) is oriented clockwise; otherwise, it is oriented counterclockwise. This can be checked with $O(1)$ triple queries to R. In the radial ordering around v_i, this corresponds to v_j being between v_k and a, or v_k being between v_j and a. □

3.2 Obtaining Hull Edges

Let P be a generalized configuration of n points, and let R be the radial system of the abstract order type χ of P. The goal is to find a set of $O(n)$ *candidate edges* that may appear on the convex hull of a realization of R (i.e., the edges of the convex hull of P, if there is no other realization of R, or the union of the edges of all important triangles). Our algorithm incrementally builds a "hull structure" (defined below) for P. Before step k, we have a current set $P_{k-1} \subseteq P$ of $k - 1$ points and a hull structure Z_{k-1} that represents the candidate edges for P_{k-1}. The algorithm selects a point $p_k \in P \setminus P_{k-1}$, adds it to P_{k-1}, and updates Z_{k-1}. A careful choice of p_k allows for updates in constant amortized time.

We begin with the description of the hull structure. Let $P_k \subseteq P$ be a set of k points ($k \geq 4$). The kth *hull structure* Z_k is an abstract representation of a graph with vertex set $V_k \subseteq P_k$ that is embedded on the sphere. That is, Z_k stores the incidences between the vertices, edges, and faces, but it does not assign coordinates to the points. Hull structures come in three types (see Figure 4), which correspond in one-to-one-fashion to the three possible configurations of important sets in Theorem 1.2:

Type 1: Z_k is a compact cycle (recall that, therefore, R restricted to V_k represents a convex $|V_k|$-gon with $|V_k| \geq 4$).

Type 2: Z_k consists of a compact cycle C and a *top vertex* t that covers C. The 3-cycles incident to t are called *top triangles*. A top triangle τ is marked either *unexamined*, *dirty*, or *empty*. Initially, τ is unexamined. Later, τ is marked either dirty or empty. "Dirty" indicates that τ cannot contain a convex hull vertex in its interior. "Empty" means that τ is a candidate for an important triangle. We orient each top triangle so that all other vertices of Z_k are to the exterior.

Type 3: Z_k is the union of two vertex-disjoint 3-cycles T_1 and T_2, called *independent triangles*. T_1 and T_2 are directed so that each has all of P_k to the interior. Moreover, the edges between the vertices of T_1 and T_2 appear as in Figure 4.

Let R_k be the restriction of R to P_k. We maintain the following invariant: (a) if R_k has exactly one important set of size at least four, Z_k is of Type 1 and represents the counterclockwise convex hull boundary; (b) if R_k has two disjoint important triangles, Z_k is of Type 3, and the important triangles are exactly the independent triangles; (c) if R_k has several important triangles with a common vertex, Z_k is of Type 2 and all important triangles appear as top triangles; (d) if R_k has exactly one important triangle, Z_k is of Type 2 or 3, with the important triangle as a top triangle (Type 2) or as an independent triangle (Type 3). Furthermore, if Z_k is of Type 2, no convex hull vertex for P lies inside a dirty triangle, and each point of P_k lies either in C or in a dirty triangle.

Initially, we pick 5 arbitrary points from P. Among those, there must be a compact 4-cycle Z_4 (see, e.g., Figure 4 in Aichholzer et al.[1]), which can be found in constant time. Our initial

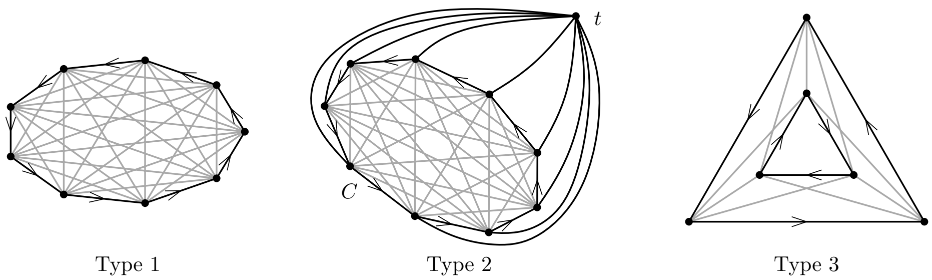

Figure 4: The three different types of hull structures.

hull structure Z_4 is of Type 1, with vertex set $V_4 = P_4$. We next describe the insertion step for each possible type. For the running time analysis, we subdivide the algorithm into *phases*. Each phase is of Type 1, 2, or 3, and a new phase begins each time the type of the hull structure changes.

3.2.1 Type 1

We pick an arbitrary vertex c of Z_{k-1}, and we inspect $R(c)$ to determine in constant time whether c has an incident edge emanating to the outside of Z_{k-1}. If not, the edges incident to c in Z_{k-1} are on the convex hull of P, and we are done; see below. Otherwise, let $p_k \in P \setminus P_{k-1}$ be the endpoint of such an edge. We set $P_k = P_{k-1} \cup \{p_k\}$, and we walk along Z_{k-1} (starting at c) to find the interval I of vertices for which the edge to p_k emanates to the outside (this can be checked in $O(1)$ triple queries). There are two cases: (i) if $I = Z_{k-1}$ (i.e., p_k covers Z_{k-1}), then Z_k is the hull structure of Type 2 with compact cycle Z_{k-1}, top vertex p_k, and all top triangles marked unexamined; (ii) if $I = \langle c_i, \dots, c_j \rangle$ is a proper subinterval of Z_{k-1}, the next hull structure Z_k is of Type 1 with vertex sequence $\langle p_k, c_j, \dots, c_i \rangle$ (since R is realizable, we have $c_j \neq c_i$).

Lemma 3.4. *We either obtain an edge from which the convex hull of P can be determined uniquely, or Z_k is a valid hull structure for P_k.*

Proof. By the invariant, Z_{k-1} is the convex hull of P_{k-1}. If c has no incident edges emanating to the outside, the edges e and f incident to c in Z_{k-1} lie on the convex hull of P, as e and f lie on the convex hull of P_{k-1} and all points of P lie in the wedge spanned by them. Furthermore, P_{k-1}, and hence P, has at least one point outside the triangle spanned by e and f, so P has a convex hull with at least four vertices. Then the convex hull of P is unique, by Theorem 1.2.

Otherwise, if p_k does not cover Z_{k-1}, the lemma follows from simple geometry. If p_k covers Z_{k-1}, then p_k must lie on every convex hull of P_k. Moreover, a candidate edge of P_k is either a candidate edge of P_{k-1} or connects p_k to an extreme point of P_{k-1}. Thus, all important sets of P_k are top triangles of Z_k. □

Lemma 3.5. *A Type 1 phase that begins with a hull structure of size m and lasts for ℓ insertions takes $O(m + \ell)$ time. Furthermore, the next phase (if any) is of Type 2, beginning with a hull structure of size at most $m + \ell$.*

Proof. Once a vertex is deleted from the hull structure, it does not reappear, so the total number of distinct vertices is at most $m + \ell$. The time to insert non-covering vertices can be charged to the deleted vertices. It takes $O(m + \ell + m)$ time to identify a covering vertex, but then the phase is over. □

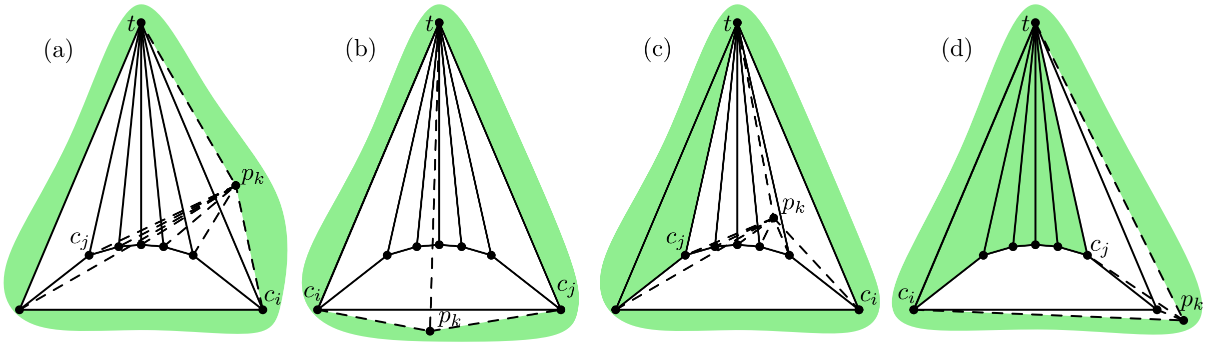

Figure 5: Z_{k-1} is of Type 2 and p_k is not covering: if p_k forms a non-crossed 4-cycle, Z_k is of Type 1 (a, b); if not, Z_k is of Type 2 with p_k on the compact cycle (c, d). The algorithm will later discover that the triangle $\langle t, c_{j+1}, c_j \rangle$ in (c) is not important since it is inside a convex quadrilateral.

3.2.2 Type 2

We begin with a simple observation.

Observation 3.6. *Let Z_{k-1} be a Type 2 hull structure with compact cycle C and top vertex t. The vertices of C appear in their circular order in the clockwise radial ordering around t.*

We need to identify a suitable vertex p_k to insert. For this, we select an unexamined top triangle $\tau = \langle t, c_{i+1}, c_i \rangle$, and we test whether c_i has an incident edge that emanates to the inside of τ. If yes, let $v \in P \setminus P_{k-1}$ be an endpoint of such an edge and check whether $c_i v$ crosses the edge tc_{i+1}. If so, the vertices of τ lie inside a convex quadrilateral, and by Lemma 3.2 there is no convex hull vertex inside τ. We mark τ dirty and proceed to the next unexamined triangle. If not, we set $p_k = v$ and $P_k = P_{k-1} \cup \{p_k\}$. If c_i has no incident edge emanating to the inside of τ, we perform the analogous steps on c_{i+1}. If c_{i+1} also has no such incident edge, we mark τ empty and proceed to the next unexamined triangle. (The empty triangle τ might still be crossed by an edge incident to t.)

Lemma 3.7. *We either find a new vertex p_k, or all candidate edges for P lie in Z_{k-1}. Furthermore, no dirty triangle contains a possible convex hull vertex of P.*

Proof. All top triangles that are newly marked dirty are part of a compact 4-cycle, so no dirty top triangle can contain a possible convex hull vertex of P in its interior. Suppose we fail to find a vertex p_k and there is a candidate edge e for P not in Z_{k-1}. Then e must have one endpoint $v \in P \setminus P_{k-1}$, since otherwise e would be a candidate edge for P_{k-1} and part of Z_{k-1}, by the invariant. Now, v cannot lie in C or in a dirty triangle, by the invariant. Also, v cannot lie in an empty triangle, since this would have been detected by the algorithm. Thus, v must hide in an unexamined triangle, but there are no such triangles left. □

With p_k at hand, we inspect the boundary of C to find the interval I of vertices for which the edge to p_k emanates to the outside of C. First, if p_k does not cover C, i.e., if $I = \langle c_i, \dots, c_j \rangle$ is a proper subinterval of C, then p_k must lie between c_{i-1} and c_{j+1} in the clockwise order around t, as in any realization one of the cases in Figure 5 applies. If p_k is between c_{i-1} and c_i or between c_j and c_{j+1}, then either $\langle p_k, t, c_{i-1}, c_i \rangle$ or $\langle t, p_k, c_j, c_{j+1} \rangle$ is a compact 4-cycle containing P_k, and we make it the next hull structure Z_k of Type 1; see Figure 5(a). The green areas in the figures are the only regions where we might still find candidate edges. Otherwise, if $i + 1 = j$ and the edge tp_k crosses $c_i c_{i+1}$, the compact 4-cycle $\langle t, c_j, p_k, c_i \rangle$ contains P_k and becomes the

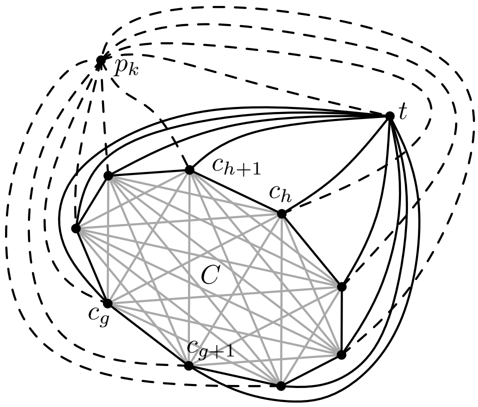

Figure 6: Z_{k-1} is of Type 2 and p_k is covering.

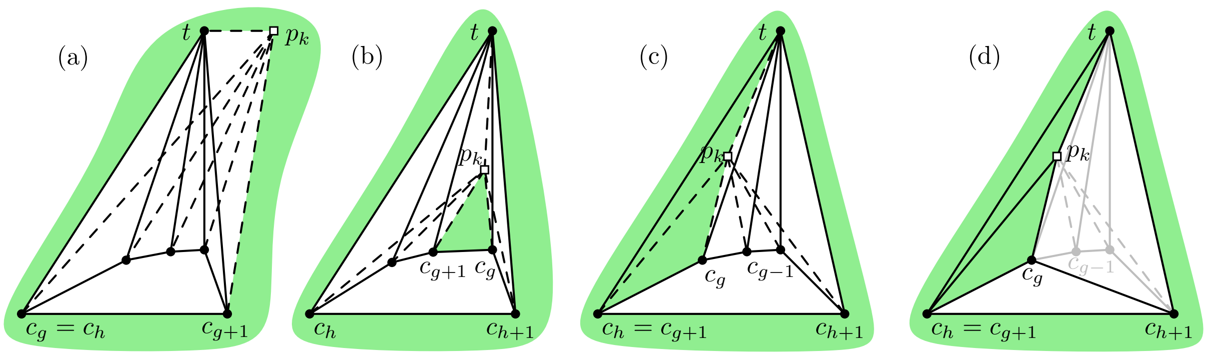

Figure 7: Z_{k-1} is of Type 2 and p_k (box) is covering: if t and p_k are between the same vertices in each other's rotation, Z_k is of Type 1 (a); if these vertices are disjoint, Z_k is of Type 3 (b); if t and p_k have a common neighbor c_j in the other's rotation (c), the new top vertex c_h of Z_k structure requires the construction of a new compact cycle (d).

next Type 1 hull structure Z_k; see Figure 5(b). In any other case (i.e., p_k lies between c_i and c_j in clockwise order around t and if $i+1=j$ then tp_k does not cross c_ic_{i+1}), Z_k is of Type 2 and obtained from Z_{k-1} by removing the top triangles between c_i and c_j and adding the top triangles $\langle t, p_k, c_i\rangle$ and $\langle t, c_j, p_k\rangle$; see Figure 5(c) and (d). If c_ip_k intersects an edge of Z_{k-1}, then $\langle t, p_k, c_i\rangle$ lies in a compact 4-cycle and is marked dirty. Otherwise, it is marked unexamined. We handle $\langle t, c_j, p_k\rangle$ similarly.

Second, suppose p_k covers C and let g, h be so that p_k is between c_g and c_{g+1} in clockwise order around t and t lies between c_h and c_{h+1} in clockwise order around p_k. Observation 3.6 ensures that these edges are well-defined; see Figure 6. Now there are three cases. First, if $g=h$, then one of $\langle c_g, c_{g+1}, t, p_k\rangle$ or $\langle c_g, c_{g+1}, p_k, t\rangle$ defines a compact 4-cycle containing P_k, so Z_k is of Type 1 and consists of this cycle; see Figure 7(a). Second, if $\{g, g+1\} \cap \{h, h+1\} = \emptyset$, then Z_k is of Type 3, with independent triangles $\langle p_k, c_g, c_{g+1}\rangle$ and $\langle t, c_h, c_{h+1}\rangle$; see Figure 7(b). Third, suppose that $h = g+1$ or $g = h+1$, say, $h = g+1$. Then Z_k is of Type 2, with top vertex c_j and compact cycle $\langle t, p_k, c_g, c_{h+1}\rangle$. The top triangle $\langle c_h, c_{h+1}, c_g\rangle$ is dirty, the other top triangles are unexamined; see Figure 7(c–d).

Lemma 3.8. *The resulting hull structure is valid for P_k.*

Proof. First suppose the vertices of C whose edges to p_k emanate to the outside form a proper subinterval $\langle c_i, \ldots, c_j\rangle$ of C. Suppose further that p_k appears between c_ℓ and $c_{\ell+1}$ in the clockwise order around t and consider some realizing abstract order type χ_k for P_k. In the induced realization χ_{k-1} with P_{k-1}, the top triangle $\tau = \langle t, c_{\ell+1}, c_\ell \rangle$ of Z_{k-1} is either the outer face or a bounded triangle.

If τ is the outer face, suppose first that the set $\{t, c_{\ell+1}, c_\ell, p_k\}$ forms a compact 4-cycle C'. Since τ is the outer face of P_{k-1}, we know that C' is the outer face of P_k. Since $|C'| = 4$, it is the only important set by Theorem 1.2 and we can set $Z_k = C'$. Equivalently, our algorithm decides this case as follows. If $C' = \langle t, p_k, c_\ell, c_{\ell+1} \rangle$, we must have $\ell = j$. If $C' = \langle t, c_\ell, c_{\ell+1}, p_k \rangle$, we must have $\ell = i - 1$. If $C' = \langle t, c_\ell, p_k, c_{\ell+1} \rangle$, we must have $i = \ell$, $j = \ell + 1$ and tp_k crossing $c_i c_{i+1}$. In either case, C' contains P_k and constitutes the unique convex hull of P_k. We can thus set $Z_k = C'$. Now, if $\{t, c_{\ell+1}, c_\ell, p_k\}$ does not form a compact 4-cycle, the interior vertex must be c_ℓ or $c_{\ell+1}$ (as p_k is on the outer face by choice of ℓ and not covering). Thus, $\ell \in \{i, j-1\}$, and the edges of Z_{k-1} incident to $c_{i+1}, \dots, c_{j-1}$ cannot be candidates (being inside a compact 4-cycle), while the only candidate edges incident to p_k are $p_k t$, $p_k c_i$ and $p_k c_j$.

Similarly, if τ is bounded, we must have $i \leq \ell < j$, and p_k must form a compact cycle with $\langle c_j, \dots, c_i \rangle$. The edges of Z_{k-1} incident to $c_{i+1}, \dots, c_{j-1}$ are in a compact 4-cycle. The only possible candidate edges incident to p_k are $p_k t$, $p_k c_i$, or $p_k c_j$ (the other such edges are crossed by edges of Z_{k-1}). Thus, in the last two cases Z_k is a valid Type 2 structure, and our algorithm covers all cases.

If p_k covers C, our algorithm distinguishes all possible rotations around t and p_k. Recall that p_k is inside the 3-cycle $\langle t, c_{g+1}, c_g \rangle$, and t is inside $\langle p_k, c_{h+1}, c_h \rangle$.

(a) If $g = h$, then $\{p_k, t, c_g, c_{g+1}\}$ is in convex position: if there were a realization with p_k or t in the interior, the rotations at p_k and t would be different; if c_g or c_{g+1} was in the interior, there would be a third vertex on C showing that p_k or t does not cover C. Furthermore, if the edge $p_k t$ crossed $c_g c_{g+1}$, the rotations at t and p_k would be different. Hence, either $\langle c_g, c_{g+1}, t, p_k \rangle$ or $\langle c_g, c_{g+1}, p_k, t \rangle$ makes a compact 4-cycle, containing all of P_k.

(b) If the two 3-cycles are disjoint, then the edge $p_k c_h$ crosses the *fan* $t \to \langle c_{g+1}, \dots, c_{h-1} \rangle$ (i.e., all edges from t to $c_{g+1}, \dots, c_{h-1}$), and $p_k c_{h+1}$ crosses the fan $t \to \langle c_{h+2}, \dots, c_g \rangle$. Furthermore, tc_g crosses $p_k \to \langle c_{h+1}, \dots, c_{g-1} \rangle$, and tc_{g+1} crosses $p_k \to \langle c_{g+2}, \dots, c_h \rangle$. Hence, $\langle t, c_h, c_{h+1} \rangle$ and $\langle p_k, c_g, c_{g+1} \rangle$, are the only cells without crossed edges and the only candidates for the convex hull of P_k. Thus, Z_k is a valid Type 3 structure for P_k.

(c) If, w.l.o.g., $h = g + 1$, then $\langle t, p_k, c_g, c_{h+1} \rangle$ forms a compact 4-cycle, so no vertex inside it can be extremal. Similarly, $\langle c_g, c_h, c_{h+1}, c_{g-1} \rangle$ contains the triangle $\langle c_g, c_h, c_{h+1} \rangle$, which is rightfully marked dirty. The possible candidate edges for P_k are the uncrossed edges of Z_{k-1} or the uncrossed edges between p_k and Z_{k-1}, and they are all included in Z_k. □

Lemma 3.9. *A Type 2 phase that begins with a hull structure of size m and lasts for ℓ insertions takes $O(m + \ell)$ time. Furthermore, if the next phase (if any) is of Type 1, it begins with a hull structure of size at most* 4.

Proof. The second claim follows by inspection. For the first claim, note that each top triangle is marked dirty or empty at most once, and that each insertion creates a constant number of new top triangles. The total running time for the insertion operations can be charged to marked and removed top triangles. □

3.2.3 Type 3

Let $T_1 = \langle a, b, c \rangle$ and $T_2 = \langle a', c', b' \rangle$ be the two independent triangles of Z_{k-1}, labeled such that the edges aa', bb' and cc' are uncrossed in the subdrawing with these six vertices. Further, let p_k be an arbitrary vertex of $P \setminus P_{k-1}$. We set $P_k = P_{k-1} \cup \{p_k\}$, and we distinguish three cases. First, if p_k is inside both T_1 and T_2, then $Z_k = Z_{k-1}$. Second, suppose that p_k is outside, say,

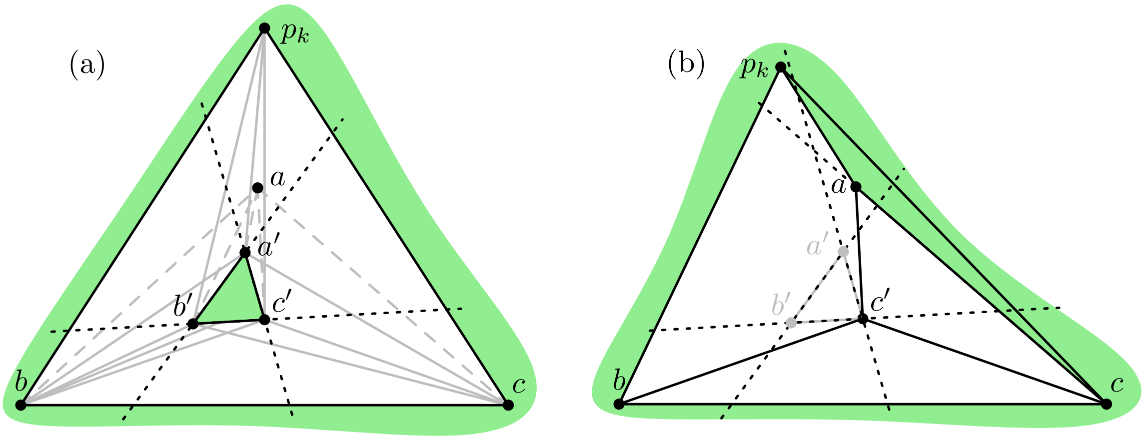

Figure 8: If a vertex of an independent triangle is in a compact 4-cycle (e.g., $\langle p_k, a', c', c\rangle$), then Z_k if of Type 3 (a). Otherwise, Z_k is of Type 2 with top vertex c (b).

T_1, and that $\{p_k, a, b, c\}$ forms a compact 4-cycle C. (Hence, p_k is inside T_2; recall that "inside" and "outside" is defined by the cycle's orientation.) Then $Z_k = C$ is of Type 1. Third, suppose that p_k is outside T_1 but $\{p_k, a, b, c\}$ does not form a compact 4-cycle. W.l.o.g., suppose further that a is inside the triangle $\langle p_k, b, c\rangle$. There are two subcases (see Figure 8): (a) if a lies inside a compact 4-cycle, we replace a by p_k in T_1 to obtain an independent 3-cycle that, together with T_2, defines Z_k, again of Type 3; (b) otherwise, a is an element of a compact 4-cycle C that involves p_k, one vertex of T_2 and one other vertex of T_1. Then, Z_k is a Type 2 hull structure with compact cycle C whose top vertex is the vertex of T_1 that is not an element of C. The top triangles incident to the vertex of T_2 are marked dirty, the remaining top triangles are marked unexamined.

Lemma 3.10. *The resulting structure Z_k is a valid hull structure for P_k.*

Proof. By the invariant, T_1 and T_2 are the only possible important triangles for P_{k-1}. Suppose that T_1 and T_2 are labeled such that $\langle a, b, b', a'\rangle$ is a compact 4-cycle. If p_k is inside both independent triangles, then p_k lies inside a compact 4-cycle, and cannot have an incident candidate edge. If p_k forces four extreme points, P_k lies in the corresponding compact 4-cycle, and we are done. Otherwise, let p_k be outside, say, $T_1 = \langle a, b, c\rangle$. Any new candidate edge for P_k must be incident to p_k. Furthermore, no new candidate edge is incident to T_2, since such an edge would intersect T_1. Thus, we have to consider the potential convex hulls formed by p_k and the vertices of T_1.

Suppose first that, w.l.o.g., a is contained in a compact 4-cycle; see Figure 8(a). This cycle must be $\langle p_k, c, c', a'\rangle$ or $\langle p_k, b, b', a'\rangle$. The only possible important triangle incident to p_k is $T_1' = \langle p_k, b, c\rangle$, and the 3-cycle $\langle p_k, b', c'\rangle$ contains a', so T_1' and T_2 form the independent 3-cycles of a Type 3 hull structure. There cannot be another important triangle, as such a triangle would have to contain a candidate edge from Z_{k-1} and thus would be incident to a.

If there is no vertex of T_1 contained in a compact 4-cycle, then an edge from p_k to a vertex of T_2 crosses an edge of T_2, say $a'b'$; see Figure 8(b). In this case, $\langle p_k, a, c', b\rangle$ forms a compact 4-cycle C, and all points from P_k lie either in C or in the triangles $\langle c, a, c'\rangle$ or $\langle c, c', b\rangle$. The latter two triangles are contained in the compact 4-cycles $\langle c, a, a', c'\rangle$ and $\langle c, c', b', b\rangle$, and hence cannot be important. Thus, all possible candidate edges are represented in Z_k. $\square$

Observation 3.11. *A Type 3 phase with ℓ insertions takes $O(\ell)$ time. If the next phase (if any) is of Type 2, it begins with a hull structure with at most 5 vertices, if it is of Type 1, it* *begins with a hull structure of size* 4.

3.2.4 Correctness and Running Time

To wrap up, we get the following lemma:

Lemma 3.12. *The final hull structure Z_n contains all candidate edges for R, and it can be obtained in $O(n)$ time.*

Proof. Correctness follows from Lemmas 3.4, 3.8, and 3.10, which show that the invariant is maintained throughout the construction. By Lemmas 3.5 and 3.9, the total time for a phase of Type 1 and Type 2 is proportional to the number of insertion operations plus the initial size of the hull structure. By Observation 3.11, the total time for a Type-3-phase is proportional to the number of insertion operations. By Lemma 3.9 and Observation 3.11, every phase of Type 1 begins with a hull structure of constant size. By Observation 3.11, every phase of Type 2 that follows a phase of Type 3 has constant size. By Lemma 3.5 and the fact that every phase of Type 1 begins with a structure of constant size, the size of the hull structure at the beginning of a phase of Type 2 that follows a phase of Type 1 can be charged to the number of insertions in that Type-1-phase. The total number of insertions is n (as the invariant ensures that in a hull structure of Type 1 or 2, every point of $P_k \setminus V_k$ is in a compact cycle or in a dirty triangle). □

3.3 Obtaining the Actual Hulls from a Hull Structure

After having obtained Z_n, it remains to identify the faces that are important sets. If Z_n is of Type 1, then it is the only important set of R. If this is not the case, we want to obtain all the important triangles of R, i.e., all convex hulls of abstract order types realizing the radial system.

Lemma 3.13. *Given a Type 2 hull structure, we can decide in linear time which top triangles are important triangles of R.*

Proof. While we would only need to check top triangles that are not dirty, we do not use this fact in the proof. Again, due to Lemma 1.3, the important triangles are exactly the top triangles that are not contained in the interior of a compact 4-cycle. Since t is an extreme point, any such compact 4-cycle contains t. Let $D = \langle t, p, q, r\rangle$ be such a compact 4-cycle, containing a top triangle $\tau = \langle t, c_{i+1}, c_i\rangle$. Note that, in any abstract order type of the radial system, D is in convex position and contains τ. The case where the edge c_ic_{i+1} is crossed by some edge ts is evident from the radial ordering around t. But even if this is not the case, observe that there is a convex quadrilateral D' containing τ that has either c_i or c_{i+1} as a vertex. W.l.o.g., let $D' = \langle t, p, q, c_i\rangle$.

We first claim that if D' exists, then there is a convex quadrilateral $\tilde{D} = \langle t, \tilde{p}, \tilde{q}, c_i\rangle$ such that $\tilde{D}$ contains τ and such that $\tilde{p}$ and $\tilde{q}$ are consecutive in the radial ordering around t. Such a quadrilateral clearly exists when $p = c_{i+1}$ and tq crosses c_ic_{i+1}, so suppose this is not the case. If there is no vertex between p and q in the clockwise radial ordering around t, the claim is true with $\tilde{p} = p$ and $\tilde{q} = q$. Otherwise, let u be a vertex between p and q. Note that $u \neq c_{i+1}$. If u is in convex position with t, c_i, and q, we can replace p by u and obtain another quadrilateral containing τ. Otherwise, we replace q by u, also obtaining another convex quadrilateral containing τ. By this process, we eventually find $\tilde{D}$.

It now remains to show how to rule out all top triangles that are contained in a compact 4-cycle. For these, we know by the previous claim that we only need to check consecutive pairs in the radial ordering around t. Let $\langle p_1, \ldots, p_{n-1}\rangle$ be the radial ordering around t (with p_1

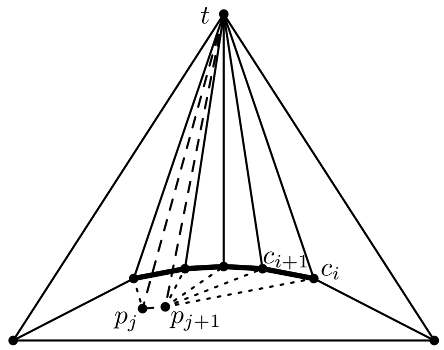

Figure 9: If there is a compact 4-cycle $\langle t, p_j, p_{j+1}, c_i\rangle$, then $c_i c_{i+1}$ is found by incrementally removing all edges of C that form a similar quadrilateral (marked bold) with p_j and p_{j+1} when starting from the edge that is intersected by the triangle $\langle t, p_j, p_{j+1}\rangle$. Note that this subset is realized in the same way in any realization of R as it is inside a compact cycle.

being an arbitrary point). The compact cycle C has exactly one edge that is intersected by the triangle $\langle t, p_1, p_2\rangle$. Let $c_j c_{j+1}$ be that edge. If $(c_j, c_{j+1}) = (p_2, p_1)$, then there is no vertex in $C \setminus \{c_j, c_{j+1}\}$ that forms a compact 4-cycle with $\{t, c_j, c_{j+1}\}$, and we continue. Otherwise, we remove $c_j c_{j+1}$ and look at its neighbors $c_{j-1}c_j$ and $c_{j+1}c_{j+2}$. If a neighboring edge forms a compact 4-cycle with p_1p_2, we remove it as well. We continue this process (i.e., checking the edges of C adjacent to the previously removed ones for convex position with p_1p_2) until no edge is removed. We then continue with the pair p_2p_3, iteratively removing edges from C that are adjacent to previously removed ones if they are in convex position with p_2p_3. We incrementally continue this process for all pairs $p_\ell p_{\ell+1}$ and claim that the edges from C we did not remove are exactly the ones that form an important triangle with t.

Suppose there exists an edge $c_i c_{i+1}$ of C that is contained in a quadrilateral formed with $p_j p_{j+1}$ but is not removed by this process. The triangle $\langle t, p_j, p_{j+1}\rangle$ intersects the cycle C (recall that all vertices not in $C \cup \{t\}$ lie inside C). As this edge is intersected, there is a quadrilateral in convex position containing that edge. Thus, there is a convex quadrilateral formed with all edges between the intersected one and $c_i c_{i+1}$, a contradiction (see Figure 9). Hence, we are left with exactly those edges that form an important triangle with t. □

Lemma 3.14. *For a Type 3 hull structure, we can decide in linear time which of the two independent triangles are important triangles of R.*

Proof. Let $T_1 = \langle a, b, c\rangle$ and $T_2 = \langle a', c', b'\rangle$ be the two independent triangles of the Type 3 hull structure. (Recall that they are labeled such that the edges aa', bb' and cc' are uncrossed in the subdrawing with these six vertices.) We first check whether there is a partition of the vertices not on T_1 or T_2 into sets P_a, P_b and P_c as required by Lemma 1.3 with T_2 on the convex hull. We do not verify part (3) in Lemma 1.3 yet; hence this step is easily done in linear time. If such a partition does not exist, there is some point p not within this partition and we know that exactly one of T_1 and T_2 is important. Hence, T_1 is contained in a compact 4-cycle Q if and only if T_2 is important.

Otherwise, suppose that such a partition does exist. We next check the second condition in Lemma 1.3. For, say, ab, let P_c be the points that are in the corresponding partition. We have to check whether there is a pair (v, w) in P_c that forms a compact 4-cycle with ab. We consider the elements in the clockwise radial ordering around a between c' and c, and compare it to the counterclockwise radial ordering around b, also between c' and c. We proceed analogously for the other edges of T_1. If each pair of these orders is consistent, we know that T_1 is an important triangle under the assumption that T_2 is the convex hull of P; if the assumption is not true, then T_1 has to be the convex hull of P anyway. So suppose the orders are conflicting for, w.l.o.g., ab. This means that there is a compact 4-cycle Q with ab as an edge. We know that Q has to be realized by four points in convex position. Hence, if T_1 is contained in Q, then T_2 is the important triangle. Otherwise, since Q separates T_1 from T_2, T_2 is contained in Q and T_1 is the important triangle. □

For each important set we obtained for the radial system R, its chirotope is now given by Lemma 3.3. This proves Theorem 1.4.

Recall that we assumed that there is at least one realization of R. We can now check this assumption in the following way. We build the dual pseudo-line arrangement using an arbitrary chirotope we obtained for R using Lemma 3.3. This whole process takes $O(n^2)$ time.[4, 7] If it fails then R has no realization. Otherwise, the dual pseudo-line arrangement explicitly gives the rotation system of the corresponding abstract order type, which we now compare to R.

Corollary 3.15. *Testing whether a set of radial orderings is the radial system of an abstract order type can be done in $O(n^2)$ time.*

We can give matching lower bounds for these subtly different settings.

Proposition 3.16. *Given a radial system R of an order type, we need $\Omega(n)$ queries to R in the worst case to determine $|T(R)|$.*

Proof. We show the lower bound by an adversary argument. Intuitively, an adversary can place any unconsidered point to "destroy" the hull structure defined by the points already queried by an algorithm. However, as we are also given the indices of each point in the rotation around another one, the adversary must not place a point in a way that alters these indices significantly. Thus, our proof uses the following setting. Consider $n-1$ points in convex position and a point t such that the hull structure of these n points is of Type 2 with t as top vertex. Let a, b, and c be three consecutive vertices in the rotation around t. If the adversary moves b over the edge ac, then only $R(a)$ and $R(c)$ will change: in $R(a)$, the elements b and c swap their position, and the analogous happens in $R(c)$. Also, note that, since the swapped elements are adjacent in the rotations, only the indices of these two vertices change. So the adversary has $n-1$ points that could be moved inside the compact cycle of the resulting hull structure, and for each of these $n-1$ points, an algorithm has to determine the position in the rotation around one of its neighbors. Hence, we need at least a linear number of queries. Finally, we remark that all such abstract order types are actually realizable. □

Proposition 3.17. *Given a radial system R, we need $\Omega(n^2)$ queries to the radial system to determine whether $T(R) \neq \emptyset$.*

Proof. Recall Observation 2.2. The adversary starts by presenting the radial system of an (arbitrary) abstract order type. If an algorithm does not inspect the relative order of any adjacent pair in the rotation around any point, the adversary can swap exactly these pairs. The resulting radial system cannot be the one of an abstract order type (and not even of a good drawing). The quadratic lower bound follows. □

Note that Proposition 3.17 applies to checking whether, for a set R of radial orderings, $T(R) \neq \emptyset$, while Proposition 3.16 applies to determine $|T(R)|$ under the assumption that $|T(R)| \geq 1$, in the same way as Theorem 1.4 provides the hull structure of R under the assumption that R is the radial system of an abstract order type, while Corollary 3.15 is for checking whether a set of radial orderings is the radial system of an abstract order type.

We can apply our insights to obtain all important sets of a given chirotope.

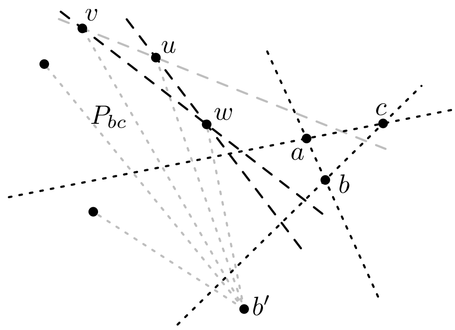

Figure 10: If two points v and w are in convex position with bc, then for any point b' in the wedge at b there is a consecutive pair for which this is also the case.

Theorem 3.18. *Given an abstract order type, a hull structure of its radial system can be found in $O(n \log n)$ time. Further, the faces in the hull structure that can become convex hulls can be reported in the same time.*

To show this theorem, we use the following lemma.

Lemma 3.19. *Let $\langle a, b, c\rangle$ be an empty triangle in an abstract order type χ on a set P. Let P_b be the set of points to the right of bc and to the right of ab, and let P_{bc} be the set of points to the left of bc. If there exist two points $v, w \in P_{bc}$ such that the line vw does not intersect the edge bc (i.e., the four points are in convex position), then, for any point $b' \in P_b \cup \{b\}$, there are two points $v', w' \in P_{bc}$ that are in convex position with bc and are consecutive in the radial ordering around b' (among the elements of P_{bc}).*

Proof. See Figure 10. Observe first that if there is a point of P_{bc} that is between a and b or c (if $b' = b$) in the radial ordering around b', then a and this point are in convex position with bc (recall that abc is empty). We consider the linear order given by the radial ordering of b' with a as the last element. W.l.o.g., let v precede w in that linear order. Let u be a point between v and w (if no such point exists, we are done). Suppose u and bc are on the same side of vw. Then the line (or pseudo-line) uw does not intersect the edge bc. Otherwise, if u and bc are on different sides of vw, then the line vu does not intersect the edge bc. Hence, this line also does not intersect bc and the two points are closer to each other in the linearized order around b'. □

In particular, note that if no two points of P_{bc} are in convex position with bc, then $P_{bc} = P_a \cup \{a\}$.

Proof of Theorem 3.18. For the given abstract order type on a set P, construct the convex hull $\mathrm{CH}(P)$ of P in $O(n \log n)$ time.[5] If it has more than three vertices, we are done. Otherwise, let $\langle a, b, c\rangle$ be the convex hull.

We first test for the case where there is another important triangle $\langle u, v, w\rangle$ that does not share a vertex with $\mathrm{CH}(P)$. For this we use Lemma 1.3. Radially sort the vertices around a, b, and c. Consider the clockwise order around a and the counterclockwise order around b, which can both be interpreted as linear orders starting with c. The last vertex where the prefixes of these two orders match is a vertex of $\langle u, v, w\rangle$, say, w for the following reason. Let p and q be the first mismatching pair (hence, $\mathrm{CH}(\{a, b, p, q\})$ is a quadrilateral). Suppose first that p and q either precede w in that order, or w is one of them. Then $\mathrm{CH}(\{a, b, p, q\})$ is a quadrilateral that contains w, a contradiction. Hence, suppose both p and q succeed w in that order, and

[5] Knuth[13] discusses how to obtain the convex hull of abstract order types in $O(n \log n)$ time. It is also straight-forward to adapt standard algorithms like Graham's scan.

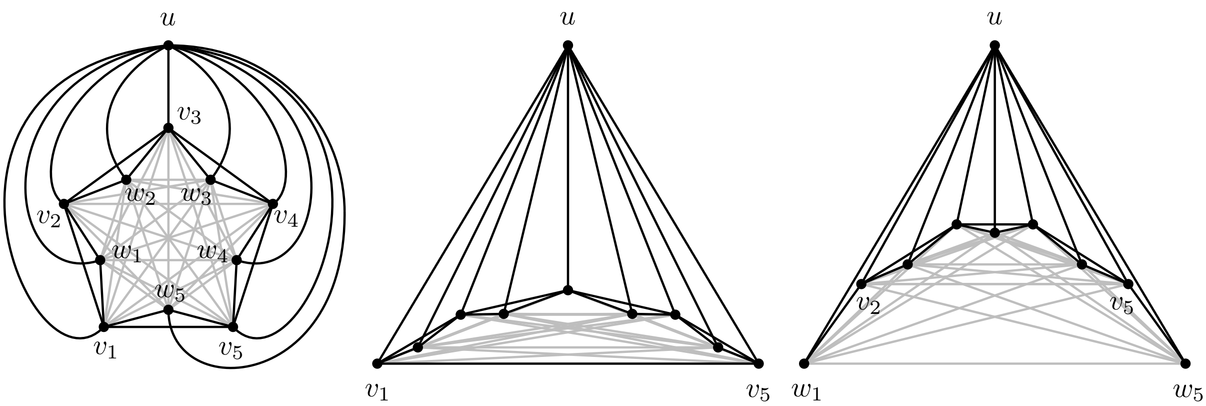

Figure 11: The construction of R_5 on the left, and point set order type realizations of two induced radial systems after removing either w_5 or v_1 on the right.

there is another point r succeeding w before one of p and q. Then $\mathrm{CH}(\{a, b, r\})$ has u and v in its interior, but does not contain w, leading again to a contradiction. The analogous holds for the other pairs of extreme points. If this method returns three points u, v and w, we can check whether the radial orderings around u, v, and w match the ones around the extreme points for the corresponding subsets defined in Lemma 1.3. We need $O(n)$ time for the partitioning, as by Lemma 3.19, we only have to check points that are adjacent in the radial orderings around a, b, and c. The check at u, v, and w also takes $O(n)$ time. If the outcome is positive, we have a valid hull structure of Type 3, and both independent triangles can become convex hulls.

Suppose there are important triangles that share a vertex. We guess the covering extreme point a. (If the guess is not correct, the following process has to be repeated at most twice for b and c.) The important triangles incident to a can be found in the following way. We obtain the radial ordering around a, as well as the convex hull of $P \setminus \{a\}$. This gives us a structure that is very similar (if not equivalent) to a Type 2 hull structure (the "compact cycle" may have only three vertices). We can apply Lemma 3.13 to obtain the important triangles for this set. □

4 Minimal non-realizable Radial Systems of Arbitrary Size

For any $k \geq 3$ we describe a radial system R_k over $n = 2k + 1$ vertices which is not realizable as an abstract order type, while every radial system induced by any strict subset of the vertices can be realized, even as a point set order type. This shows that realizability of radial systems cannot be decided by checking realizability of all induced radial systems up to any fixed constant size.

Theorem 4.1. *For any $k \geq 3$ there exists a radial system R_k over $n = 2k + 1$ vertices that is not realizable as an abstract order type, while every radial system induced by any strict subset of the vertices can be realized as a point set order type.*

Proof. Throughout, we refer to Figure 11, which illustrates the construction of R_5. We start with a so-called *double circle* with a total of $2k$ vertices. Imagine a regular k-gon with vertices $v_1, \ldots, v_k$, and k additional vertices $w_1, \ldots, w_k$ that are placed inside the k-gon and arbitrarily close to the midpoints of its k edges, where we place w_i next to the edge v_iv_{i+1}. The radial system for these $2k$ vertices is obtained by drawing all edges as straight lines and by observing the radial orderings of the edges around each vertex. We add one additional vertex u, which can be thought of being outside of the initial k-gon. More precisely, u comes directly between

v_{i-1} and v_{i+1} in the radial ordering around any vertex v_i, and it comes directly between v_i and v_{i+1} in the radial ordering around any vertex w_i.

First, observe that edges of type $v_i w_i$ and $w_i v_{i+1}$ cannot be boundary edges of the convex hull in any realization of R_k since they always will be in the interior of the k-gon $v_1, \ldots, v_k$. Second, all pairs of edges that cross in Figure 11 also cross in any other good drawing of R_k, and hence they also cannot be convex hull edges. (See the paragraph on good drawings in Section 1.) This leaves us only with edges of type uv_i as potential convex hull edges. However, since there is no cycle that contains only such edges, there is no viable candidate for the convex hull of R_k, which concludes the proof of the first part of the theorem.

We now show that any strict subset of the vertices induces a radial system that can be realized as a point set order type. We distinguish the following three cases. If we remove the vertex u then, by definition, we already have an appropriate straight-line drawing of the remaining vertices and edges. If we remove any vertex w_i, say w_k, then we can draw $v_1, \ldots, v_k$ as a convex k-gon in such a way that all edges except for $v_1 v_k$ face the vertex u, which means the remaining vertices $w_1, \ldots, w_{k-1}$ can be added easily. If we remove any vertex v_i, say v_1, then we reuse the drawing from the previous case for the $(k-1)$-gon $v_2, \ldots, v_k$ and the interior vertices $w_2, \ldots, w_{k-1}$. Observe that the two remaining vertices w_1 and w_k do not have to be placed inside this $(k-1)$-gon, and hence that it is simple to position them appropriately. □

The above proof also works for general good drawings of the complete graph.

5 Conclusion

Problem 5.1. *Can we reconstruct an order type of the vertices of a simple polygon when given only the radial orderings of visible vertices around each vertex (similar to Chen and Wang[5], but without angles)?*

This question is closely related to characterizing visibility graphs of simple polygons, which is still open. It is known that there are infinitely many minimal forbidden induced subgraphs of visibility graphs. See the book of Ghosh[9] and references therein.

Acknowledgments. This work was initiated during the *ComPoSe Workshop on Order Types and Rotation Systems* held in February 2015 in Strobl, Austria. We thank the participants for valuable discussions. We would also like to thank the anonymous reviewers for multiple suggestions that helped to improve the presentation in the paper.

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