Long Alternating Paths Exist

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Abstract

Let P be a set of $2n$ points in convex position, such that n points are colored red and n points are colored blue. A non-crossing alternating path on P of length ℓ is a sequence $p_1, \dots, p_\ell$ of ℓ points from P so that (i) all points are pairwise distinct; (ii) any two consecutive points p_i, p_{i+1} have different colors; and (iii) any two segments $p_i p_{i+1}$ and $p_j p_{j+1}$ have disjoint relative interiors, for $i \neq j$.

We show that there is an absolute constant $\varepsilon > 0$, independent of n and of the coloring, such that P always admits a non-crossing alternating path of length at least $(1+\varepsilon)n$. The result is obtained through a slightly stronger statement: there always exists a non-crossing bichromatic separated matching on at least $(1+\varepsilon)n$ points of P. This is a properly colored matching whose segments are pairwise disjoint and intersected by common line. For both versions, this is the first improvement of the easily obtained lower bound of n by an additive term linear in n. The best known published upper bounds are asymptotically of order $4n/3 + o(n)$.

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1 Introduction

We study a family of problems that were discovered independently in two different (but essentially equivalent) settings. Researchers in discrete and computational geometry found a geometric formulation, while researchers in computational biology and stringology studied circular words. Around 1989, Erdős asked the following geometric question [4, p. 409]: given a set P of n red and n blue points in convex position, how many points of P can always be collected by a non-intersecting polygonal path π with vertices in P such that the vertex-color along π alternates between red and blue. Taking every other segment of π, we obtain a properly colored set of pairwise disjoint segments with endpoints in P. A closely related

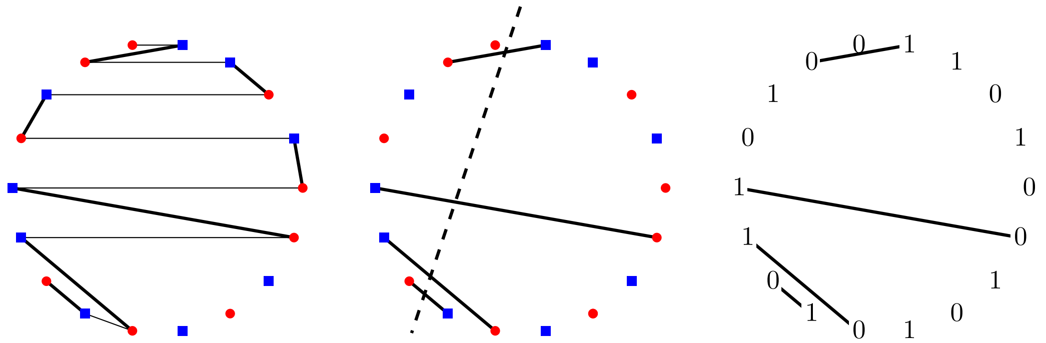

Figure 1 Left: a set P of 18 points in convex position, 9 of them red and 9 of them blue, with an alternating path of length 15 (this is not a longest such path). Taking every other segment, we obtain a properly colored disjoint matching on P. Middle: a separated matching on P, with a dashed line that intersects all matching edges (this is not a maximum such matching). Right: an antipalindromic subsequence on a circular word of 18 bits, 9 of them 0 and 9 of them 1.

problem asks for a large *separated matching*, a collection of such segments with the extra property that all of them are intersected by a common line. This is equivalent to finding a long *antipalindromic subsequence* in a circular sequence of $2n$ bits, where n bits are 0 and n bits are 1, see Figure 1. This formulation was stated in 1999 in a paper on protein folding [10]. Similar questions were also studied for *palindromic* subsequences [14]. One such question is equivalent to finding many disjoint monochromatic segments with endpoints in P, a problem that was also studied by the geometry community.

An easy lower bound for alternating paths is n, and the best known lower bound is $n + \Omega(\sqrt{n})$ [11]. We increase this to $cn + o(n)$, for a constant $c > 1$. Similarly, for the other mentioned problems, we improve the lower bounds by an additive term of εn, for some fixed $\varepsilon > 0$. Also here, this constitutes the first $\Omega(n)$ improvement over the trivial lower bounds.

The (geometric) setting. We have a set P of $2n$ points $p_0, p_1, \ldots, p_{2n-1}$ in convex position, numbered in clockwise order. The points in P are colored red and blue, so that there are exactly n red points and n blue points. The goal is to find a long *non-crossing alternating path* in P. That is, a sequence $\pi : q_0, q_1, \ldots, q_{\ell-1}$ of points in P such that (i) each point from P appears at most once in π; (ii) π is *alternating*, i.e., for $i = 0, \ldots, \ell - 2$, we have that q_i is red and q_{i+1} is blue or that q_i is blue and q_{i+1} is red; (iii) π is *non-crossing*, i.e., for $i, j \in \{0, \ldots, \ell - 2\}$, $i \neq j$, the two segments $q_i q_{i+1}$ and $q_j q_{j+1}$ intersect only in their endpoints and only if they are consecutive in π, see Figure 1(left). We will also just say *alternating path* for π. Alternating paths for planar point sets in general (not just convex) position have been studied in various previous papers, e.g., [1–3, 5, 6].

For most of this work, we will focus on another, closely related, structure. A *non-crossing separated bichromatic matching* M in P is a set $\{p_1 q_1, p_2 q_2, \ldots, p_k q_k\}$ of k pairs of points in P, such that (i) all points $p_1, \ldots, p_k, q_1, \ldots, q_k$ are pairwise distinct; (ii) the segments $p_i q_i$ and $p_j q_j$ are disjoint, for all $1 \leq i < j \leq k$; (iii) for $i = 1, \ldots, k$, the points p_i and q_i have different colors; and (iv) there exists a line that intersects all segments $p_1 q_1, p_2 q_2, \ldots, p_k q_k$, see Figure 1(middle). Often, we will just use the term *separated bichromatic matching* or simply *separated matching* for M.

Previous results. The following basic lemma says that a large separated matching immediately yields a long alternating path. The (very simple) proof was given by Kynčl, Pach, and

Tóth [9, Section 3].

▶ **Lemma 1.** *Suppose that a bichromatic convex point set P admits a separated matching with k segments. Then, P has an alternating path of length $2k$.*

Let $l(n)$ be the largest number such that for every set P of n red and n blue points in convex position, there is an alternating path of length at least $l(n)$. Around 1989, Erdős and others [9] conjectured that $\lim_{n\to\infty} l(n)/n = 3/2$. Abellanas, García, Hurtado, and Tejel [1] and, independently, Kynčl, Pach, and Tóth [9, Section 3] disproved this by showing the upper bound $l(n) \leq 4n/3 + O(\sqrt{n})$. Kynčl, Pach, and Tóth [9] also improved the (almost trivial) lower bound $l(n) \geq n$ to $l(n) \geq n + \Omega(\sqrt{n/\log n})$. They conjectured that in fact $l(n) = 4n/3 + o(n)$. In her PhD thesis [11] (see also [8, 12, 13]), Mészáros improved the lower bound to $l(n) \geq n + \Omega(\sqrt{n})$, and she described a wide class of configurations where every separated matching has at most $2n/3 + O(\sqrt{n})$ edges. This also implies the upper bound $l(n) \leq 4n/3 + O(\sqrt{n})$ mentioned above [1, 9]. It was announced to us in personal communication that E. Csóka, Z. Blázsik, Z. Király, and D. Lenger constructed configurations with an upper bound of $cn + o(n)$ on the size of the largest separated matching, where $c = 2 - \sqrt{2} \approx 0.5858$ [7].

Our results. We improve the almost trivial lower bound $n/2$ for separated matchings to $n/2 + \varepsilon n$.

▶ **Theorem 2.** *There is a fixed $\varepsilon > 0$ such that any convex point set P with n red and n blue points admits a separated matching with at least $n/2 + \varepsilon n$ edges.*

By Lemma 1, we obtain the following corollary about long alternating paths.

▶ **Theorem 3.** *There is a fixed $\varepsilon > 0$ such that any convex point set P with n red and n blue points admits an alternating path with at least $n + \varepsilon n$ vertices.*

A variant of Theorem 2 also holds for the *monochromatic* case. The definition of a *non-crossing separated monochromatic matching*, or simply *separated monochromatic matching*, is obtained from the definition of a separated bichromatic matching by changing condition (iii) to (iii') for $i = 1, \ldots, k$, the points p_i and q_i have the same color. Some of the upper bound constructions for separated bichromatic matchings apply to the monochromatic setting, also giving the upper bound $2n/3 + O(\sqrt{n})$. Here is a monochromatic version of Theorem 2. Due to space reasons, the proof of Theorem 4 has been omitted from this extened abstract. It can be found in the full version of this paper.

▶ **Theorem 4.** *There are constants $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such any convex point set P with $n \geq n_0$ points, colored red and blue, admits a separated monochromatic matching with at least $n/2 + \varepsilon n$ vertices.*

There are two differences between the statement of Theorem 2 and Theorem 4: we do not require that the number of red and blue points in P is equal (and hence the size of the matching is stated in terms of vertices instead of edges), and we need a lower bound on the size of P. This is necessary, because Theorem 4 does not always hold for, e.g., $n = 4$. It was announced to us in a personal communication that the construction of E. Csóka, Z. Blázsik, Z. Király and D. Lenger from above also gives the upper bound $cn + o(n)$ on the size of a largest separated monochromatic matching, where $c = 2 - \sqrt{2} \approx 0.5858$ [7].

Our results in the setting of finite words. As we already said, the problems in this paper were independently discovered by researchers in computational biology and stringology. In a study on protein folding algorithms, Lyngsø and Pedersen [10] formulated a conjecture that is equivalent to saying that the bound in Theorem 2 can be improved to $2n/3$ (for n divisible by 3). Müllner and Ryzhikov [14, p. 461] write that this conjecture "has drawn substantial attention from the combinatorics of words community". For the convenience of readers from this community, we rephrase our theorems for separated matchings in the finite words setting. We use the terminology of Müllner and Ryzhikov [14], without introducing it here. The following corresponds to Theorem 2.

▶ **Theorem 5.** *There is a fixed $\varepsilon > 0$ such that for any even $n \in \mathbb{N}$, every binary circular word of length n with equal number of zeros and ones has an antipalindromic subsequence of length at least $n/2 + \varepsilon n$.*

The following corresponds to Theorem 4.

▶ **Theorem 6.** *There are constants $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ so that for any $n \in \mathbb{N}$, $n \geq n_0$, every binary circular word of length n has a palindromic subsequence of length at least $n/2 + \varepsilon n$.*

2 Existence of large separated bichromatic matchings

In this section, we prove our main result: large separated bichromatic matchings exist.

2.1 Runs and separated matchings

A *run* of P is a maximal sequence $p_i, p_{i+1}, \ldots, p_{i+\ell}$ of consecutive points with the same color.[1] That is, for $j = i, \ldots, i + \ell - 1$, the color of p_j and of p_{j+1} are the same, and the colors of p_{i-1} and p_i and the colors of $p_{i+\ell}$ and $p_{i+\ell+1}$ are different. The number of runs is always even. Kynčl, Pach, and Tóth showed that if P contains t runs, then P admits an alternating path of length $n + \Omega(t)$ [9, Lemma 3.2]. We will need the following analogous result for separated matchings. The proof can be found in the full version.

▶ **Theorem 7.** *Let $c_1 = 1/32$ and $t \geq 4$. Let P be a bichromatic convex point set with $2n$ points, n red and n blue, and suppose that P has t runs. Then, P admits a separated matching with at least $n/2 + c_1 t^2/n$ edges.*

2.2 Chunks, partitions, and configurations

Let $k \in \{1, \ldots, n\}$. A *k-chunk* is a sequence of consecutive points in P with exactly k points of one color and less than k points of the other color. Hence, a k-chunk has at least k and at most $2k - 1$ points. A *clockwise k-chunk with starting point p_i* is the shortest k-chunk that starts from p_i in clockwise order. A *counterclockwise k-chunk* with starting point p_i is defined analogously, going in the counterclockwise direction. For a k-chunk C, we denote by $r(C)$ the number of red points and by $b(C)$ the number of blue points in C. We call C a *red chunk* if $r(C) = k$ (and hence $b(C) < k$) and a *blue chunk* if $b(C) = k$ (and hence $r(C) < k$). The *index* of C is $b(C)/k$ for a red chunk and $r(C)/k$ for a blue chunk. Thus, the index of C lies between 0 and $(k - 1)/k$, and it measures how "mixed" C is.

[1] When calculating with indices of points in P, we will always work modulo $2n$.

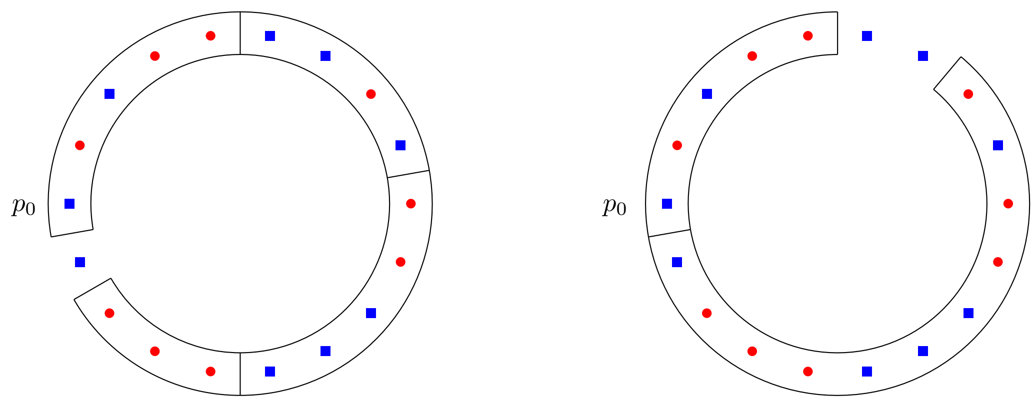

Figure 2 A set of 18 points and its $(3,0)$-partition (left) and $(3,1)$-partition (right). In the $(3,0)$-partition, the first chunk is red with index $2/3$, the second chunk is blue with index $1/3$, the third chunk is blue with index $2/3$, and the fourth chunk is red with index 0. The average red index is $1/3$, the average blue index is $1/2$. The index of the $(3,0)$-partition is $1/2$. The $(3,1)$-partition has one clockwise 3-chunk and one counterclockwise 6-chunk. The 3-chunk is red with index $2/3$, the 6-chunk is red with index $5/6$. The average red index is $3/4$, the average blue index is 0 The index of the $(3,1)$-partition is $3/4$.

Next, let $k \in \{1,\dots,n\}$ and $\lambda \in \mathbb{N} \cup \{0\}$. We define a (k,λ)-*partition*. Suppose that k is odd. First, we construct a maximum sequence $C_0, C_1, \dots$ of clockwise disjoint k-chunks, as follows: we begin with the clockwise k-chunk C_0 with starting point p_0, and we let ℓ_0 be the number of points in C_0. Next, we take the clockwise k-chunk C_1 with starting point p_{ℓ_0}, and let ℓ_1 be the number of points in C_1. After that, we take the clockwise k-chunk C_2 with starting point $p_{\ell_0+\ell_1}$, and so on. We stop once we reach the last k-chunk that does not overlap with C_0. Next, we construct a maximum sequence $D_0, D_1, \dots$ of *counterclockwise* $(k+3)$-chunks, starting with the point p_{2n-1}, in an analogous manner. Let λ' be the minimum of λ and the number of $(k+3)$-chunks D_i. Now, to obtain the (k,λ)-partition, we take λ' counterclockwise $(k+3)$-chunks $D_0, \dots, D_{\lambda'-1}$ and a maximum number of clockwise k-chunks $C_0, C_1, \dots$ that do not overlap with $D_0, \dots, D_{\lambda'-1}$. If k is even, the (k,λ)-partition is defined analogously, switching the roles of the clockwise and the counterclockwise direction. There may be some points that do not lie in any chunk of the (k,λ)-partition. We call these points *uncovered*.

The *average red index* of Γ is the average index in a red chunk of Γ (0, if there are no red chunks). The *average blue index* of Γ is defined analogously. The *index* of Γ is the maximum of the average red index and the average blue index of Γ. The *max-index color* is the color whose average index achieves the index of Γ, the other color is called the *min-index color*, see Figure 2 for an illustration of the concepts so far. The following simple proposition helps us bound the number of chunks. The (somewhat technical) proof can be found in the full version.

▶ **Proposition 8.** *Let P be a convex bichromatic point set with $2n$ points, n red and n blue, and let Γ be a (k,λ)-partition of P. In Γ, there are at most $2k-2$ uncovered points, at most $k-1$ of them red and at most $k-1$ of them blue. Furthermore, let R be the number of red chunks and B the number of blue chunks in Γ, and let α be the index of Γ. Then,*

$$R + B \le \frac{2n}{k} \quad \textit{and} \quad \max\{R,B\} \le \frac{n}{k}. \tag{1}$$

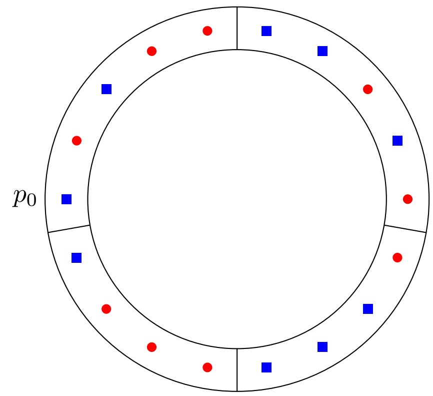

Figure 3 A set of 18 points and a 3-configuration for it. The chunk from p_0 is red with index 2/3, the next clockwise chunk is blue with index 2/3, followed by another blue chunk of index 1/3 and a final red chunk of index 1/3. The average blue index and the average red index are both 1/2. Note that the chunks are not minimal.

Furthermore, we have

$$R + B \geq \left\lfloor \frac{2n}{2k+5} \right\rfloor > \frac{2n}{7k} - 1, \quad \max\{R, B\} \geq \frac{1}{2} \left\lfloor \frac{2n}{2k+5} \right\rfloor > \frac{n}{7k} - \frac{1}{2},$$
$$\textit{and} \quad \min\{R, B\} \geq \frac{1-\alpha}{2} \left\lfloor \frac{2n}{2k+5} \right\rfloor - \frac{k-1}{k+3} > (1-\alpha)\frac{n}{7k} - 2. \quad (2)$$

If $\lambda = 0$, the lower bounds improve to

$$R + B \geq \left\lfloor \frac{2n}{2k-1} \right\rfloor \geq \frac{n}{k} - 1, \quad \max\{R, B\} \geq \frac{1}{2} \left\lfloor \frac{2n}{2k-1} \right\rfloor > \frac{n}{2k} - \frac{1}{2},$$
$$\textit{and} \quad \min\{R, B\} \geq \frac{1-\alpha}{2} \left\lfloor \frac{2n}{2k-1} \right\rfloor - \frac{k-1}{k} > (1-\alpha)\frac{n}{2k} - 2. \quad (3)$$

The purpose of the (k, λ)-partitions is to transition smoothly between the $(k, 0)$-partition and the $(k + 3, 0)$-partition. In our proof, this will enable us to gradually increase the chunk-sizes, while keeping the index under control.

A *k-configuration* of P is a partition of P into k-chunks, leaving no uncovered points, see Figure 3. In contrast to a (k, λ)-partition, the chunks in a k-configuration are not necessarily minimal. Note that while P always has a (k, λ)-partition, it does not necessarily admit a k-configuration. The average red index, the average blue index, etc. of a k-configuration are defined as for a (k, λ)-partition. The following proposition helps us bound the number of chunks in a k-configuration. The proof can be found in the full version.

▶ **Proposition 9.** *Let P be a convex bichromatic point set with $2n$ points, n red and n blue, and let Γ be a k-configuration of P. Let R be the number of red chunks, B the number of blue chunks,α the average red index and β the average blue index of Γ. Then,*

$$n = kR + \beta kB = kB + \alpha kR. \quad (4)$$

Furthermore, $R + B \geq n/k$, $\max\{R, B\} \geq n/2k$, and $\min\{R, B\} \geq (1 - \max\{\alpha, \beta\})n/2k$. Finally, $\max\{R, B\} = R$ if and only if $\alpha \geq \beta$.

In our proof, the key challenge will be to analyze k-configurations with small constant index (say, around 0.1).

2.3 From (k,λ)-partitions to k-configurations

Our first goal is to show that we can focus on (k,λ)-partitions with large k and constant, but not too large index. We begin by noting that if the $(k,0)$-partition of P for a constant k has a large index, then we can find a long alternating path in P. The proof can be found in the full version.

▶ **Lemma 10.** *Set $c_2 = 1/12800$. Let $k,n \in \mathbb{N}$ with $8k^2 \le n$. Let P be a convex bichromatic point set with $2n$ points, n red and n blue. If the $(k,0)$-partition Γ of P has index at least 0.1, then P admits a separated matching of size at least $(1/2 + c_2/k^4)n$.*

Next, we show that if the $(k,0)$-partition still has a small index for $k = \Omega(n)$, then we can find a large separated matching. The proof, which is inspired by a simlar argument of Kynčl, Pach, and Tóth [9, Lemma 3.1], can be found in the full version.

▶ **Lemma 11.** *Set $c_3 = 1/81$. Let $k,n \in \mathbb{N}$ with $k \le n$ and $6480n \le k^2$. Let P be a convex bichromatic point set with $2n$ points. If the $(k,0)$-partition Γ of P has index at most 0.1, then P admits a separated matching of size at least $(1/2 + c_3(k/n)^2)n$.*

Our goal now is to show that we can focus on k-configurations with k neither too small nor too large, and of index approximately 0.1. Here, we only sketch the argument, and we will make it more precise below, once all the lemmas have been stated formally: we choose $k_1 = O(1)$ and $k_2 = \Omega(n)$ to satisfy the previous two lemmas, and we consider the sequence of the $(k_1,0)$-partition, the $(k_1,1)$-partition, the $(k_1,2)$-partition, ..., up to the $(k_2,0)$-partition of P. By Lemma 10 and Lemma 11, we can assume that the first partition in the sequence has index less than 0.1 and the last partition in the sequence has index larger than 0.1. Thus, at some point the index has to jump over 0.1. Our definition of (k,λ)-partition ensures that this jump is gradual. The proof can be found in the full version.

▶ **Lemma 12.** *Let $k,n \in \mathbb{N}$ with $n \ge 210000k$. Let P be a convex bichromatic point set with $2n$ points, n red and n blue. Let Γ_1 be the (k,λ)-partition and Γ_2 the $(k,\lambda+1)$-partition of P. Suppose that the index of Γ_1 is at most 0.1. Then, the average red index and the average blue index of Γ_1 and Γ_2 each differ by at most 0.001.*

It follows that we can assume that we are dealing with a (k,λ)-partition of index approximately 0.1. Actually, we will see that it suffices to consider *k-configurations* of index 0.1. This will be the focus of the next section.

2.4 Random chunk-matchings in k-configurations

In this section, we will focus on convex bichromatic point sets P that admit a k-configuration Γ with special properties. Later, we will see how to reduce to this case.

Let $C_0, C_1, \dots, C_{\ell-1}$ be the chunks of the k-configuration Γ. We define a notion of *chunk-matching*, as illustrated in Figures 4 and 5. A chunk matching pairs each of the ℓ chunks with another chunk (possibly itself). Our goal is to define chunk matchings in such a way that we can easily derive from a chunk matching a separated matching between the points in P.

Formally, we define ℓ matchings $M_0, \dots, M_{\ell-1}$ by saying that for $i,j = 0,\dots,\ell-1$, the matching M_i pairs the chunks C_j and $C_{(i-j) \bmod \ell}$. Again, refer to Figures 4 and 5 for examples. The matching rule is symmetric, i.e., if C_a is matched to C_b then C_b is matched to C_a. Note that if $j \equiv (i-j) \pmod{\ell}$, the chunk C_j is matched to itself in M_i. If ℓ is even, this happens only for even i, namely for $j = i/2$ and for $j = i/2 + \ell/2$. If ℓ is odd,

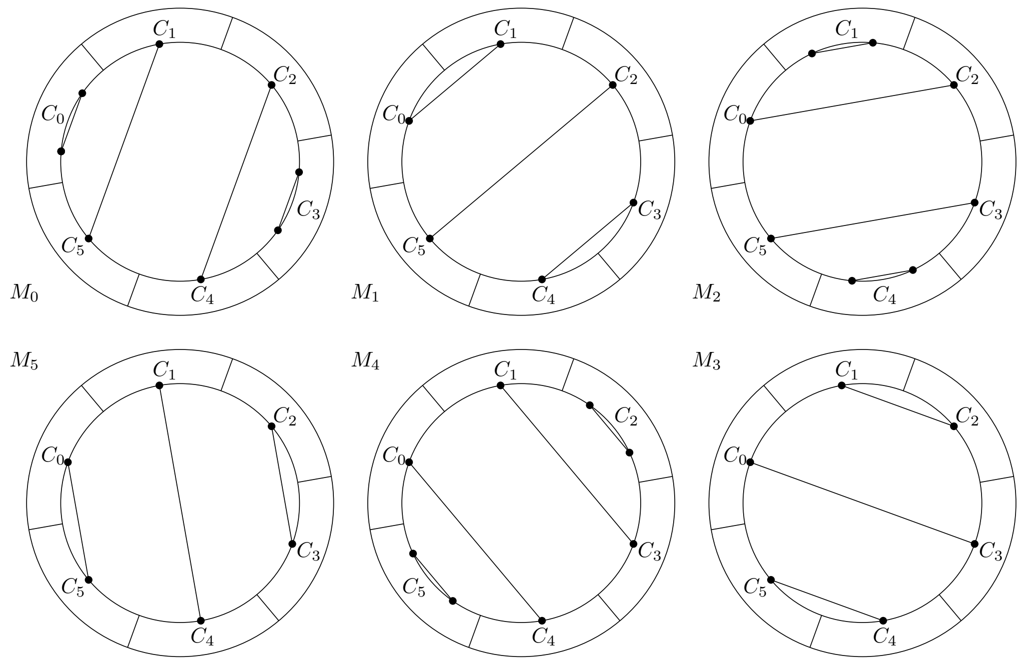

Figure 4 The six chunk matchings $M_0, \ldots, M_5$ for a set of six chunks. If i is even, the chunks $C_{i/2}$ and $C_{i/2+3}$ are matched to themselves. If i is odd, every chunk is matched to a different chunk.

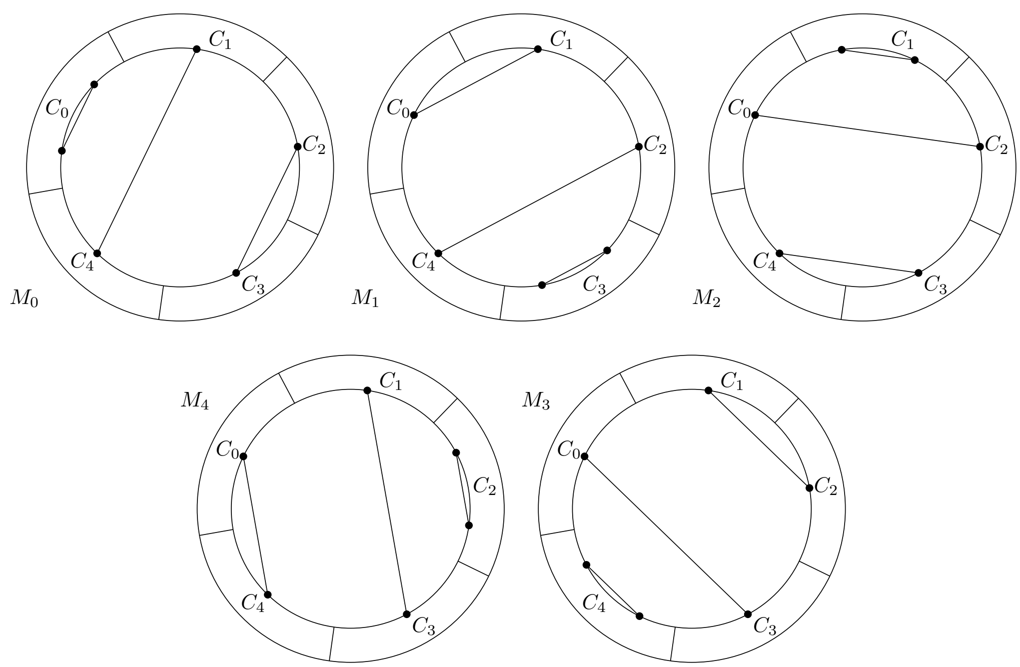

Figure 5 The five chunk matchings $M_0, M_1, \ldots, M_4$ for a set of five chunks. In matching M_i, the chunk $C_{(3i \bmod 5)}$ is matched to itself. Every other chunk is matched to a different chunk.

this happens in every matching, namely for $j \equiv (\ell+1)i/2 \pmod{\ell}$. By construction, for every M_i, if we connect the matched chunks by straight line edges, we obtain a set of plane segments such that there is one line that intersects all segments. Furthermore, every pair C_i, C_j of chunks, $0 \le i \le j \le \ell - 1$ appears in exactly one chunk matching. In essence, these matchings correspond to partitioning the chunks of Γ with a line, where the line can possibly pass through one or two chunks of Γ that are then matched to themselves.

Next, we describe how to derive from a given chunk matching M a separated matching on P, see Figure 6 for an illustration. We look at every two chunks C and D paired my M (possibly, $C = D$). If C is red and D blue, we match the k red points in C to the k blue points in D, getting k matched edges. The case that C is blue and D is red is analogous. If $C \neq D$ and both C and D are red, we could match the k red points in C to the $b(D) < k$ blue points in D, or vice versa. We choose the option that gives more edges, yielding $\max\{b(C), b(D)\}$ matched edges. The case that $C \neq D$ and both are blue is similar. Finally, suppose that $C = D$, and for concreteness, suppose that C is red. In this case, we split the points in C into two parts, containing $\lceil k/2 \rceil$ red points each (if k is odd, the median point belongs to both parts). In one part, we have at least $\lceil b(C)/2 \rceil$ blue points, and we match these blue points to the red points in the other part. This yields $\lceil b(C)/2 \rceil \ge b(C)/2$ matched edges. Thus, a chunk matching M gives a separated matching with at least

$$\frac{1}{2}\Bigg(\sum_{\substack{(C,D)\in M \\ C \text{ red}, D \text{ red}}} \max\{b(C), b(D)\} + \sum_{\substack{(C,D)\in M \\ C \text{ red}, D \text{ blue}}} k + \sum_{\substack{(C,D)\in M \\ C \text{ blue}, D \text{ red}}} k + \sum_{\substack{(C,D)\in M \\ C \text{ blue}, D \text{ blue}}} \max\{r(C), r(D)\} \Bigg) \quad (5)$$

matched edges, where the sums go over all ordered pairs of matched chunks in M, i.e., a matched pair (C, D) with $C \neq D$ appears twice (which is compensated by the leading factor of $1/2$) and a matched pair (C, C) appears once. The next lemma shows that a chunk matching that is chosen uniformly at random usually matches half the points of P.

▶ **Lemma 13.** *Let Γ be a k-configuration of P and M a random chunk matching in Γ. The expected number of matched edges in the corresponding separated matching is at least $n/2$.*

Proof. Let R be the number of red chunks in Γ and B the number of blue chunks in Γ. Let α be the average index of the red chunks, and β the average index of the blue chunks. We sum (5) over all $R + B$ possible chunk matchings and take the average. This gives the expected number of matched edges (the sums range over all ordered pairs of chunks in Γ).

$$\frac{1}{2(R+B)}\Bigg(\sum_{C \text{ red}, D \text{ red}} \max\{b(C), b(D)\} + 2\sum_{C \text{ red}, D \text{ blue}} k + \sum_{C \text{ blue}, D \text{ blue}} \max\{r(C), r(D)\} \Bigg)$$

Since there are R red chunks and B blue chunks, this is

$$= \frac{1}{2(R+B)}\Bigg(\sum_{C \text{ red}, D \text{ red}} \max\{b(C), b(D)\} + 2kRB + \sum_{C \text{ blue}, D \text{ blue}} \max\{r(C), r(D)\} \Bigg)$$

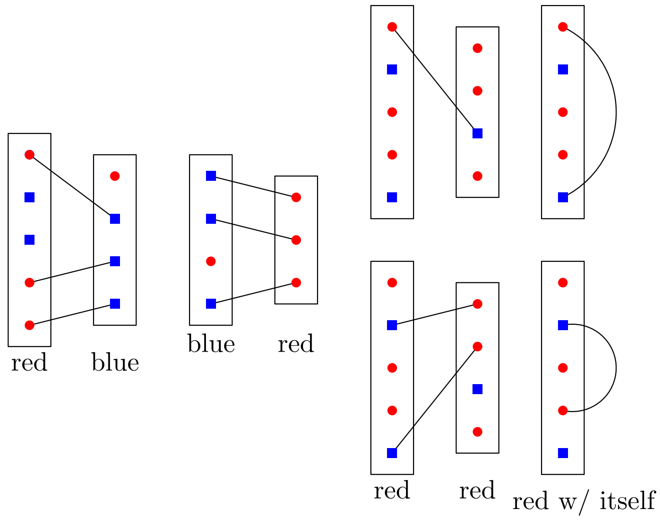

Figure 6 Going from a matched pair of chunks to a separated matching. If the two chunks have different colors, we can match k edges. If the two colors are the same, there are two reasonable options, matching the red points in one chunk with the blue points in the other chunk. We choose the one that matches more edges. A special case occurs if a chunk is matched to itself. In this case, we split the majority color into half and match between the halves.

We lower bound the maximum by the average to estimate this as

$$\geq \frac{1}{2(R+B)}\left(\sum_{C \text{ red}, D \text{ red}} \frac{b(C)+b(D)}{2} + 2kRB + \sum_{C \text{ blue}, D \text{ blue}} \frac{r(C)+r(D)}{2}\right) \tag{**}$$

Simplifying the sums, this is

$$= \frac{1}{2(R+B)}\left(R\sum_{C \text{ red}} b(C) + 2kRB + B\sum_{C \text{ blue}} r(C)\right)$$

Since the total number of blue points in red chunks is αkR and the total number of red points in blue chunks is βkB, this equals

$$= \frac{\alpha kR^2 + 2kRB + \beta kB^2}{2(R+B)}$$

Regrouping the terms and using (4), this becomes

$$= \frac{R(\alpha kR + kB) + B(\beta kB + kR)}{2(R+B)} = \frac{(R+B)n}{2(R+B)} = \frac{n}{2}.$$ ◀

2.5 Taking advantage of k-configurations

One inefficiency in the calculation in Lemma 13 is that we bound the maximum by the average in inequality (**). If these two quantities often differ significantly, we can gain an advantage over Lemma 13. This is made precise in the next lemma.

▶ **Lemma 14.** *Set $c_4 = 1/40$. Let $\delta > 0$ and let P be a convex bichromatic point set with $2n$ points, n red and n blue, and Γ a k-configuration for P with index at most 0.11 that contains at least $\delta(n/k)$ red chunks or at least $\delta(n/k)$ blue chunks with index at least 0.22. Then, P admits a separated matching of size at least $(1/2 + c_4\delta^2)n$.*

Proof. Suppose without loss of generality that there are at least $\delta(n/k)$ red chunks with index at least 0.2. Let R be the number of red chunks and B the number of blue chunks. The average red index of Γ is at most 0.11. Thus, if writing $\gamma_1(n/k)$ for the number of red chunks with index in $(0.11, 0.22)$ and $\gamma_2(n/k) \geq \delta(n/k)$ for the number of red chunks with index in $[0.22, 1)$, we have

$$0.11R \geq 0.11\gamma_1 \frac{n}{k} + 0.22\gamma_2 \frac{n}{k} = 0.11(\gamma_1 + 2\gamma_2)\frac{n}{k}.$$

It follows that $R \geq (\gamma_1 + 2\gamma_2)(n/k)$, and there must be at least $\gamma_2(n/k) \geq \delta(n/k)$ red chunks of index in $[0, 0.11]$. Now, consider the following sum over all ordered pairs (C, D) of red chunks, where one chunk (C or D) has red index at most 0.11 and the other chunk (D or C) has red index at least 0.22:

$$\frac{1}{2(R+B)}\left(\sum_C \sum_D \max\{b(C), b(D)\} - \frac{b(C)+b(D)}{2}\right)$$

Since $2\max\{a, b\} - a - b = \max\{a, b\} - \min\{a, b\}$, for all $a, b \in \mathbb{R}$, this equals

$$= \frac{1}{4(R+B)} \sum_C \sum_D (\max\{b(C), b(D)\} - \min\{b(C), b(D)\})$$

One chunk in each summand contains at least $0.22k$ blue points, the other chunk contains at most $0.11k$ blue points, so we can lower bound this as

$$\geq \frac{1}{4(R+B)} \sum_C \sum_D (0.22 - 0.11)k$$
$$\geq \frac{1}{4(R+B)} \sum_C \sum_D \frac{k}{10} \geq \frac{\delta^2(n/k)^2}{R+B}\frac{k}{20} \geq \frac{\delta^2}{40}n,$$

since we are adding over at least $2\delta^2(n/k)^2$ ordered pairs (C, D) (recall that each ordered pair (C, D) has a partner (D, C) in the sum) and since by (1), we have $R + B \leq 2n/k$. Thus, comparing with (**), the lemma follows. ◀

Lemma 14 shows that we can assume that few chunks in the k-configuration Γ of P have index larger than 0.22. In fact, suppose now that Γ contains no chunk of index at least 0.3 (this will be justified below). From now on, we will also assume that k is divisible by 3. We subdivide each chunk in our k-configuration Γ into three $(k/3)$-*subchunks*. Since all k-chunks have index less than 0.3, the subchunks have the same color as the original chunk. Let C be a k-chunk. The *middle subchunk* of C, denoted by C_M, is the $(k/3)$-subchunk of C that lies in the middle of the three subchunks. Now, we consider the middle subchunks. If the middle subchunks of the max-index color contain many points of the min-index color, we can gain an advantage by considering two *cross-matchings* between chunks of the max-index color.

▶ **Lemma 15.** *Set $c_5 = 1/4$. Let $\delta > 0$ and let P be a convex bichromatic point set with $2n$ points, n red and n blue. Let Γ be a k-configuration for P such that (i) k is divisible by 3; (ii) every chunk in Γ has index less than 0.3; and (iii) the middle subchunks of the max-index color contain in total at least δn points of the min-index color. Then P admits a separated matching of size at least $(1/2 + c_5\delta)n$.*

Proof. Suppose that the max-index color is red. We take a random chunk matching M of Γ, and we derive a separated matching from M as described above. However, when

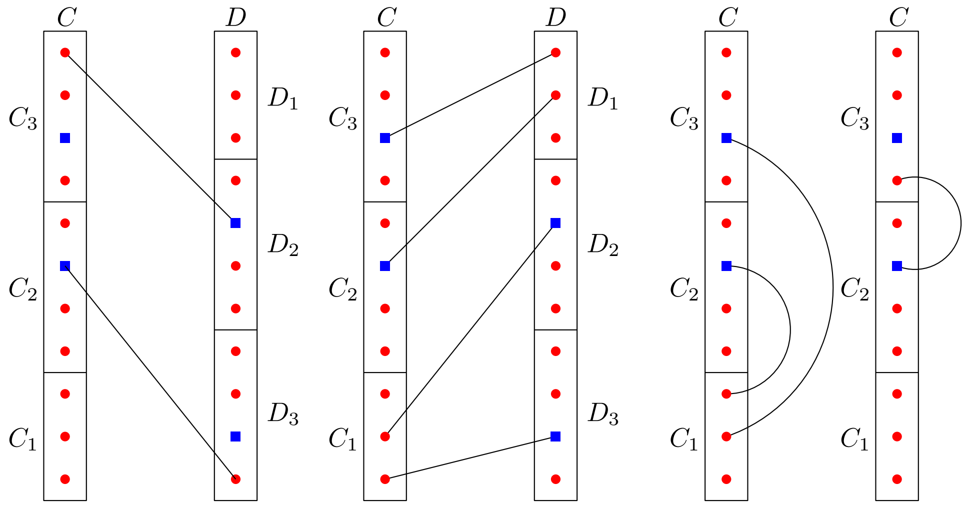

Figure 7 The two different separated matchings between two distinct red chunks (left) and the same red chunk (right).

considering a pair (C, D) of two red chunks, we proceed slightly differently. First, suppose that $C \neq D$, and let C_1, C_2, C_3 be the three subchunks of C, and D_1, D_2, D_3 be the three subchunks of D (in clockwise order). We have $r(C_i) = r(D_i) = k/3$, for $i = 1, 2, 3$; and $b(C_1) + b(C_2) + b(C_3) < k/3$ and $b(D_1) + b(D_2) + b(D_3) < k/3$. We consider two separated matchings between C and D (see Figure 7(left): (a) match all blue points in C_1 and C_2 to red points in D_3 and all blue points in D_1 and D_2 to red points in C_3; and (b) match all blue points in D_2 and D_3 to red points in C_1 and all blue points in C_2 and C_3 to red points in D_1. We take the better of the two matchings. The number of matched edges matched(C, D) is lower-bounded by the average, so

$$\begin{aligned}\text{matched}(C, D) &\geq \frac{1}{2}\left(b(C_1) + b(C_2) + b(D_1) + b(D_2) + b(D_3) + b(D_2) + b(C_2) + b(C_3)\right) \\ &= \frac{1}{2}\left(b(C) + b(D) + b(C_2) + b(D_2)\right). \qquad (6)\end{aligned}$$

Second, if $C = D$, we subdivide C into the three subchunks C_1, C_2, C_3 with $(C_1) = r(C_2) = r(C_3) = k/3$ and $b(C_1) + b(C_2) + b(C_3) < k/3$. Again, we consider two different matchings for C (see Figure 7(right): (a) match the blue points in C_1 and C_2 to the red points in C_3, and (b) match the blue points in C_2 and C_3 to the red points in C_1. Again, the number of matched edges matched(C, C) is at least

$$\text{matched}(C, C) \geq \frac{1}{2}(b(C_1) + b(C_2) + b(C_2) + b(C_3)) = \frac{1}{2}(b(C) + b(C_2)). \qquad (7)$$

Now, we set R to the number of red chunks and B to the number of blue chunks in Γ. Then, in a random chunk matching, the expected number of edges in the separated matchings between the pairs (C, D) of red chunks is

$$\frac{1}{2(R+B)}\left(\sum_{C \neq D, C, D \text{ red}} \text{matched}(C, D) + \sum_{C \text{ red}} 2\,\text{matched}(C, C)\right). \qquad (8)$$

Note that in the first sum, each unordered pair $\{C, D\}$ of distinct red chunks appears twice, even though it appears once in a random chunk matching. This is compensated by the leading factor of 1/2, which again leads to a coefficient of 2 for the expected number of edges in the separated matching in a chunk that is paired with itself. Using (6, 7), we can write

$$(8) \geq \frac{1}{2(R+B)} \left(\sum_{C \text{ red}} \sum_{D \text{ red}} \frac{b(C)+b(D)+b(C_M)+b(D_M)}{2} \right),$$

where we sum over all ordered pairs (C, D) of red chunks and C_M and D_M denote the middle chunks of C and D. Now we compare with (**).

$$\frac{1}{2(R+B)} \left(\sum_{C \text{ red}} \sum_{D \text{ red}} \frac{b(C)+b(D)+b(C_M)+b(D_M)}{2} - \frac{b(C)+b(D)}{2} \right)$$
$$= \frac{1}{2(R+B)} \left(\sum_{C \text{ red}} \sum_{D \text{ red}} \frac{b(C_M)+b(D_M)}{2} \right)$$

In the sum, every middle chunk C_M and every middle chunk D_M appears exactly R times, and by assumption, the total number of blue points in the red middle chunks is at least δn. Thus, this is lower-bounded as

$$\geq \frac{1}{2(R+B)} R\delta n \geq \frac{1}{4R} R\delta n = \frac{\delta}{4} n,$$

since red is the max-index color and hence by Proposition 9, we have $B \leq R$ and $R+B \leq 2R$. Thus, the lemma follows. ◀

Finally, we consider the case that the middle subchunks of the max-index color contain relatively few points. Since the index of Γ is relatively small, it means that the indices of the middle subchunks of the max-index color have a large variance. As in Lemma 14, this leads to a large separated matching. The proof is very similar to the proof of Lemma 14, and it can be found in the full version.

▶ **Lemma 16.** *Set $\delta = 10^{-4}$ and $\varepsilon = 10^{-5}$. Let P be a convex bichromatic point set with $2n$ points, n red and n blue, and let Γ be a k-configuration for P such that (i) k is divisible by 3; (ii) Γ has index at least 0.09; and (iii) every chunk in Γ has index less than 0.3. Then, if the middle subchunks of the max-index color contain in total at most δn points of the min-index color, P admits a separated matching of size at least $(1/2 + \varepsilon)n$.*

2.6 Putting it together

From Theorem 7, it follows that if P has at least four runs, there is always a separated matching with strictly more than $n/2$ edges. Moreover, if P has two runs, then P has a separated matching with $n > n/2$ edges. Therefore, the following theorem implies Theorem 2.

▶ **Theorem 17.** *There exist constants $\varepsilon_* > 0$ and $n_0 \in \mathbb{N}$ with the following property: let P be a convex bichromatic point set with $2n \geq 2n_0$ points, n red and n blue. Then, P admits a separated matching on at least $(1 + \varepsilon_*)n$ vertices.*

Proof. Set $n_0 = 10^{100}$ and $\varepsilon = 10^{-5}$, as in Lemma 16. Let k_1 the smallest integer larger than $10^3\varepsilon^{-3} = 10^{18}$ that is divisible by 3. Since $n \geq 10^{100} \geq 8k_1^2$, Lemma 10 shows that if the $(k_1, 0)$-partition Γ_1 of P has index at least 0.1, the theorem follows with $\varepsilon_* = \Omega(1/k_1^4) = \Omega(1)$. Thus, we may assume the following claim:

▷ Claim 18. The $(k_1, 0)$-partition Γ_1 of P has index less than 0.1, where k_1 is a fixed constant with $k_1 \geq 10^3\varepsilon^{-3} = 10^{18}$.

Next, let k_2 be the largest integer in the interval $[10^{-4}\varepsilon^3 n, 10^{-3}\varepsilon^3 n]$ that is divisible by 3. Since $n \geq 10^{100}$, it follows that k_2 exists. Furthermore, since $n \geq k_2$ and $6480n \leq 10^{-8}\varepsilon^6 n^2 \leq k_2^2$, Lemma 11 implies that if the $(k_2, 0)$-partition Γ_2 of P has index at most 0.1, the theorem follows with $\varepsilon_* = \Omega((k_2/n)^2) = \Omega(1)$. Hence, we may assume the following claim:

▷ Claim 19. The $(k_2, 0)$-partition Γ_2 of P has index more than 0.1, where k_2 is the largest integer in the interval $[10^{-4}\varepsilon^3 n, 10^{-3}\varepsilon^3 n]$ that is divisible by 3.

We now interpolate between Γ_1 and Γ_2. Consider the sequence of (k, λ)-partitions of P for the parameter pairs

$$(k_1, 0), (k_1, 1), \ldots, (k_1, \lambda(k_1)), (k_1 + 3, 0), (k_1 + 3, 1), \ldots,$$
$$(k_1 + 3, \lambda(k_1 + 3)), (k_1 + 6, 0), \ldots, (k_2, 0),$$

where $\lambda(k)$ denotes the largest λ for which the (k, λ)-partition of P still contains a k-chunk. Let (k_*, λ_*) be the first parameter pair for which the index of the (k_*, λ_*)-partition Γ_3 of P is larger than 0.1. This parameter pair exists, because $(k_2, 0)$ is a candidate.

▷ Claim 20. The (k_*, λ_*)-partition Γ_3 of P has index in $[0.1, 0.101]$. Here, k_* is divisible by 3 and lies in the interval $[10^3\varepsilon^{-3}, 10^{-3}\varepsilon^3 n]$.

Proof. The claim on k_* and the fact that Γ_3 has index at least 0.1 follow by construction. Furthermore, let (k_{**}, λ_{**}) be such that Γ_3 is the $(k_{**}, \lambda_{**} + 1)$ partition of P (we either have $k_{**} = k_*$ and $\lambda_{**} = \lambda_* - 1$; or $k_{**} = k_* - 1$ and $\lambda_{**} = \lambda(k_{**})$). Since $210000k_{**} \leq 10^6 \cdot 10^{-3}\varepsilon^3 n \leq n$, Lemma 12 implies that the index of Γ_3 is at most 0.101. ◀

We rearrange P to turn Γ_3 into a k_*-configuration Γ_4 of a closely related point set P_2.

▷ Claim 21. There exists a convex bichromatic point set P_2 with $2n$ points, n red and n blue, and a k_*-configuration Γ_4 of P_2 such that (i) P_2 differs from P in at most $10^{-1}\varepsilon^3 n$ points; and (ii) the index of Γ_4 lies in $[0.097, 0.103]$.

Proof. We remove from P all the uncovered points of Γ_3 as well as 3 points of the majority color from each $(k_* + 3)$-chunk of Γ_3 (and, if necessary, up to 3 points of the minority color, to keep chunk structure valid). If we consider a single red $(k_* + 3)$-chunk C and denote the original number of blue points in C by $b(C)$ and the resulting number of blue points by $b'(C)$, then the index of C changes by at most

$$\left| \frac{b(C)}{k_* + 3} - \frac{b'(C)}{k_*} \right| = \left| \frac{k_* b(C) - (k_* + 3)b'(C)}{k_*(k_* + 3)} \right| \leq \frac{|b(C) - b'(C)|}{k_* + 3} + \frac{3b'(C)}{k_*(k_* + 3)} \leq \frac{6}{k_* + 3},$$

since $|b(C) - b(C')| \leq 3$ and $b'(C) \leq k_*$. A similar bound holds for a blue $(k_* + 3)$-chunk.

By (1), there are at most $2n/k_* \leq 2 \cdot 10^{-3}\varepsilon^3 n$ many $(k_* + 3)$-chunks, and by Proposition 8, there at most $2k_* - 1 \leq 2 \cdot 10^{-3} \cdot \varepsilon^3 n$ uncovered points, so in total we remove at most $14 \cdot 10^{-3}\varepsilon^3 n \leq 10^{-1}\varepsilon^3 n$ points. We arrange these points into as many pure chunks of k_* red points or of k_* blue points as possible. This creates at most $10^{-1}\varepsilon^3(n/k_*)$ new k^*-chunks, all of which have index 0. Now, less than k_* red points and less than k_* blue points remain. By (2), there are at least

$$(1 - 0.101)\frac{n}{7k_*} - 2 \geq 10^{-1} \cdot 10^3\varepsilon^{-3} - 2 \geq 10^3$$

chunks of each color in Γ_3. Thus, we can partition the remaining red points into at most 10^3 groups of size at most $10^{-3}k_*$ and add each group to a single blue chunk; and similarly for the remaining blue points. This changes the index of each chunk by at most 10^{-3}.

We call the resulting rearranged point set P_2 and the resulting k_*-configuration Γ_4. As mentioned, P_2 was obtained from P by moving at most $10^{-1} \cdot \varepsilon^3 n$ points. We change the index of any existing chunk by at most $6/(k^* + 3) + 10^{-3} \leq 2 \cdot 10^{-3}$. Furthermore, we create at most $10^{-1}\varepsilon^3(n/k_*)$ new k_*-chunks (all of index 0) and by (2), we have at least $(1 - 0.101)n/(7k_*) - 2 \geq (10^{-1} - 10^{-2} \cdot \varepsilon^3)(n/k_*)$ original chunks of each color in Γ_3. Thus, if we denote by α the average index of the existing red chunks after the rearrangement, by R the number of existing red chunks, and by R' the number of new red chunks, the average red index of Γ_4 can differ from α by at most

$$\alpha - \frac{R}{R + R'}\alpha = \alpha\frac{R'}{R + R'} \leq \alpha\frac{R'}{R} \leq 0.102\frac{10^{-1}\varepsilon^3}{10^{-1} - 10^{-2}\varepsilon^3} \leq 10^{-3},$$

and similarly for the average blue index of Γ_5. It follows that Γ_4 has index in $[0.097, 0.103]$. ◀

Now, using Lemma 14 with $\delta = 10^{-1}\varepsilon$, we get that if the k^*-configuration Γ_4 contains at least $\delta(n/k_*)$ red chunks or at least $\delta(n/k_*)$ blue chunks with index at least 0.22, then the rearranged point set P_2 admits a separated matching of size at least

$$\left(\frac{1}{2} + \frac{1}{40} \cdot 10^{-2}\varepsilon^2\right) n \geq \left(\frac{1}{2} + 10^{-4} \cdot \varepsilon^2\right) n.$$

By Claim 21, P_2 differs from P by at most $10^{-1}\varepsilon^3 n$ points. Since $\varepsilon = 10^{-5}$, it follows that after deleting all matching edges incident to a rearranged point, we obtain the theorem. Thus, we may assume the following claim:

▷ Claim 22. At most $10^{-1} \cdot \varepsilon(n/k_*)$ red chunks and at most $10^{-1} \cdot \varepsilon(n/k_*)$ blue chunks in Γ_4 have index more than 0.22.

We again rearrange the point set P_2 to obtain a point set P_3 and a k^*-configuration Γ_5 for P_3 such that every k^*-chunk in Γ_5 has index less than 0.3.

▷ Claim 23. There exists a convex bichromatic point set P_3 with $2n$ points, n red and n blue, and a k_*-configuration Γ_5 of P_3 such that (i) P_3 differs from P_2 in at most $2 \cdot 10^{-1}\varepsilon n$ points; (ii) the index of Γ_5 is at least 0.096; (iii) all chunks in Γ_5 have index less than 0.3; and (iv) k_* is divisible by 3.

Proof. We remove all the blue points from red chunks of index at least 0.22 and all the red points from all blue chunks of index at least 0.22. These are at most $2 \cdot 10^{-1} \cdot \varepsilon n$ points in total. By removing these points, we decrease the index of at most $10^{-1}\varepsilon(n/k_*)$ existing chunks of each color to 0. By Proposition 9, there are at least

$$(1 - 0.103)\frac{n}{2k_*} \geq 10^{-1} \cdot \frac{n}{k_*} \tag{9}$$

existing chunks of each color, so this step decreases the average index by at most ε.

We rearrange the deleted points into as many pure chunks with k_* red points or with k_* blue points as possible. Less than k_* red points and less than k_* blue points remain. By (9), there are at least $10^{-1}(n/k_*) \geq 10^3$ chunks of each color, so we group the remaining points into blocks of size $10^{-3} \cdot k_*$ and distribute the blocks over the existing red and blue chunks. This increases the average index of the existing chunks by at most 10^{-3}.

Finally, we create at most $10^{-1} \cdot \varepsilon(n/k_*)$ new chunks of each color (all with index 0), and the existing number of chunks of the max-index color of Γ_4 is at least $n/2k_*$, by Proposition 9. Suppose for concreteness that the max-index color of Γ_4 is red, and let R be the number of existing red chunks, R' the number of new red chunks, and α the average index of the existing red chunks after the rearrangement. Then, the average red index after the rearrangement differs from α be at most

$$\alpha - \frac{R}{R+R'}\alpha \leq \alpha\frac{R'}{R} \leq 0.104 \cdot \frac{10^{-1}\varepsilon}{1/2} \leq \varepsilon.$$

Thus, the red index in the resulting k_*-configuration Γ_5 is at least $0.097 - 2\varepsilon \geq 0.096$. This implies that the index of Γ_5 is at least 0.096. ◀

Now, we consider the k^*-configuration Γ_5. By Lemma 16, if in Γ_5 the middle-chunks of the max-index color contain in total at most $10^{-4}n$ points of the min-index color, we get a separated matching for P_3 of size at least $(1/2+\varepsilon)n$. By deleting all the matching edges that are incident to the at most $2 \cdot 10^{-1}\varepsilon n + 10^{-1}\varepsilon^3 n \leq 0.3\varepsilon n$ points that were moved to obtain P_3 from P, the theorem follows. Similarly, if in Γ_5 the middle-chunks of the max-index color contain in total more than $10^{-4}n$ points of the min-index color, by Lemma 15, we get a separated matching for P_3 of size at least $(1/2+10^4/4)n \geq (1/2+\varepsilon)n$. Again, we obtain the theorem after deleting edges that are incident to the rearranged points. ◀

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