

# Interference Minimization in Asymmetric Sensor Networks<sup>\*</sup>

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**Abstract.** A fundamental problem in wireless sensor networks is to connect a given set of sensors while minimizing the *receiver interference*. This is modeled as follows: each sensor node corresponds to a point in  $\mathbb{R}^d$  and each *transmission range* corresponds to a ball. The receiver interference of a sensor node is defined as the number of transmission ranges it lies in. Our goal is to choose transmission radii that minimize the maximum interference while maintaining a strongly connected asymmetric communication graph.

For the two-dimensional case, we show that it is NP-complete to decide whether one can achieve a receiver interference of at most 5. In the one-dimensional case, we prove that there are optimal solutions with non-trivial structural properties. These properties can be exploited to obtain an exact algorithm that runs in quasi-polynomial time. This generalizes a result by Tan et al. to the asymmetric case.

## 1 Introduction

Wireless sensor networks constitute a popular paradigm in mobile networks: several small independent devices are distributed in a certain region, and each device has limited computational resources. The devices can communicate through a wireless network. Since battery life is limited, it is imperative that the overhead for the communication be kept as small as possible. Thus, the literature on sensor networks contains many strategies to reduce the number of communication links while maintaining desirable properties of the communication networks. The term *topology control* refers to the general paradigm of dropping edges from the communication graph in order to decrease energy consumption.

Traditionally, topology control focused on properties such as sparsity, dilation, or congestion of the communication graph. This changed with the work of Burkhart et al. [3], who pointed out the importance of explicitly considering the *interference* caused by competing senders. By reducing the interference, we can

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avoid costly retransmission of data due to data collisions, leading to increased battery life. At the same time, we need to ensure that the resulting communication graph remains connected.

There are many different ways to formalize the problem of interference minimization [4,6,8,9,11]. Usually, the devices are modeled as points in  $d$ -dimensional space, and the transmission ranges are modeled as  $d$ -dimensional balls. Each point can choose the radius of its transmission range, and different choices of transmission ranges lead to different reachability structures. There are two ways to interpret the resulting communication graph. In the *symmetric* case, the communication graph is undirected, and it contains an edge between two points  $p$  and  $q$  if and only if both  $p$  and  $q$  lie in the transmission range of the other point [6,8,9,11]. For a valid assignment of transmission ranges, we require that the communication graph is connected. In the *asymmetric* case, the communication graph is directed, and there is an edge from  $p$  to  $q$  if and only if  $p$  lies in the transmission range of  $q$ . We require that the communication graph is strongly connected, or, in a slightly different model, that there is one point that is reachable from every other point through a directed path [4].

In both the symmetric and the asymmetric case, the (*receiver-centric*) *interference* of a point is defined as the number of transmission ranges that it lies in [12]. The goal is to find a valid assignment of transmission ranges that makes the maximum interference as small as possible. We refer to the resulting interference as *minimum interference*. The minimum interference under the two models for the asymmetric case differs by at most one: if there is a point reachable from every other, we can increase its transmission range to include all other points. As a result, the communication graph becomes strongly connected, while the minimum interference increases by at most one. All of these models have been also considered in a non-euclidean setting, in which the problems studied in this paper cannot be approximated efficiently under standard assumptions [1].

Let  $n$  be the number of points. In the symmetric case, one can always achieve interference  $O(\sqrt{n})$ , and this is sometimes necessary [5,12]. In the one-dimensional case, there is an efficient approximation algorithm with approximation factor  $O(n^{1/4})$  [12]. Furthermore, Tan et al. [13] prove the existence of optimal solutions with interesting structural properties in one dimension. This can be used to obtain a nontrivial exact algorithm for this case. In the asymmetric case, the interference is significantly smaller: one can always achieve interference  $O(\log n)$ , which is sometimes optimal (e.g., [7]). The one-dimensional model is also called the *highway model* [11]. Rickenbach et al. [11] cite the “intuition that already one-dimensional networks exhibit most of the complexity of finding minimum-interference topologies” as their motivation to study this model. Based on our experiences with the model, we can only support this claim.

There exist several other interference models. For example, the *sender-centric* model of Burkhart et al. [3] aims to minimize the maximum interference caused by any *edge* of the communication graph (i.e., the total number of points that lie in a certain region defined by the edge). Moscibroda and Wattenhofer [9] present a more general model that works in abstract metric spaces and that can distin-

guish between active and passive nodes. Johannsson and Carr-Motyčková [6] define a notion of *average path interference*, where the interference is taken as the average of the interferences over all interference-optimal paths in the network.

*Our results.* We consider interference minimization in asymmetric wireless sensor networks in one and two dimensions. We show that for two dimensions, it is NP-complete to find a valid assignment that minimizes the maximum interference. In one dimension we consider our second model requiring one point that is reachable from every other point through a directed path. Generalizing the result by Tan et al. [13], we show that there is an optimal solution that exhibits a certain binary tree structure. By means of dynamic programming, this structure can be leveraged for a nontrivial exact algorithm. Unlike the symmetric case, this algorithm always runs in quasi-polynomial time  $2^{O(\log^2 n)}$ , making it unlikely that the one-dimensional problem is NP-hard. Nonetheless, a polynomial time algorithm remains elusive.

## 2 Preliminaries and Notation

We now formalize our interference model for the planar case. Let  $P \subset \mathbb{R}^2$  be a planar  $n$ -point set. A *receiver assignment*  $N : P \rightarrow P$  is a function that assigns to each point in  $P$  the furthest point that receives data from  $P$ . The resulting (asymmetric) *communication graph*  $G_P(N)$  is the directed graph with vertex set  $P$  and edge set  $E_P(N) = \{(p, q) \mid \|p - q\| \leq \|p - N(p)\|\}$ , i.e., from each point  $p \in P$  there are edges to all points that are at least as close as the assigned receiver  $N(p)$ . The receiver assignment  $N$  is *valid* if  $G_P(N)$  is strongly connected.

For  $p \in \mathbb{R}^2$  and  $r > 0$ , let  $B(p, r)$  denote the closed disk with center  $p$  and radius  $r$ . We define  $B_P(N) = \{B(p, d(p, N(p))) \mid p \in P\}$  as the set that contains for each  $p \in P$  a disk with center  $p$  and  $N(p)$  on the boundary. The disks in  $B_P(N)$  are called the *transmission ranges* for  $N$ . The *interference* of  $N$ ,  $I(N)$ , is the maximum number of transmission ranges that cover a point in  $P$ , i.e.,  $I(N) = \max_{p \in P} |\{p \in B \mid B \in B_P(N)\}|$ . In the *interference minimization problem*, we are looking for a valid receiver assignment with minimum interference.

## 3 NP-completeness in Two Dimensions

We show that the following problem is NP-complete: given a planar point set  $P$ , does there exist a valid receiver assignment  $N$  for  $P$  with  $I(N) \leq 5$ ? It follows that the minimum interference for planar point sets is NP-hard to approximate within a factor of  $6/5$ .

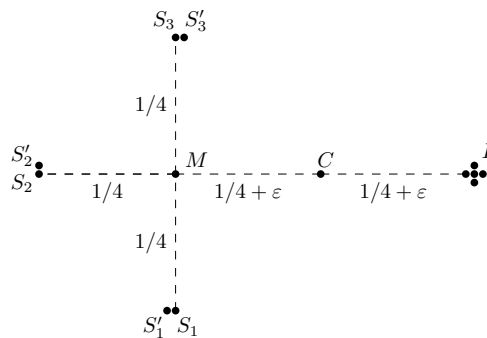
The problem is clearly in NP. To show that interference minimization is NP-hard, we reduce from the problem of deciding whether a grid graph of maximum degree 3 contains a Hamiltonian path: a *grid graph*  $G$  is a graph whose vertex set  $V \subset \mathbb{Z} \times \mathbb{Z}$  is a finite subset of the integer grid. Two vertices  $u, v \in V$  are adjacent

in  $G$  if and only if  $\|u - v\|_1 = 1$ , i.e., if  $u$  and  $v$  are neighbors in the integer grid. A *Hamiltonian path* in  $G$  is a path that visits every vertex in  $V$  exactly once. Papadimitriou and Vazirani showed that it is NP-complete to decide whether a grid graph  $G$  of maximum degree 3 contains a Hamiltonian cycle [10]. Note that we may assume that  $G$  is connected; otherwise there can be no Hamiltonian path.

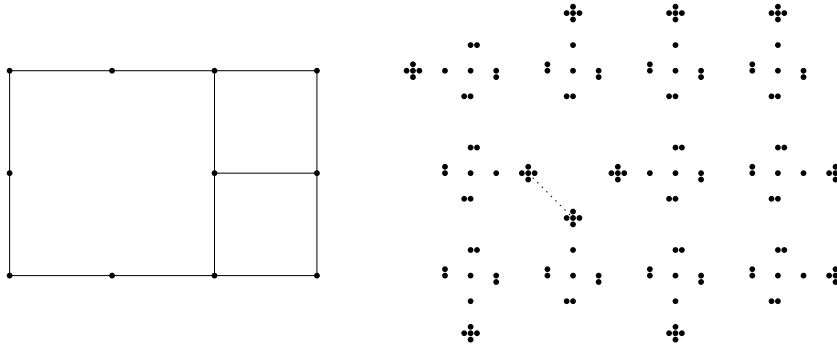
Our reduction proceeds by replacing each vertex  $v$  of the given grid graph  $G$  by a *vertex gadget*  $P_v$ ; see Fig. 1. The vertex gadget consists of 13 points, and it has five parts: (a) the *main point*  $M$  with the same coordinates as  $v$ ; (b) three *satellite stations* with two points each:  $S_1, S'_1, S_2, S'_2, S_3, S'_3$ . The coordinates of the  $S_i$  are chosen from  $\{v \pm (0, 1/4), v \pm (1/4, 0)\}$  so that there is a satellite station for each edge in  $G$  that is incident to  $v$ . If  $v$  has degree two, the third satellite station can be placed in any of the two remaining directions. The  $S'_i$  lie at the corresponding clockwise positions from  $\{v \pm (\varepsilon, 1/4), v \pm (1/4, -\varepsilon)\}$ , for a sufficiently small  $\varepsilon > 0$ ; (c) the *connector*  $C$ , a point that lies roughly at the remaining position from  $\{v \pm (0, 1/4), v \pm (1/4, 0)\}$  that is not occupied by a satellite station, but an  $\varepsilon$ -unit further away from  $M$ . For example, if  $v + (0, 1/4)$  has no satellite station, then  $C$  lies at  $v + (0, 1/4 + \varepsilon)$ ; and (d) the *inhibitor*, consisting of five points  $I_c, I_1, \dots, I_4$ . The point  $I_c$  is the center of the inhibitor and  $I_1$  is the point closest to  $C$ . The position of  $I_c$  is  $M + 2(C - M) + \varepsilon(C - M)/\|C - M\|$ , that is, the distance between  $I_c$  and  $C$  is an  $\varepsilon$ -unit larger than the distance between  $C$  and  $M$ :  $\|M - C\| + \varepsilon = \|C - I_c\|$ . The points  $I_1, \dots, I_4$  are placed at the positions  $\{I_c \pm (0, \varepsilon), I_c \pm (\varepsilon, 0)\}$ , with  $I_1$  closest to  $C$ .

Given a grid graph  $G$ , the reduction can be carried out in polynomial time: just replace each vertex  $v$  of  $G$  by the corresponding gadget  $P_v$ ; see Fig. 2 for an example. Let  $P = \bigcup_{v \in G} P_v$  be the resulting point set. Two satellite stations in  $P$  that correspond to the same edge of  $G$  are called *partners*. First, we investigate the interference in any valid receiver assignment for  $P$ .

**Lemma 3.1.** *Let  $N$  be a valid receiver assignment for  $P$ . Then in each vertex gadget, the points  $I_c$  and  $M$  have interference as least 5, and the points  $S_1, S_2$ , and  $S_3$  have interference at least 3.*

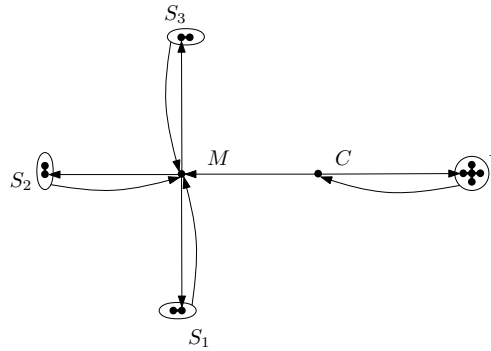


**Fig. 1.** The vertex gadget.



**Fig. 2.** An example reduction.

*Proof.* For each point  $p \in P$ , the transmission range  $B(p, d(p, N(p)))$  must contain at least the nearest neighbor of  $p$ . Furthermore, in each satellite station and in each inhibitor, at least one point must have an assigned receiver outside of the satellite station or inhibitor; otherwise, the communication graph  $G_P(N)$  would not be strongly connected. This forces interference of 5 at  $M$  and at  $I_c$ : each satellite station and  $C$  must have an edge to  $M$ , and  $I_1, \dots, I_4$  all must have an edge to  $I_c$ . Similarly, for  $i = 1, \dots, 3$ , the main point  $M$  and the satellite  $S'_i$  must have an edge to  $S_i$ ; see Fig. 3.  $\square$



**Fig. 3.** The nearest neighbors in a vertex gadget.

Let  $N$  be a valid receiver assignment, and let  $P_v$  be a vertex gadget in  $P$ . An *outgoing* edge for  $P_v$  is an edge in  $G_P(N)$  that originates in  $P_v$  and ends in a different vertex gadget. An *incoming* edge for  $P_v$  is an edge that originates in a different gadget and ends in  $P_v$ . A *connecting* edge for  $P_v$  is either an outgoing or an incoming edge for  $P_v$ . If  $I(N) \leq 5$  holds, then Lemma 3.1 implies that a connecting edge can be incident only to satellite stations.

**Lemma 3.2.** *Let  $N$  be a valid receiver assignment for  $P$  with  $I(N) \leq 5$ . Let  $P_v$  be a vertex gadget of  $P$  and  $e$  an outgoing edge from  $P_v$  to another vertex gadget  $P_w$ . Then  $e$  goes from a satellite station of  $P_v$  to its partner satellite station in  $P_w$ . Furthermore, in each satellite station of  $P_v$ , at most one point is incident to outgoing edges.*

*Proof.* By Lemma 3.1, both  $M$  and  $I_c$  in  $P_v$  have interference at least 5. This implies that neither  $M$ , nor  $C$ , nor any point in the inhibitor of  $P_v$  can be incident to an outgoing edge of  $P_v$ : such an edge would increase the interference at  $M$  or at  $I_c$ . In particular, note that the distance between the inhibitors in two distinct vertex gadgets is at least  $\sqrt{2}/2 - O(\varepsilon) > 1/2 + O(\varepsilon)$ , the distance between  $M$  and its corresponding inhibitor; see the dotted line in Fig 2.

Thus, all outgoing edges for  $P_v$  must originate in a satellite station. If there were a satellite station in  $P_v$  where both points are incident to outgoing edges, the interference at  $M$  would increase. Furthermore, if there were a satellite station in  $P_v$  with an outgoing edge that does not go the partner station, this would increase the interference at the main point of the partner vertex gadget, or at the inhibitor center  $I_v$  of  $P_v$ .  $\square$

Next, we show that the edges between the vertex gadgets are quite restricted.

**Lemma 3.3.** *Let  $N$  be a valid receiver assignment for  $P$  with  $I(N) \leq 5$ . For every vertex gadget  $P_v$  in  $P$ , at most two satellite stations in  $P_v$  are incident to connecting edges in  $G_P(N)$ .*

*Proof.* By Lemma 3.2 connecting edges are between satellite stations and by Lemma 3.1, the satellite points  $S_i$  in  $P_v$  have interference at least 3.

First, assume that all three satellite stations in  $P_v$  have outgoing edges. This would increase the interference at all three  $S_i$  to 5. Then,  $P_v$  could not have any incoming edge from another vertex gadget, because this would increase the interference for at least one  $S_i$  (note that due to the placement of the  $S'_i$ , every incoming edge causes interference at an  $S_i$ ). If  $P_v$  had no incoming edge,  $G_P(N)$  would not be strongly connected. It follows that  $P_v$  has at most two satellite stations with outgoing edges.

Next, assume that two satellite stations in  $P_v$  have outgoing edges. Then, the third satellite station of  $P_v$  cannot have an incoming edge, as the two outgoing edges already increase the interference at the third satellite station to 5.

Hence, we know that every vertex gadget  $P_v$  either (i) has connecting edges with all three partner gadgets, exactly one of which is outgoing, or (ii) is connected to at most two other vertex gadgets. Take a vertex gadget  $P_v$  of type (i) with partners  $P_{u_1}, P_{u_2}, P_w$ . Suppose that  $P_v$  has incoming edges from  $P_{u_1}$  and  $P_{u_2}$  and that the outgoing edge goes to  $P_w$ . Follow the outgoing edge to  $P_w$ . If  $P_w$  is of type (i), follow the outgoing edge from  $P_w$ ; if  $P_w$  is of type (ii) and has an outgoing edge to a vertex gadget we have not seen yet, follow this edge. Continue this process until  $P_v$  is reached again or until the next vertex gadget has been visited already. This gives all vertex gadgets that are reachable from  $P_v$  on a directed path. However, in each step there is only one choice for the

next vertex gadget. Thus, the process cannot discover  $P_{u_1}$  and  $P_{u_2}$ , since both of them would lead to  $P_v$  in the next step, causing the process to stop. It follows that at least one of  $P_{u_1}$  or  $P_{u_2}$  is not reachable from  $P_v$ , although  $G_P(N)$  should be strongly connected. Therefore, all vertex gadgets in  $G_P(N)$  must be of type (ii), as claimed in the lemma.  $\square$

We can now prove the main theorem of this section.

**Theorem 3.4.** *Given a point set  $P \subset \mathbb{R}^2$ , it is NP-complete to decide whether there exists a valid receiver assignment  $N$  for  $P$  with  $I(N) \leq 5$ .*

*Proof.* Using the receiver assignment  $N$  as certificate, the problem is easily seen to be in NP. To show NP-hardness, we use the polynomial time reduction from the Hamiltonian path problem in grid graphs: given a grid graph  $G$  of maximum degree 3, we construct a planar point set  $P$  as above. It remains to verify that  $G$  has a Hamiltonian path if and only if  $P$  has a valid receiver assignment  $N$  with  $I(N) \leq 5$ .

Given a Hamilton path  $H$  in  $G$ , we construct a valid receiver assignment  $N$  for  $P$  as follows: in each vertex gadget, we set  $N(M) = C$ ,  $N(C) = M$ , and  $N(I_1) = C$ . For  $i = 1, \dots, 3$  we set  $N(S'_i) = S_i$  and  $N(I_{i+1}) = I_c$ . Finally, we set  $N(I_c) = I_1$ . This essentially creates the edges from Fig. 3, plus the edge from  $M$  to  $C$ . Next, we encode  $H$  into  $N$ : for each  $S_i$  on an edge of  $H$ , we set  $N(S_i)$  to the corresponding  $S_i$  in the partner station. For the remaining  $S_i$ , we set  $N(S_i) = M$ . Since  $H$  is Hamiltonian,  $G_P(N)$  is strongly connected (note that each vertex gadget induces a strongly connected subgraph). It can now be verified that  $M$  and  $I_c$  have interference 5;  $I_2, I_3, I_4$  have interference 2; and  $I_1$  has interference 3. The point  $C$  has interference between 2 and 4, depending on whether  $S_1$  and  $S_3$  are on edges of  $H$ . The satellites  $S_i$  and  $S'_i$  have interference at most 5 and 4, respectively.

Now consider a valid receiver assignment  $N$  for  $P$  with  $I(N) \leq 5$ . Let  $F$  be the set of edges in  $G$  that correspond to pairs of vertex gadgets with a connecting edge in  $G_P(N)$ . Let  $H$  be the subgraph that  $F$  induces in  $G$ . By Lemma 3.3,  $H$  has maximum degree 2. Furthermore, since  $G_P(N)$  is strongly connected, the graph  $H$  is connected and meets all vertices of  $G$ . Thus,  $H$  is a Hamiltonian path (or cycle) for  $G$ , as desired.  $\square$

**Remark.** A similar result to Theorem 3.4 also holds for symmetric communication graphs networks [2].

## 4 The One-Dimensional Case

For the one-dimensional case we minimize receiver interference under the second model discussed in the introduction: given  $P \subset \mathbb{R}$  and a receiver assignment  $N : P \rightarrow P$ , the graph  $G_P(N)$  now has a directed edge from each point  $p \in P$  to its assigned receiver  $N(p)$ , and no other edges.  $N$  is *valid* if  $G_P(N)$  is acyclic and if there is a sink  $r \in P$  that is reachable from every point in  $P$ . The sink has no outgoing edge. The interference of  $N$ ,  $I(N)$ , is defined as before.

#### 4.1 Properties of Optimal Solutions

We now explore the structure of optimal receiver assignments. Let  $P \subset \mathbb{R}$  and  $N$  be a valid receiver assignment for  $P$  with sink  $r$ . We can interpret  $G_P(N)$  as a directed tree, so we call  $r$  the *root* of  $G_P(N)$ . For a directed edge  $pq$  in  $G_P(N)$ , we say that  $p$  is a *child* of  $q$  and  $q$  is the *parent* of  $p$ . We write  $p \rightsquigarrow_N q$  if there is a directed path from  $p$  to  $q$  in  $G_P(N)$ . If  $p \rightsquigarrow_N q$ , then  $q$  is an *ancestor* of  $p$  and  $p$  a *descendant* of  $q$ . Note that  $p$  is both an ancestor and a descendant of  $p$ . Two points  $p, q \in P$  are *unrelated* if  $p$  is neither an ancestor nor a descendant of  $q$ . For two points  $p, q$ , we define  $((p, q)) = (\min\{p, q\}, \max\{p, q\})$  as the open interval bounded by  $p$  and  $q$ , and  $[[p, q]] = [\min\{p, q\}, \max\{p, q\}]$  as the closure of  $((p, q))$ . An edge  $pq$  of  $G_P(N)$  is a *cross edge* if the interval  $((p, q))$  contains at least one point that is not a descendant of  $p$ .

Our main structural result is that there is always an optimal receiver assignment for  $P$  without cross edges. A similar property was observed by Tan et al. for the symmetric case [13].

**Lemma 4.1.** *Let  $N^*$  be a valid receiver assignment for  $P$  with minimum interference. There is a valid assignment  $\tilde{N}$  for  $P$  with  $I(\tilde{N}) = I(N^*)$  such that  $G_P(\tilde{N})$  has no cross edges.*

*Proof.* Pick a valid assignment  $\tilde{N}$  with minimum interference that minimizes the total length of the cross edges

$$C(\tilde{N}) := \sum_{pq \in C} \|p - q\|,$$

where  $C$  are the cross-edges of  $G_P(\tilde{N})$ . If  $C(\tilde{N}) = 0$ , we are done. Thus, suppose  $C(\tilde{N}) > 0$ . Pick a cross edge  $pq$  such that the hop-distance (i.e., the number of edges) from  $p$  to the root is maximum among all cross edges. Let  $p_l$  be the leftmost and  $p_r$  the rightmost descendant of  $p$ .

**Proposition 4.2** *The interval  $[p_l, p_r]$  contains only descendants of  $p$ .*

*Proof.* Since  $p_l$  and  $p_r$  each have a path to  $p$ , the interval  $[p_l, p_r]$  is covered by edges that begin in proper descendants of  $p$ . Thus, if  $[p_l, p_r]$  contains a point  $z$  that is not a descendant of  $p$ , then  $z$  would be covered by an edge  $p_1p_2$  with  $p_1$  a proper descendant of  $p$ . Thus,  $p_1p_2$  would be a cross edge with larger hop-distance to the root, despite the choice of  $pq$ .  $\square$

Let  $R$  be the points in  $((p, q))$  that are not descendants of  $p$ . Each point in  $R$  is either unrelated to  $p$ , or it is an ancestor of  $p$ . Let  $z \in R$  be the point in  $R$  that is closest to  $p$  (i.e.,  $z$  either lies directly to the left of  $p_l$  or directly to the right of  $p_r$ ). We now describe how to construct a new valid assignment  $\hat{N}$ , from which we will eventually derive a contradiction to the choice of  $\tilde{N}$ . The construction is as follows: replace the edge  $pq$  by  $pz$ . Furthermore, if (i)  $q \rightsquigarrow_{\tilde{N}} z$ ; (ii) the last edge  $z'z$  on the path from  $q$  to  $z$  crosses the interval  $[p_l, p_r]$ ; and (iii)  $z'z$  is not a cross-edge, we also change the edge  $z'z$  to the edge that connects  $z'$  to the closer of  $p_l$  or  $p_r$ .



**Proposition 4.3**  $\widehat{N}$  is a valid assignment.

*Proof.* We must show that all points in  $G_P(\widehat{N})$  can reach the root. At most two edges change:  $pq$  and (potentially)  $z'z$ . First, consider the change of  $pq$  to  $pz$ . This affects only the descendants of  $p$ . Since  $z$  is not a descendant of  $p$ , the path from  $z$  to the root does not use the edge  $pq$ , and hence all descendants of  $p$  can still reach the root. Second, consider the change of  $z'z$  to an edge from  $z'$  to  $p_l$  or  $p_r$ . Both  $p_l$  and  $p_r$  have  $z$  as ancestor (since we introduced the edge  $pz$ ), so all descendants of  $z'$  can still reach the root.  $\square$

**Proposition 4.4** We have  $I(N^*) = I(\widehat{N})$ .

*Proof.* Since the new edges are shorter than the edges they replace, each transmission range for  $\widehat{N}$  is contained in the corresponding transmission range for  $\widetilde{N}$ . The interference cannot decrease since  $N^*$  is optimal.  $\square$

**Proposition 4.5** We have  $C(\widehat{N}) < C(\widetilde{N})$ .

*Proof.* First, we claim that  $\widehat{N}$  contains no new cross edges, except possibly  $pz$ : suppose  $ab$  is a cross edge of  $G_P(\widehat{N})$ , but not of  $G_P(\widetilde{N})$ . This means that  $((a, b))$  contains a point  $x$  with  $x \rightsquigarrow_{\widetilde{N}} a$ , but  $x \not\rightsquigarrow_{\widehat{N}} a$ . Then  $x$  must be a descendant of  $p$  in  $G_P(\widetilde{N})$  and in  $G_P(\widehat{N})$ , because as we saw in the proof of Claim 4.3, for any  $y \in P \setminus [p_l, p_r]$ , we have that if  $y \rightsquigarrow_{\widetilde{N}} a$ , then  $y \rightsquigarrow_{\widehat{N}} a$ .

Hence,  $((a, b))$  and  $[p_l, p_r]$  intersect. Since  $ab$  is a cross edge, the choice of  $pq$  now implies that  $[p_l, p_r] \subseteq ((a, b))$ . Thus,  $z$  lies in  $[[a, b]]$ , because  $z$  is a direct neighbor of  $p_l$  or  $p_r$ . We claim that  $b = z$ . Indeed, otherwise we would have  $z \rightsquigarrow_{\widetilde{N}} a$  (since  $ab$  is not a cross edge in  $G_P(\widetilde{N})$ ), and thus also  $z \rightsquigarrow_{\widehat{N}} a$ . However, we already observed  $x \rightsquigarrow_{\widehat{N}} p$ , so we would have  $x \rightsquigarrow_{\widehat{N}} a$  (recall that we introduce the edge  $pz$  in  $\widehat{N}$ ). This contradicts our choice of  $x$ .

Now it follows that  $ab = az$  is the last edge on the path from  $p$  to  $z$ , because if  $a$  were not an ancestor of  $p$ , then  $ab$  would already be a cross-edge in  $G_P(\widetilde{N})$ . Hence, (i)  $a$  is an ancestor of  $q$ ; (ii)  $az$  crosses the interval  $[p_l, p_r]$ ; and (iii)  $az$  is not a cross edge in  $\widetilde{N}$ . These are the conditions for the edge  $z'z$  that we remove from  $\widetilde{N}$ . The new edge  $e$  from  $a$  to  $p_l$  or  $p_r$  cannot be a cross edge, because  $ab$  is not a cross edge in  $G_P(\widehat{N})$  and  $e$  does not cover any descendants of  $p$ .

Hence,  $G_P(\widehat{N})$  contain no new cross-edges, except possibly  $pz$  which replaces the cross edge  $pq$ . By construction,  $\|p - z\| < \|p - q\|$ , so  $C(\widehat{N}) < C(\widetilde{N})$ .  $\square$

Propositions 4.3–4.5 yield a contradiction to the choice of  $\widetilde{N}$ . It follows that we must have  $C(\widetilde{N}) = 0$ , as desired.  $\square$

Let  $P \subset \mathbb{R}$ . We say that a valid assignment  $N$  for  $P$  has the *BST-property* if the the following holds for any vertex  $p$  of  $G_P(N)$ : (i)  $p$  has at most one child  $q$  with  $p < q$  and at most one child  $q$  with  $p > q$ ; and (ii) let  $p_l$  be the leftmost and  $p_r$  the rightmost descendant of  $p$ . Then  $[p_l, p_r]$  contains only descendants of  $p$ . In other words:  $G_P(N)$  constitutes a binary search tree for the (coordinates of the) points in  $P$ . A valid assignment without cross edges has the BST-property. The following is therefore an immediate consequence of Lemma 4.1.

**Theorem 4.6.** *Every  $P \subset \mathbb{R}$  has an optimal valid assignment with the BST-property.*  $\square$

## 4.2 A Quasi-Polynomial Algorithm

We now show how to use Theorem 4.6 for a quasi-polynomial time algorithm to minimize the interference. The algorithm uses dynamic programming. A subproblem  $\pi$  for the dynamic program consists of four parts: (i) an interval  $P_\pi \subseteq P$  of *consecutive* points in  $P$ ; (ii) a root  $r_\pi \in P_\pi$ ; (iii) a set  $I_\pi$  of *incoming interference*; and (iv) a set  $O_\pi$  of *outgoing interference*.

The objective of  $\pi$  is to find an optimal valid assignment  $N$  for  $P_\pi$  subject to (i) the root of  $G_N(P_\pi)$  is  $r$ ; (ii) the set  $O_\pi$  contains all transmission ranges of  $B_{P_\pi}(N)$  that cover points in  $P \setminus P_\pi$  plus potentially a transmission range with center  $r_\pi$ ; (iii) the set  $I_\pi$  contains transmission ranges that cover points in  $P_\pi$  and have their center in  $P \setminus P_\pi$ . The interference of  $N$  is defined as the maximum number of transmission ranges in  $B_{P_\pi}(N) \cup I_\pi \cup O_\pi$  that cover any given point of  $P_\pi$ . The transmission ranges in  $O_\pi \cup I_\pi$  are given as pairs  $(p, q) \in P^2$ , where  $p$  is the center and  $q$  a point on the boundary of the range.

Each range in  $O_\pi \cup I_\pi$  covers a boundary point of  $P_\pi$ . Since it is known that there is always an assignment with interference  $O(\log n)$  (see [12] and Observation 5.1), no point of  $P$  lies in more than  $O(\log n)$  ranges of  $B_P(N^*)$ . Thus, we can assume that  $|I_\pi \cup O_\pi| = O(\log n)$ , and the total number of subproblems is  $n^{O(\log n)}$ .

A subproblem  $\pi$  can be solved recursively as follows. Let  $A$  be the points in  $P_\pi$  to the left of  $r_\pi$ , and  $B$  the points in  $P_\pi$  to the right of  $r_\pi$ . We enumerate all pairs  $(\sigma, \rho)$  of subproblems with  $P_\sigma = A$  and  $P_\rho = B$ , and we connect the roots  $r_\sigma$  and  $r_\rho$  to  $r_\pi$ . Then we check whether  $I_\pi, O_\pi, I_\sigma, O_\sigma, I_\rho,$  and  $O_\rho$  are *consistent*. This means that  $O_\sigma$  contains all ranges from  $O_\pi$  with center in  $A$  plus the range for the edge  $r_\sigma r_\pi$  (if it does not lie in  $O_\pi$  yet). Furthermore,  $O_\sigma$  may contain additional ranges with center in  $A$  that cover points in  $P_\pi \setminus A$  but not in  $P \setminus P_\pi$ . The set  $I_\sigma$  must contain all ranges in  $I_\pi$  and  $O_\rho$  that cover points in  $A$ , as well as the range from  $O_\pi$  with center  $r_\pi$ , if it exists and if it covers a point in  $A$ . The conditions for  $\rho$  are analogous.

Let  $N_\pi$  be the valid assignment for  $\pi$  obtained by taking optimal valid assignments  $N_\sigma$  and  $N_\rho$  for  $\sigma$  and  $\rho$  and by adding edges from  $r_\sigma$  and  $r_\rho$  to  $r_\pi$ . The interference of  $N_\pi$  is then defined with respect to the ranges in  $B_{P_\pi}(N_\pi) \cup I_\pi$  plus the range with center  $r_\pi$  in  $O_\pi$  (the other ranges of  $O_\pi$  must lie in  $B_{P_\pi}(N_\pi)$ ). We take the pair  $(\sigma, \rho)$  of subproblems which minimizes this interference. This step takes  $n^{O(\log n)}$  time, because the number of subproblem pairs is  $n^{O(\log n)}$  and the overhead per pair is polynomial in  $n$ .

The recursion ends if  $P_\pi$  contains a single point  $r_\pi$ . If  $O_\pi$  contains only one range, namely the edge from  $r_\pi$  to its parent, the interference of  $\pi$  is given by  $|I_\pi| + 1$ . If  $O_\pi$  is empty or contains more than one range, then the interference for  $\pi$  is  $\infty$ .

To find the overall optimum, we start the recursion with  $P_\pi = P$ ,  $O_\pi = I_\pi = \emptyset$  and every possible root, taking the minimum of all results. By implementing the recursion with dynamic programming, we obtain the following result.

**Theorem 4.7.** *Let  $P \subset \mathbb{R}$  with  $|P| = n$ . The optimum interference of  $P$  can be found in time  $n^{O(\log n)}$ .  $\square$*

Theorem 4.7 can be improved slightly. The number of subproblems depends on the maximum number of transmission ranges that cover the boundary points of  $P_\pi$  in an optimum assignment. This number is bounded by the optimum interference of  $P$ . Using exponential search, we get the following theorem.

**Theorem 4.8.** *Let  $P \subset \mathbb{R}$  with  $|P| = n$ . The optimum interference  $\text{OPT}$  for  $P$  can be found in time  $n^{O(\text{OPT})}$ .  $\square$*

## 5 Further Structural Properties in One Dimension

In this section, we explore further structural properties of optimal valid receiver assignments for one-dimensional point sets. It is well known that for any  $n$ -point set  $P$ , there always exists a valid assignment  $\tilde{N}$  with  $I(\tilde{N}) = O(\log n)$ . Furthermore, there exist point sets such that any valid assignment  $N$  for them must have  $I(N) = \Omega(\log n)$  [12]. For completeness, we include proofs for these facts in Section 5.1. In Section 5.2, Below we show that there may be an arbitrary number of left-right turns in an optimal solution. To the best of our knowledge, this result is new, and it shows that in a certain sense, Theorem 4.6 cannot be improved.

### 5.1 Nearest Neighbor Algorithm and Lower Bound

First, we prove that we can always obtain interference  $O(\log n)$ , a fact used in Section 4.2. This is achieved by the *Nearest-Neighbor-Algorithm* (NNA) [7, 12]. It works as follows.

At each step, we maintain a partition  $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$  of  $P$ , such that the convex hulls of the  $S_i$  are disjoint. Each set  $S_i$  has a designated sink  $r_i \in S_i$  and an assignment  $N : S_i \rightarrow S_i$  such that the graph  $G_{S_i}(N_i)$  is acyclic and has  $r_i$  as the only sink. Initially,  $\mathcal{S}$  consists of  $n$  singletons, one for each point in  $P$ . Each point in  $P$  is the sink of its set, and the assignments are trivial.

Now we describe how to go from a partition  $\mathcal{S} = \{S_1, \dots, S_k\}$  to a new partition  $\mathcal{S}'$ . For each sink  $r_i \in S_i$ , we define the successor  $Q(r_i)$  as the closest point to  $r_i$  in  $P \setminus S_i$ . We will ensure that this closest point is unique in every round after the first. In the first round, we break ties arbitrarily. Consider the directed graph  $R$  that has vertex set  $P$  and contains all edges from the component graphs  $G_{S_i}(N_i)$  together with edges  $r_i Q(r_i)$ , for  $i = 1, \dots, k$ . Let  $S'_1, S'_2, \dots, S'_{k'}$  be the components of  $R$ . Each such component  $S'_j$  contains exactly one cycle, and each such cycle contains exactly two sinks  $r_a$  and  $r_{a+1}$ . Pick  $r'_j \in \{r_a, r_{a+1}\}$  such that the distances between  $r'_j$  and the closest points in the neighboring components

$S'_{j-1}$  and  $S'_{j+1}$  are distinct (if they exist). At least one of  $r_a$  and  $r_{a+1}$  has this property, because  $r_a$  and  $r_{a+1}$  are distinct. Suppose that  $r'_j = r_a$  (the other case is analogous). We make  $r_a$  the new sink of  $S'_j$ , and we let  $N'_j$  be the union of  $r_{a+1}Q(r_{a+1})$  and the assignments  $N_i$  for all components  $S_i \subseteq S_j$ . Clearly,  $N'_j$  is a valid assignment for  $S'_j$ . We set  $\mathcal{S}' = \{S'_1, \dots, S'_{k'}\}$ . This process continues until a single component remains.

**Observation 5.1** *The nearest neighbor algorithm ensures interference at most  $\lceil \log n \rceil + 2$ .*

*Proof.* Since each component in  $\mathcal{S}$  is combined with at least one other component of  $\mathcal{S}$ , we have  $k' \leq \lfloor k/2 \rfloor$ , so there are at most  $\lceil \log n \rceil$  rounds.

Now fix a point  $p \in P$ . We claim that in the interference of  $p$  increases by at most 1 in each round, except for possibly two rounds in which the interference increases by 2. Indeed, in the first round, the interference increases by at most 2, since each point connects to its nearest neighbor (the increase by 2 can happen if there is a point with two nearest neighbors). In the following rounds, if  $p$  lies in the interior of a connected component  $S_i$ , its interference increases by at most 1 (through the edge from  $r_i$  to  $Q(r_i)$ ). If  $p$  lies on the boundary of  $S_i$ , its interference may increase by 2 (through the edge between  $r_i$  and  $Q(r_i)$  and the edge that connects a neighboring component to  $p$ ). In this case, however,  $p$  does not appear on the boundary of any future components, so the increase by 2 can happen at most once.  $\square$

Next, we show that interference  $\Omega(\log n)$  is sometimes necessary. We make use of the points sets  $P_i$  constructed in Section 5.

**Corollary 5.2** *For every  $n$ , there exists a point set  $Q_n$  with  $n$  points such that every valid assignment for  $N$  has interference  $\lceil \log n \rceil$ .*

*Proof.* Take the point set  $P_{\lceil \log n \rceil}$  from Section 5 and add  $n - 2^{\lceil \log n \rceil}$  points sufficiently far away. The bound on the interference follows from Proposition 5.3.  $\square$

## 5.2 Bends

In Theorem 4.6 we proved that there always exists an optimal solution with the BST-property. Now, we will show that the structure of an optimal solution cannot be much simpler than that. Let  $P \subset \mathbb{R}$  be finite and let  $N$  be a valid receiver assignment for  $P$ . A *bend* in  $G_P(N)$  is an edge between two non-adjacent points. We will show that for any  $k$  there is a point set  $Q_k$  such that any optimal assignment for  $Q_k$  has at least  $k$  bends.

For this, we inductively define sets  $P_0, P_1, \dots$  as follows. For each  $P_i$ , let  $\ell_i$  denote the diameter of  $P_i$ .  $P_0$  is just the origin (and  $\ell_0 = 0$ ). Given  $P_i$ , we let  $P_{i+1}$  consist of two copies of  $P_i$ , where the second copy is translated by  $2\ell_i + 1$  to the right, see Fig. 4. By induction, it follows that  $|P_i| = 2^i$  and  $\ell_i = (3^i - 1)/2$ .

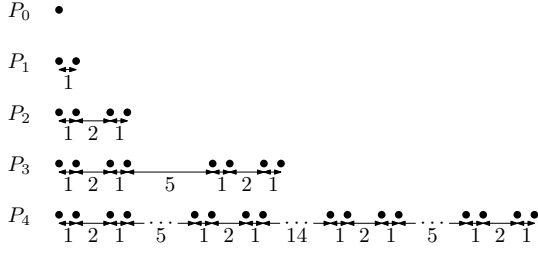


Fig. 4. Inductive construction of  $P_i$ .

**Proposition 5.3** *Every valid assignment for  $P_i$  has interference at least  $i$ .*

*Proof.* The proof is by induction on  $i$ . For  $P_0$  and  $P_1$ , the claim is clear.

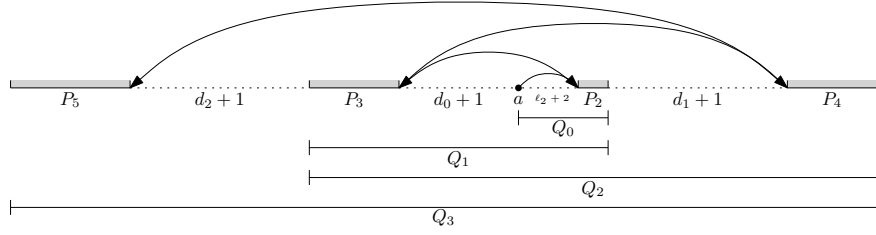
Now consider a valid assignment  $N$  for  $P_i$  with sink  $r$ . Let  $Q$  and  $R$  be the two  $P_{i-1}$  subsets of  $P_i$ , and suppose without loss of generality that  $r \in R$ . Let  $E$  be the edges that cross from  $Q$  to  $R$ . Fix a point  $p \in Q$ , and let  $q$  be the last vertex on the path from  $p$  to  $r$  that lies in  $Q$ . We replace every edge  $ab \in E$  with  $a \neq q$  by the edge  $aq$ . By the definition of  $P_i$ , this does not increase the interference. We thus obtain a valid assignment  $N' : Q \rightarrow Q$  with sink  $q$  such that  $I(N) \geq I(N') + 1$ , since the ball for the edge between  $q$  and  $R$  covers all of  $Q$ . By induction, we have  $I(N') \geq i - 1$ , so  $I(N) \geq i$ , as claimed.  $\square$

**Lemma 5.4.** *For  $i \geq 1$ , there exists a valid assignment  $N_i$  for  $P_i$  that achieves interference  $i$ . Furthermore,  $N_i$  can be chosen with the following properties: (i)  $N_i$  has the BST-property; (ii) the leftmost or the rightmost point of  $P_i$  is the root of  $G_{P_i}(N_i)$ ; (iii) the interference at the root is 1, the interference at the other extreme point of  $P_i$  is  $i$ .*

*Proof.* We construct  $N_i$  inductively. The point set  $P_1$  has two points at distance 1, so any valid assignment has the claimed properties.

Given  $N_i$ , we construct  $N_{i+1}$ : recall that  $P_{i+1}$  consists of two copies of  $P_i$  at distance  $\ell_i + 1$ . Let  $L$  be the left and  $R$  the right copy. To get an assignment  $N_{i+1}$  with the leftmost point as root, we use the assignment  $N_i$  with the left point as root for  $L$  and for  $R$ , and we connect the root of  $R$  to the rightmost point of  $L$ . This yields a valid assignment. Since the distance between  $L$  and  $R$  is  $\ell_i + 1$ , the interference for all points in  $R$  increases by 1. The interferences for  $L$  do not change, except for the rightmost point, whose interference increases by 1. Since  $|L| \geq 2$ , the desired properties follow by induction. The assignment with the rightmost point as root is constructed symmetrically.  $\square$

The point set  $Q_k$  is constructed recursively.  $Q_0$  consists of a single point  $a = 0$  and a copy  $R_2$  of  $P_2$  translated to the right by  $\ell_2 + 1$  units. Let  $d_{k-1}$  be the diameter of  $Q_{k-1}$ . To construct  $Q_k$  from  $Q_{k-1}$ , we add a copy  $R_{k+2}$  of  $P_{k+2}$ , at distance  $d_{k-1} + 1$  from  $Q_k$ . If  $k$  is odd, we add  $R_{k+2}$  to the left, and if  $k$  is even, we add  $R_{k+2}$  to the right; see Fig. 5.



**Fig. 5.** The structure of  $Q_3$ . The arrows indicate the bends of an optimal assignment.

**Theorem 5.5.** *We have the following properties: (i) the diameter  $d_k$  is  $(3^{k+3} - 2^{k+3} - 1)/2$ ; (ii) the optimum interference of  $Q_k$  is  $k + 2$ ; and (iii) every optimal assignment for  $Q_k$  has at least  $k$  bends.*

*Proof.* By construction, we have  $d_0 = 9$  and  $d_k = 2d_{k-1} + 1 + \ell_{k+2}$ , for  $k \geq 1$ . Solving the recursion yields the claimed bound.

In order to prove (ii), we first exhibit an assignment  $N$  for  $Q_k$  that achieves interference  $k + 2$ . We construct  $N$  as follows: first, for  $i = 2, \dots, k + 1$ , we take for  $R_i$  the assignment  $N_i$  from Lemma 5.4 whose root is the closest point of  $P_i$  to  $a$ . Then, we connect  $a$  to the closest point in  $R_2$ , and for  $i = 2, \dots, k + 1$ , we connect the root of  $R_i$  to the root of  $R_{i+1}$ . Using the properties from Lemma 5.4, we can check that this assignment has interference  $k + 2$ .

Next, we show that all valid assignments for  $Q_k$  have interference at least  $k + 2$ . Let  $N$  be an assignment for  $Q_k$ . Let  $p$  be the leftmost point of  $R_{k+2}$ , and let  $q$  be the last point on the path from  $p$  to the root of  $N$  that lies in  $R_{k+2}$ . We change the assignment  $N$  such that all edges leaving  $R_{k+2}$  now go to  $q$ . This yields a valid assignment  $\tilde{N}$  for  $R_{k+2}$  with root  $q$ . Thus,  $I(\tilde{N}) \geq k + 2$ , by Proposition 5.3. Hence, by construction,  $I(N) \geq I(\tilde{N}) \geq k + 2$ , since  $d_k \geq \ell_{k+2}$ .

For (iii), let  $N$  be an optimal assignment for  $Q_k$ . We prove by induction that the root of  $N$  lies in  $R_{k+2}$ , and that  $N$  has  $k$  bends, all of which originate outside of  $R_{k+2}$ . As argued above, we have  $I(N) = k + 2$ . As before, let  $p$  be the leftmost point of  $R_{k+2}$  and  $q$  the last point on the path from  $p$  to the root of  $G_{Q_k}(N)$ . Suppose that  $q$  is not the root of  $N$ . Then  $q$  has an outgoing edge that increases the interference of all points in  $R_{k+2}$  by 1. Furthermore, by constructing a valid assignment  $\tilde{N}$  for  $R_{k+2}$  as in the previous paragraph, we see that the interference in  $N$  of all edges that originate from  $P_{k+2} \setminus q$  is at least  $k + 2$ . It follows that  $I(N) \geq k + 3$ , although  $N$  is optimal.

Thus, the root  $r$  of  $N$  lies in  $R_{k+2}$ . Let  $b$  be a point outside  $R_{k+2}$  with  $N(b) \in R_{k+2}$ . The outgoing edge from  $b$  increases the interference of all points in  $Q_k \setminus R_{k+2}$  by 1. Furthermore, we can construct a valid assignment  $\hat{N}$  for  $Q_k \setminus R_{k+2}$  by redirecting all edges leaving  $Q_{k-1}$  to  $b$ . By construction,  $I(\hat{N}) \leq k + 1$ , so by (ii),  $\hat{N}$  is optimal for  $Q_{k-1}$  with interference  $k + 1$ . By induction,  $\hat{N}$  has its root in  $R_{k+1}$  and has at least  $k - 1$  bends, all of which originate outside  $R_{k+1}$ . Thus,  $b$  must lie in  $R_{k+1}$ . Since  $b$  was arbitrary, it follows that all bends of  $\hat{N}$  are also bends of  $N$ . The edge from  $b$  in  $N$  is also a bend, so the claim follows.  $\square$

## 6 Conclusion

We have shown that interference minimization in two-dimensional planar sensor networks is NP-complete. In one dimension, there exists an algorithm that runs in quasi-polynomial time, based on the fact that there are always optimal solutions with the BST-property. Since it is generally believed that NP-complete problems do not have quasi-polynomial algorithms, our result indicates that one-dimensional interference minimization is probably not NP-complete. However, no polynomial-time algorithm for the problem is known so far. Furthermore, our structural result in Section 5 indicates that optimal solutions can exhibit quite complicated behavior, so further ideas will be necessary for a better algorithm. In two dimensions naturally approximation algorithms (or approximation lower bounds

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